AN INDEX OF MATHEMATICAL TABLES

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FOREWORD

I am glad to have this opportunity of expressing my appreciation of the energy and initiative which have resulted in the collection and organization of the very large amount of information contained in this *Index of Mathematical Tables*, and in its preparation for publication under the difficulties and distractions of war-time conditions.

This *Index* will be of great value in a wide variety of investigations, both to users and to makers of mathematical tables. Its value to users of tables is plain; its value to makers of tables is no less, in showing what has been done and so avoiding duplication, and by giving sources of already available information which may be used to shorten the production of new tables. Further, it is to be hoped that by making clear just what are the main functions of which tables are at present inadequate or entirely lacking, it may stimulate work in the direction of filling the main gaps.

Thus the authors of this *Index* by their industry, by the wide range of their search, and by their presentation of the information they have gained, and Dr. Comrie by his enterprise in undertaking the publication of this *Index*, have placed all concerned with any but the most elementary mathematical tables in their debt, and I express to them the hope that the results of their labour and their care will have the wide recognition and success which they deserve.

D. R. HARTREE

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1945 February

PUBLISHER'S PREFACE

My interest was naturally aroused when I learned that the three authors of this *Index* were working on a subject dear to my own heart—one on which I have laboured for a quarter of a century. I realised immediately that, with their combined mathematical knowledge, their team work, and the opportunities for research that come a little more easily in universities than to one responsible for a team of thirty computers busily engaged on war work, they had the means of bringing to early fruition what might take me a decade. I therefore resolved to offer every assistance in my power. Of the nature of that assistance the authors themselves have spoken in generous terms. It consisted mainly in seeing that the standard of printing was up to the standard of authorship. Photographic reproduction of typescript was rejected from the beginning, and a four-figure bill for typesetting faced without hesitation.

In one respect I have failed. The manuscript was sent to the printers in the summer of 1943, under a promise of quick handling. But the printers fell on evil days: they lost most of their staff, and they had to accept a large quota of government work to keep even the staff who remained. In spite of all my urgings, and several personal visits to Edinburgh, what we expected to take a few months took more than two years, although the standard of composition was never allowed to fall. The consequences of this regrettable but uncontrollable delay were serious. Firstly the *Index* was not available to war workers apart from some in a limited circle; secondly the incorporation of some of the new material which became available while the *Index* was in the press necessitated costly alterations and corrections.

Should there be found any who complain about the price of this book, let them be informed that the price of any book is a function (although not a tabulated one to be found in this *Index*) of the cost of production and of the estimated sales. Here the former is high, and the latter, from the specialised nature of the subject, cannot be expected to be high. Indeed the publisher faces no prospects other than financial loss, which he does calmly with the knowledge that he has served science by helping this work to see the light of day in its present dignified form.

The more I saw of this *Index*, the greater became my admiration for the meticulous care that had been lavished on it, and the discriminating judgment shown in every minute particular. Even with my own knowledge and experience of tables, I found it difficult to detect flaws or suggest improvements. As a lover of tables, I doff my hat to the authors.

Naturally the *Index* will need revision in a few years, the more so as it has just come to our knowledge that continental workers have been busy on similar compilations, although none (as far as I know) has reached the press yet. The incentive to face revision and reprinting will come largely from the reception afforded to this first edition—and in particular from those who value the work sufficiently to have a copy on their own desks rather than rely on more remote library copies.

L. J. COMRIE

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1945 August

AUTHORS' PREFACES

I hope that this *Index* will be a useful servant to the many scientists the world over who are working to build up a better civilization. To them this work is respectfully dedicated.

This book was not designed to be encyclopædic; it was hoped that it would be a working tool for the working scientist. Most tables of only historical interest have therefore been omitted. The scope, construction and arrangement are fully explained in the Introduction and the explanation need not be repeated here. In view of the magnitude of our undertaking, I trust that readers will be generous and point out corrections and supply us with information that might profitably be included in future issues.

The general outline of this undertaking was conceived in times of peace. It was felt then, as now, that the growth of the application of mathematics to industry runs parallel with the solving of partial differential equations and the numerical tabulation of their solutions. The book was planned in a leisurely way and a certain amount of compilation was carried out. With the outbreak of war the development of the work received a serious blow, but it continued under difficult conditions. I was called away for full-time war service and have not been able to contribute much to the subsequent growth. Fletcher and Miller, however, with a singleness of mind that rises above any praise I can give, steadily continued the accumulation and painstaking ordering of information. They continued their work despite lecturing to classes swollen much beyond their peace-time size, the distraction of civil defence duties and other difficulties. They have, however, the satisfaction of knowing that their work has already considerably expedited the completion of several important scientific investigations.

Nevertheless, in spite of the efforts of Fletcher, Miller and myself, this book would never have been published without the helpful and kindly support of many people who have hovered in the wings. Firstly there is Dr. L. J. Comrie, of Scientific Computing Service, who was even more emphatic than we were about the necessity for immediate printing, and backed his opinion by financing the cost of publication. He has been a tower of strength. I feel that we have been very fortunate in finding a publisher who is himself an expert in the subject and a typographical enthusiast into the bargain. The number of asterisks in Part II has been considerably reduced by his kindness in lending us many volumes from his very valuable private collection of tables and from that of Scientific Computing Service. I should also like to place on record our indebtedness to the University of Liverpool for the many facilities offered to us, particularly in the Harold Cohen Library. The Library of the Liverpool University Mathematical Laboratory, dedicated to the memory of our one-time friend and colleague, Dr. S. F. Grace, has been invaluable.

At one time it seemed possible that this work would have been published during the war, when Ministry of Supply sanction for publication would have been necessary. This permission is no longer required, but I should like to record my appreciation of the helpful interest shown in this undertaking by some senior members of the Civil Service.

I should like to place on record, too, our indebtedness to our distinguished collaborator, Professor D. R. Hartree, F.R.S., for his Foreword.

Lastly, may I add a special word of praise for the printers, Messrs. Neill & Co., of Edinburgh, for the excellence of their work.

L. R.

AUTHORS' PREFACES

Some mathematical functions whose applications to science and technology are well known have been extensively and even excessively tabulated. Others whose potential value is less widely realized are tabulated in books or periodicals not easily accessible. We hope that this *Index* will be of value in indicating where information may be obtained. A complete classification is well-nigh impossible, but enough has been done, we feel, to justify publication. Suggestions for the improvement of the work will be gladly received.

It is with grateful thanks that we acknowledge the invaluable help of the staff of the Harold Cohen Library of the University of Liverpool. In particular, we are greatly indebted to Mr. K. Povey and Mr. H. G. Ward for much courteous help and friendly advice, and to Miss E. Whelan for her assiduity in locating and borrowing for us from other libraries many of the works listed in Part II.

Our principal indebtedness outside Liverpool has been to the facilities which, as has been mentioned above, Dr. Comrie placed at our disposal; the *Index* is an altogether fuller and better work than it could have been without his unstinting assistance. We have also made substantial use, both in person and by post, of the Library of the Royal Astronomical Society, and we offer our best thanks to Miss E. Wadsworth for her efficient help in this regard. Leisure to visit other libraries has been regrettably small, but we contrived visits to the Cambridge University Library, to the great benefit of our undertaking. We have also availed ourselves occasionally of the privilege of drawing upon notes made by Dr. Comrie in the Crawford Library of the Royal Observatory, Edinburgh.

It is also our pleasant duty to thank all who have kindly lent us tables or drawn our attention to them; in particular, Dr. W. G. Bickley, Dr. C. E. P. Brooks, Mr. C. R. Cosens, Dr. A. T. Doodson, Dr. R. O. Griffith, Mr. S. Johnston, Professor E. H. Neville, Mr. D. H. Sadler, Dr. A. J. Thompson and Mr. M. J. Tickle.

It is not possible to acknowledge adequately the encouragement and help which we have received from our co-author at all stages of the work; without Professor Rosenhead's collaboration it is certain that the *Index* would not have seen the light of day. We must also express our gratitude to our colleagues in the Department of Applied Mathematics, Dr. C. Strachan and Mr. S. D. Daymond. The former while in charge of a depleted Department has made every effort to allow us adequate opportunity of completing the work, while the latter gave timely help in copying out manuscript. For valuable co-operation in the early stages of sorting out material and of compilation we are indebted to a former pupil, Mr. C. W. Jones, M.Sc.

Up to this point our Preface stands nearly as we finished it in December 1943, some months after the rest of the original manuscript had been sent to the publishers. During the passage of the book through the press we have had various opportunities of verifying or correcting, and even to some extent of adding to, the information contained in it.

We wish especially to supplement the acknowledgments made in our Introduction to *Mathematical Tables and other Aids to Computation* (as this quarterly is now entitled) and to Professor R. C. Archibald. Only two numbers of the periodical had appeared when type-setting of our book commenced. In the meantime we have received seven more numbers, including a 104-page special number (No. 7, 1944 July) containing *A Guide to Tables of Bessel Functions*, by H. Bateman and R. C. Archibald; this constitutes the second instalment of the American scheme begun (see page 3 of our Introduction) with D. H. Lehmer's *Guide to Tables in the Theory of Numbers* (1941).

Other publications have naturally appeared during the passage of the book through the press, and it seems desirable that our readers should have some indication of the

way in which we have dealt with the editorial difficulties arising from so copious an accession to sources of information at a late stage in our own work.

We have pointed out in the Introduction that it would be almost impossible to make an index of mathematical tables "complete" to any particular date. Certainly we have not tried to do so; our Bibliography contains over 2000 items, but we have consciously excluded about a thousand more. It may, however, be useful to state that, as it was found desirable to draw a line somewhere, we have ignored everything published after the end of 1944. The more important pieces of information reaching us from any source in 1943 and 1944, sometimes in relation to works of earlier date, have been incorporated, but often very briefly; for example, by a mere reference to an item in Part II or to a page in *M.T.A.C.* Our object has been to keep our *Index* up to date in essentials and at least to put our readers on the track of all important non-secret tables known to us, without embarrassing our publishers and printers by undue disturbance of type; even so, we must express our appreciation of the latitude allowed to us, especially in the Bessel Function sections of Part I and in the Bibliography in Part II. In some instances it has seemed best to place additional information at the end of a section; in such cases the articles so supplemented are marked with a dagger (†), whose significance is recalled by a footnote on its first appearance in the sections; see, for example, Section 5.

The first proofs of our Bessel Function sections and of Part II were sent to Professor Archibald, and we wish to record our gratitude to him for offering a number of very helpful suggestions, including some which assisted in resolving acute bibliographical difficulties.

We have also been able to utilize, though necessarily for a limited time, additional library resources during the passage of the book through the press.

It is a particular pleasure to acknowledge the assistance so freely given to one of us in the University Library and the Mathematical Institute of the University of Edinburgh, and in the Library of the Royal Society of Edinburgh. While both modern and old works were examined, we cannot refrain from particular allusion to the latter. In connection with the history of mathematical tables the principal libraries of London, Cambridge and Oxford have been well worked over by such writers as De Morgan, Glaisher and Henderson. In spite of some use in books on, for example, Napier and the history of logarithms, material existing in the treasure-houses of Scotland has been comparatively neglected, although its richness and importance are beyond question. Should any extensive new treatment of the history and bibliography of old mathematical tables be undertaken in this island, it is much to be hoped that it will rest on northern as well as southern documentation.

It is also a pleasure to acknowledge the help received by one of us in examining a number of volumes in the Libraries of Trinity College, Cambridge, and of the Royal Society.

Several visits to libraries were facilitated by grants, for which we tender our sincere thanks, from the Joint Committee on Research of the University of Liverpool.

We are indebted to Mrs. Fletcher for invaluable assistance in the construction of the Index to Part I and in the proof-reading of the entire volume, as well as for copying out part of the final manuscript.

A. F.
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1945 June

CONTENTS

INTRODUCTION

The tabulation of mathematical functions in substantially the modern manner may be said for practical purposes to have begun with the tables of natural trigonometrical functions made in the sixteenth century and the tables of logarithms constructed in the first half of the seventeenth century. During approximately the first quarter of the nineteenth century tables of transcendental functions, such as the error integral, gamma function, elliptic integrals and Bessel functions, began to be made, and from that time the number of different functions tabulated has steadily grown. More recently the whole subject of tabulation, and indeed of numerical mathematics in general, has been revolutionized by the growth of the use of calculating machines. Machines of early types were occasionally used to assist tabulation in the nineteenth century; now the machine has become the primary means of calculation, and few extensive tables can have been made in the last twenty-five years without mechanical aid.

The effect of this mechanization has been profound. It has permitted the execution of calculations that previously could never have been undertaken. As far as method of computation is concerned, it has brought about a greatly increased use of natural, as opposed to logarithmic, values of functions. It has also made easier the adequate checking of tables, for instance by the method of differencing. Perhaps the most important distinction between the best recent tables of elementary functions and their predecessors of some three centuries ago lies in the far greater reliability of the modern tables. It is true that by the middle of the last century the slow and laborious process of continual re-editing had enabled the ordinary seven-decimal table of common logarithms of five-figure numbers to be cleared of error, but this was a case in which special efforts had been made. While the avoidance of all error is never to be achieved without care and vigilance, it is not a superhuman task with modern equipment to make an error-free table, to considerably more than seven figures if more are required.

The exploitation of modern mechanical resources has been taken up principally by Comrie and by the British Association Mathematical Tables Committee in England, by Peters in Germany, by Davis and his team and, most recently, by the New York W.P.A. workers in America. Since their first publication in 1939, the latter, a body of arithmeticians some hundreds strong, working under the ægis of the Work Projects Administration for the City of New York, have poured out important fundamental tables of both elementary and transcendental functions at an unprecedented rate. Recently, however, the number of W.P.A. workers has been drastically reduced.

In these circumstances the user of mathematical tables, be he mathematician, astronomer, physicist, engineer, bio-mathematician, or other calculator, is liable to find the existing sources of information about mathematical tables (if indeed he knows of their existence) more or less out of date. In general he may be tempted to think that the older a table is, that is, the more inaccessible and inaccurate it probably is, the easier it is to find information about it. The inquirer will in many cases fail to discover that a reliable table has been published in the last ten or twenty years and is still in print.

In describing briefly the chief existing sources of information, we shall make references to works by means of the list given in Part II, which should be consulted for further details. First, there are several useful articles on mathematical tables which, unfortunately in some respects, are contained in general encyclopædias. These range from De Morgan *1842b*, *1846*, through the still useful De Morgan

I

1861, to Glaisher 1888 and the important Glaisher 1911*b*, and the less extensive Henderson 1929. Glaisher's monumental report in B.A. 1873 deals almost entirely with tables of elementary functions, but it is invaluable for the older tables of this kind. Some of the more important are also described in Knott 1915, while Mehmke 1902 and Mehmke–d'Ocagne 1909 have been found very useful. A working list of tables is given by H. Bell and J. R. Milne in Horsburgh 1914. Errata in this are given in Comrie 1929*a*. Various indexes more restricted in subject or period exist. Thus on logarithms we have the invaluable Henderson 1926, as well as Glaisher 1911*a* and a number of earlier works, such as the well-known Hutton 1811 and several somewhat inaccessible papers by Bierens de Haan. Mathematical works on special functions give varying amounts of information about tables, ranging from the full account of tables of Bessel functions in Watson 1922 down to brief references. Von Mises 1928 gives details of a certain number of tables of Bessel, elliptic, and Legendre functions. Davis 1933, 1935 give accounts of earlier tables in the fields with which these volumes are specially concerned, as well as general lists of references. Comrie 1929 gives a working list of tables of elementary functions, and useful information is given in his reports (Comrie 1925, 1932*a*). Archibald 1933–36 and *M.T.A.C.* have been found of great use. Mathematical information about both books and papers may frequently be found in the *Jahrbuch über die Fortschritte der Mathematik* and similar abstracts, and bibliographical information in the usual library sources.

It will be seen that a considerable mass of information awaits some sort of unified treatment. A complete account would be a very large task. In 1939 one of us (A. F., at the suggestion of L. R.) made a rough subject index of the tables contained in the Library of the Mathematical Laboratory of the University of Liverpool, which contains a high proportion of recent publications. An existing card-index catalogue of Mathematical Laboratory tables compiled by J. C. P. M. was also supplemented by cards made in the first place from references to a number of the more important and modern tables given in various places, e.g. in the bibliographies mentioned above. It was then decided to endeavour to construct a working index of tables, primarily on the basis of the Mathematical Laboratory Library and the Harold Cohen Library of the University of Liverpool, in which a fair amount of general mathematical literature was available. Initially the collaborators were L. R., A. F. and J. C. P. M. It was hoped that existing bibliographies could be relied upon to miss few items up to about 1930; a number of periodicals were searched for items after that date. The departure of L. R. on national service in the summer of 1940, coupled with the increased demands on A. F. and J. C. P. M. in respect of ordinary university duties, dimmed the prospects of quick progress; on the other hand the need for an index was not diminished, and in October 1940, at L. R.'s request, A. F. and J. C. P. M. agreed, notwithstanding the difficulties, to try to complete at any rate the most vital parts of the original scheme. The field of mathematical tables had already been divided into 24 sections, not differing greatly from those finally adopted in Part I, and now A. F. and J. C. P. M., with the assistance of Mr. C. W. Jones, went through the entries in the card index and noted on a separate sheet for each section the items which apparently concerned that section. The various sections were then divided between A. F. and J. C. P. M. for compilation; each section was to be compiled by one of us and criticized by the other. A preliminary version of Part II was also duplicated for use in giving references in the form there adopted.

In the course of the work it appeared that a certain number of references had failed to be picked up in the initial process, since acquaintance with fresh literature was continually bringing others to light. It has not been possible to make a search of all the literature available, but it is believed that few items in the chief English

mathematical periodicals published since 1850 have escaped inclusion. It would in any case have been impossible to make the *Index* complete; while most of the more important modern American, German and French tables were available in Liverpool, and others could be borrowed, a number of, for the most part, less important ones were inaccessible; German and French works could not be ordered, and even American ones could be obtained only with difficulty and delay. Nevertheless it seemed well worth while to proceed with the work, and indeed a "complete" index, if such could be made, would be very unwieldy. None of the indexes mentioned above is complete, but many of them had been found very useful within their limitations of range or date, and it seemed unlikely that a comprehensive and up-to-date index, deliberately selective in parts and doubtless suffering from occasional unwitting omissions in others, would fail to be of real help for the tasks of the moment, particularly since there was every prospect that the percentage of first-hand information given would be fairly high.

In the middle of the work it was learned that in the United States a "Committee on Mathematical Tables and Aids to Computation" of the National Research Council had adopted a large plan, involving the search of many books and series of periodicals by numerous collaborators. As it was not expected that the whole American scheme could be quickly carried out, and as in any case the plan was unlikely to coincide with ours, it was resolved to persevere with the production of a working index. The portion of the American scheme which deals with the theory of numbers has been completed, much in advance of the rest, and the resulting volume (D. H. Lehmer 1941) will be found invaluable in this special field, with which our own *Index* is little concerned. In January 1943 the same committee commenced publication of a valuable quarterly journal entitled *Mathematical Tables and Aids to Computation* (see Part II).

In the spring of 1942, Dr. L. J. Comrie, who was not then aware of our efforts, began work on a proposed *Computer's Guide to Mathematical Tables*. We learn from him that his conception of Part I was similar in principle to ours, but that in his Part II, which would be as large as this volume, he intended to give descriptions, and often illustrations, of the more important tables. A special feature would have been an assessment of the accuracy and reliability of each table, with lists of errors and the results of checkings and comparisons, a subject to which Dr. Comrie has devoted much attention during the last 20 years, and on which he is a recognized authority. We realize that we have not exhausted all that can usefully be said about the tables in our *Index*, and that there is ample room for the individual description of many tables, which is beyond the scope of our work.

The work of compiling the various sections of Part I was eventually completed, and the initial list of references was combined with two supplementary lists to form one list in Part II. Before going on to describe the arrangement of the *Index*, it seems advisable to say a few words about its scope.

The *Index* here presented is an index of *mathematical* tables. Tables containing experimentally determined numbers, even as physical and other constants, have almost always been excluded. Tables of functions capable of a purely mathematical definition and occurring in a physical or similar context have been included when they seemed to be of some mathematical interest. Weight has, however, been given to the importance of the physical or other problem involved; thus some account has been given of the combinations of complete elliptic integrals which occur in connection with the mutual inductance of coaxial circles and the self-inductance of cylindrical coils, while a number of combinations of the same integrals which occur in various less standard connections have been referred to only very briefly. In the case of some of the more standard tables of dynamical astronomy, such as tables of elliptic

motion, which are of a purely mathematical character but are hardly likely to occur in other connections, and tables of parabolic motion, which may or may not be in purely mathematical form, we have refrained from giving an account in Part I, but we have allowed the details about a number of works containing such tables to stand in Part II, and we have given a list of them later in this Introduction.

While the *Index* is concerned mainly with mathematical tables, some attention has been given to other matter relating to numerical mathematics. For example, Section 5 gives an extensive account of available values of mathematical constants, while Section 24 gives some information about numerical harmonic analysis. We have, however, not collected information about graphical methods. We give later in this Introduction a few lists of works dealing with general numerical methods, with particular branches such as the numerical solution of differential equations, with calculating machines, with statistics, and so on. It should be added that we have often included details about books on computational or nearly related mathematics, when papers on the same subject would have been omitted.

We have included old tables only when they seemed to be still of interest. Details about many which we have omitted may be found in B.A. 1873 and other sources already mentioned. Those which we have included are mainly such as will be familiar, if only by reputation, to all who are interested in tables. Most of them have now been largely superseded. Perhaps only Briggs & Gellibrand 1633, which has never been equalled, and a copy of which the Mathematical Laboratory is fortunate enough to possess, is still of great working value. But other tables of the same period, such as those of Rheticus, Pitiscus, Briggs and Vlacq, remained standard for some three centuries, and are the basis of smaller tables which are still common; others, such as Napier 1614, are of great historic interest; some tables of moderate age, such as the numerous editions of Hutton 1811, are very common in this country, and contain one or two tables worth notice.

It is hoped that the above will give a fairly clear idea of the scope of the *Index*. The field covered is not susceptible to thoroughly precise definition. While many tables had a clear right to be included, there were a number of border-line cases. It is no uncommon thing for the author of a scientific paper to give a short table of some purely mathematical function which occurs in the course of the investigation; in many cases such are to be considered rather as applications of certain mathematical tables than as fresh tables meant for further use. Our judgment on questions of inclusion has necessarily been ultimately personal and subjective; we have included items when we have felt, as workers with some experience of computation and with interest in most branches of numerical mathematics, that it would be useful to have a record of the information given, taking into account such things as the nature (basic or derivative) of the function tabulated, the number of values given, the number of figures or decimals given, the availability of other tables, and the degree of difficulty involved in computing the table. We have not lightly rejected minor tables published during the last few decades, because we have felt that a table of any importance at all should be given its place in the literature of the subject and brought to general notice; it can easily be dropped by later workers.

It is hoped that any lack of uniformity in dealing with border-line items will be found by users of the *Index* to be more of the nature of occasional inclusion of items which might have been omitted without serious loss than a failure to record useful information.

While the index according to functions given in Part I is the cardinal feature of the present work, we have felt that the list of references in Part II will also be found of use, and we have gone to considerable trouble to make it as reliable as possible. Here we have included items for very varied reasons, occasionally without intending

to use the items in Part I. It has been found a great convenience to have a concise reference list of books and papers containing tables or dealing with such things as errors in particular tables or the bibliography of tables, particularly in the cases of recent items and of older items not to be found in existing lists in which they would naturally be sought. Thus we have occasionally included items for such diverse reasons as that existing lists omitted them or gave incomplete or erroneous information about them; that they were liable to be confused with other more important works by the same author, and that the statement of the various dates and titles would serve to differentiate the works; that in general, if an author has contributed, say, half a dozen works dealing definitely with tables, it is a convenience to have a list of them, even though some may be less important than others; that if a foreign table which is inaccessible appears from its title, extent, author, etc. to be likely to be of interest, it is better to keep it on record, even though for the moment we may have to content ourselves with giving merely bibliographical information about it. The bibliography of mathematical tables is perhaps more difficult than that of many other branches of mathematics. Tables of a purely mathematical character may appear in various kinds of scientific and engineering periodicals, or as independent works from engineering and nautical presses, or from small general presses which do not undertake the setting up of general mathematical works. While the percentage of such works contained in our *Index* is not large, the items are sometimes of interest, and when a systematic search or an accident of browsing has led us to a little-known table, we have considered in many cases that it would be undesirable to reject it. There is always a chance that some odd item may escape even the wide American net previously mentioned!

We now proceed to describe in detail the arrangement of the work. It will be convenient to begin with Part II, which gives an alphabetical list of references that is used in Part I and in this Introduction. The general principle is that each work is referred to by the name of the author or authors and the year of publication; when two or more works of the same authorship are published in the same year, they are distinguished by letters *a*, *b*, . . . placed after the year, e.g. Aitken 1933*a*, Aitken 1933*b*. In the case of books, titles or brief titles and place of publication are given; names of publishers are usually given for modern books, but are frequently omitted in the case of books now out of print—there may be some lack of uniformity here, the names of publishers of old books having sometimes been thought worth giving when they were at hand, but not worth hunting for. As far as periodicals are concerned, the references are mostly self-explanatory; place of publication has been given only in the case of a few somewhat obscure periodicals. Works of joint authorship have normally been listed under the name of the first author, cross-references being given under the names of other authors whenever it seemed at all likely that the item might be sought under their names. An asterisk marking an item denotes that we have not seen the work, so that any information given about it, in either Part I or Part II, is second-hand.

It is necessary to make some provision in the case of books, etc. published in more than one edition. Whether we have seen the work or not, we have always indicated the year of publication of the last edition known to us, and have tried to give the year of publication of the first edition. The bibliographical resources of the Harold Cohen Library in respect of at least English, German, French and recent American books are considerable, and we have rarely been compelled to omit information concerning the first edition; a few nautical and other technical publications have been exceptions, and we have had occasional difficulties in dealing with publications of other countries.

In choosing the primary or reference date, with which an item commences,

we have varied our practice according to circumstances. If the primary date is not followed by an asterisk, we have seen the edition of that date, and any information given in Part I refers to that edition, unless the contrary is stated. Except in a very few cases, we do not indicate whether or not we have seen other editions, and whether or not any table mentioned in Part I is contained in other editions. We merely give the dates of certain other editions, as a matter of convenience; there is no absolute presumption that they contain the same information, though they often do; in a few cases we have been able, and have thought it worth while, to warn that they do not. In a few cases, of which the tables of Barlow and of Jahnke and Emde are the chief, where the various editions differ greatly and are all of importance, we have listed them as separate works.

If the primary date is followed by an asterisk, we have not seen any edition, and the date belongs to the edition to which, in view of our second-hand information, it seems best to refer. Any information given in Part I relates to what we understand to be the contents of that edition. In general we give no information about the contents of other editions; any user of this *Index* who has access to another edition will, of course, see whether it contains what he requires.

It may be useful to state that where the bracket containing information about other editions is placed before the place of publication (and publisher, if given), it is implied that the place of publication, etc. refers also to the other editions. Compare, for example, in Part II, the items:

Brunt 1931, where both editions were published at Cambridge by the Cambridge University Press;
Hutton 1811, where all editions were published at London;
Elderton 1938, where a publishing change has occurred.

In giving effect to the general principles explained above, we have had occasional difficulty over authors and more frequent difficulty over dates. In respect of authors, we started with the wish to index tables under the names of their actual computers, but there are so many cases in which tables have been computed by one person and included in a work by another, and the difficulty of keeping track of actual computers is so great, that in the end we have probably most often indexed under the name of the author of the larger work; sometimes we have added the name of the actual computer in brackets in making references in Part I, sometimes we have given a cross-reference under the names of actual computers in Part II. Thus the table computed by Holland, Jones and Lamb is so usually and conveniently referred to by the name of Perry that we have called it Perry 1891; in Part II we have given the names of the computers, in the main item and in a cross-reference; the three names would be too bulky for convenient insertion in the tabular matter of Part I. In the case of Wolfram's table, a very important one with which the name of Wolfram must certainly be associated in some way, we have given an item Wolfram in Part II, but in Part I we have found it more convenient to describe the two eighteenth-century publications as Schulze 1778 (Wolfram) and Vega 1794 (Wolfram). In the case of Davis 1933, 1935 the number of computers is too large to make it desirable to give their names in Part II, and only occasionally have we added them in brackets in Part I. In the case of tables published in the reports of the British Association Committee, we have given a separate item in Part II where it is abundantly clear that a table is by one definite person; thus we have given an item Watson 1916*b*, which will be useful to anyone who may remember the table as "Watson's table" and wish to look up the reference; but in Part I we have referred to B.A. 1916 (Watson), though Watson 1916*b* would also have been correct. In another type of case we have referred simply to B.A. 1911, and so on, the omission of any name implying

that the authorship is not amenable to brief description, or that we have thought it safest to regard the Committee as jointly responsible, even though the table may probably be mainly the work of one man. In general, in dealing with British Association matter, we have usually found it best to give references in Part I in forms such as B.A. 1919 (for the Committee's reports) and B.A. 1, 1931 (for the separate volumes of mathematical tables), adding a name in brackets on occasion. Glaisher's monumental report in B.A. 1873 is so well known that we have not usually cumbered the reference with his name; the reader will continually meet references to B.A. 1873, meaning the report of Glaisher (and of the other members of the Committee, as far as they contributed).

It seems unnecessary to do more than give the above indications of our attitude towards name-problems; we may not have been completely consistent, but it is hoped that the cross-references will make it easy to look up references in the comparatively few awkward cases.

The dating of references has given us rather more trouble. Our original plan was to give the year of publication of each table, with distinguishing letters in the case of publications of the same authorship in the same year. The year of publication is evidently the most important date connected with a mathematical table; the influence of a general mathematical paper may commence when it is read at a meeting of a society, but a table is not usable by the mathematical public until the printed version is issued.

We have, however, made conscious modifications in two ways, affecting a few tables. In the first place, there are a few publications to which years are so conveniently attached in a certain way that it would cause intolerable awkwardness or even confusion to refer to them by year of publication. Thus our references to British Association reports are dated by the year of meeting (up to some 12 years ago publication usually occurred in the following year) and our references to the *Nautical Almanac* are dated by the year for which ephemerides are given (publication occurs a year or two earlier).

In the second place, if we attributed a year to a publication by an author, and subsequently decided to refer to another publication of the same year by the same author, it was not always convenient to go through many sections of Part I, adding say the letter *a* to the year in every reference to the first table. Thus in some cases we have, for example, two items 1930 and 1930*a*, instead of 1930*a* and 1930*b*. Having done this a few times in the above way, we eventually used the notation deliberately in a few cases; thus two new items by the same author in the same year might be called 1930 and 1930*a*, if the former were much more important than the latter. It might have been better to have used this device from the beginning.

With the above exceptions, we have tried to keep to our original scheme, but in a number of cases we have found the year of publication doubtful. The difficulty arises to only a small extent with books. We refer to a well-known set of tables as Dale 1903; the preface is dated December 1903 and publication would appear from the *English Catalogue of Books* to have occurred in January 1904 (the year 1905 mentioned as that of first publication in at least one impression of the book itself is a misprint); we have accepted the date usually given in the literature of the subject. Our copy of New York W.P.A. 1940*a* has 1939 on the title-page and 1940 on the binding; we find that the New York workers list this as a publication of 1939, but we have not thought it necessary to rearrange a large number of references. Other difficulties of the same kind need not detain us here. We have not thought it profitable to introduce any special notation for works published round about the change of the year. In the last resort, the year given is one which we have chosen, perhaps with some arbitrariness, for the purpose of making references. In one or two cases where

fresh information came to hand at a late stage, we have added a note about it in Part II, without altering our reference date.

A more serious difficulty has been found in the case of some papers in periodicals. We have given the actual year of publication of the paper whenever we could, but in some cases we have had to use the date given on the title-page of the completed volume, although the part containing the paper may have been published in an earlier year. Possible errors of one year arising in this way were eliminated as far as was possible without undue expenditure of effort, but a number must remain. Errors of several years were possible in the case of papers published in a few periodicals which complete volumes at rather wide intervals, and here we did our best (not always with success) to find other evidence of date of publication. It may be added that even at the present time there are periodicals which give information about date of publication which is inadequate, or, worse, seriously misleading.

We now proceed to describe the arrangement of Part I, which is the central feature of the present *Index*. This consists of 24 sections, each devoted to tables of a particular group of functions.

Each section consists of a number of articles, and each article deals with tables of one function or of a strictly limited group of functions. The articles are numbered in a decimal notation, the integral part being the number of the section. In any section the articles which have the same digit in the first decimal place may be regarded as forming a sub-section, and we have usually given a list of these sub-sections in the introduction to the section. See, for example, the introductory article, 10·0, of Section 10.

The majority of the articles consist essentially of lists of tables which give the same function (or sometimes very closely related functions). Modifications have had to be made in some articles, but in general the arrangement is as follows. The order of the items is normally that of the number of decimals or figures given. Each item normally occupies one line, and gives, in order from left to right, information about:

(1) number of decimals or figures,
(2) interval and range of argument,
(3) facilities for interpolation,
(4) authorship and date, in the form used in Part II.

We proceed to consider these in turn.

It has usually seemed more convenient to arrange items in order of number of decimals or figures than in order of date or smallness of interval, and we need not dwell on this point. We have distinguished carefully between numbers of decimal places and numbers of significant figures; thus a table of common logarithms is almost invariably to a given number of decimals (dec. or d.), while a table of common antilogarithms is almost invariably to a given number of figures (fig. or f.). Thus our usual beginning of a line is something like, say, "7 dec." or "5 fig." When it has been necessary to save space, we have used "d." instead of "dec." and "f." instead of "fig."

There are a number of points to notice in this connection. We have not usually taken account of dots and similar devices for indicating half-units in the last place (i.e. for giving about an extra third of a place). Where, say, high and low dots are used to give a distinction between three types of rounding off, we have often thought it worth while to count this as an extra half-place; in particular, all the tables by Milne-Thomson and Comrie in MT.C. 1931 which possess this feature have been described as, say, "4½ dec." or "4½ fig." A most important point, namely our use of the semicolon, as in "4; 5 d.", will be explained later.

We have tried to make our descriptions of numbers of decimals or figures as accurate as possible without sacrificing conciseness. At the same time, we have had to disregard minutiæ. Thus some authors dislike changing the number of decimals in the middle of a column or line, so that when they are in principle tabulating to a fixed number of significant figures, they give a different number (usually one more) for part of a column or line round each change. Or a function may be tabulated to a given number of significant figures, except that fewer figures may be given in the immediate neighbourhood of a zero of the function. A few table-makers tend to be somewhat erratic in number of places given, without apparent reason; if a mainly 10-decimal table has a small proportion of 9-decimal values, we normally use "9–10 dec." or "about 10 dec.", but if the proportion is very small indeed we may say simply "10 dec."

We have very occasionally (chiefly in dealing with tables of natural tangents and secants in Section 7) used, for example, "8:" in relation to a table which we have not seen, to denote that the table is reputed to be, broadly speaking, an 8-figure one, but that we have no information whether the number of decimals, or the number of figures, or neither, is kept constant as the function tends, say, to infinity.

Where all the values given for a function are exact, we have usually said so. If we state a number of decimals, it does not rule out the possibility that many of the values may be exact. See for example Art. 16·11, where obviously nearly all the approximate tables mentioned give $P_2(x)$ exactly, and some give further Legendre polynomials exactly. When values are given to the nearest integer, we have usually said this, though on occasion we have used "0 dec." for conciseness. In a few places we have even used "-2 dec." to denote that values are given to the nearest hundred, and so on.

When values such as $0^\circ{\cdot}01$, 1^g, $1'$, $0''{\cdot}1$, $0^m{\cdot}1$, $0^s{\cdot}01$, etc. stand in the first column, the table in question gives values to hundredths of a degree, to grades, to minutes of arc, to tenths of a second of arc, to tenths of a minute of time, to hundredths of a second of time, etc.

Expressions such as, for example, "3 dec. or 4 fig." mean that the table is to 3 decimals, except that, when the function is greater than 10, only 4 figures are given. We might also say that the table is to 4 figures, except that, when the function is less than 1, only 3 decimals are given. This expression is particularly useful in dealing with double-entry tables; in dealing with single-entry tables, it is often worth while to state in detail the variation of precision with argument, in the manner to be described shortly.

If an author tabulates, say, $100y$ to the nearest integer, we have normally indexed the table as one of y to 2 decimals, without further explanation. In general, we do not guarantee that it may not occasionally be necessary to change the position of the decimal point, in either argument or function or both, in order to make the table and our description of it agree.

Occasionally we have used expressions such as "30+ dec." to denote that at least 30 decimals are given throughout the table, or part of a table, under consideration. The greater the number of decimals, the less profitable it is to go into details if the facts are not simple.

We now turn to the consideration of the column, normally the second, which gives details about the argument of the table. In the simplest case, where there is a constant interval throughout an obvious range, we have simply stated the interval. See for example Art. 7·22, where "1°" means that tangents are given for every degree of the quadrant, "$0^\circ{\cdot}01$" that they are given for every hundredth of a degree throughout the quadrant, and so on. (The *arrangement* of the values may be either

"quadrantal" or "semi-quadrantal", but all the tangents mentioned are given somewhere—both tangents less than unity and tangents greater than unity.)

When we have wished to specify both interval and range, we have used the now common notation of enclosing the interval in brackets. Thus "1(1)100" means every integer between 1 and 100 inclusive, "$0(10'')45°(1°)90°$" means every multiple of 10″ from 0 to 45°, then every whole degree from 45° to 90°, and similarly in other cases. Where changes of interval are frequent, and we have not thought it worth while to give full details, we have used, for example, "0(1)100(10)1000(var.)1,000,000" to give details up to 1000 and then indicate that there are various other intervals up to 1,000,000. "1(1)100(primes)1000" or "1(1)100(primes)997" is used to indicate that the argument takes all integral values up to and including 100, then all prime values less than a thousand. Here "primes" does not denote an interval in quite the normal way, but the notation is unambiguous.

Undue importance must not be attached to the first and last arguments in our descriptions. Our descriptions of tables of sines and tangents are not influenced by such accidents as whether tabulators have or have not printed information about sines and tangents of zero and of 90°. In describing tables of Legendre polynomials $P_n(x)$ with argument x, where the function $P_1(x) = x$ hardly requires tabulation (though it sometimes achieves it!), we prefer to say, for example, "$n = 2(1)7$" or "To $n = 7$" rather than "$n = 1(1)7$"; but in other similar cases we may have included the trivial value of the parameter. On the other hand, a table giving the Bessel function $J_0(x)$ for $x = 0(\cdot 01)9\cdot 99$ would normally be so described, for $J_0(10)$ is not a trivial quantity, and $x = 0(\cdot 01)10$ would be somewhat misleading and little shorter.

Where there is no possible ambiguity about the argument, we do not always state it; we may say simply "1(1)100" instead of "$x = 1(1)100$".

Our use of the semicolon, alluded to above, has occurred mainly in connection with the first and second columns, and may be illustrated by quoting a line from the list of tables of tan x given in Art. 7·22:

8 dec.; 8 fig.	$0(0°\cdot 01)45°;\ 45°(0°\cdot 01)90°$	Δ	Buckingham 1935

Here the table gives 8 decimals from 0 to 45°, and 8 significant figures from 45° to 90°. Items will sometimes begin with such expressions as "4; 5 dec." or "5; 6 f." and so on. It is evident that the range of the argument may be divided into more than two parts in this way, and information given about the precision in each part. We have made much use of this mode of statement, but sometimes it has been necessary to allot separate lines to different parts of the range. It is clear that semicolons used in the above fashion usually occur in corresponding pairs, and cannot occur singly; if occasionally a semicolon does occur singly in some line, it is not used in the above sense. In Section 15 the semicolon is used systematically in another sense, there explained.

We now proceed to explain the symbols used in the column (usually immediately preceding the name of the author) giving information about means of interpolation. Very few comments need accompany the list of symbols placed at the end of this Introduction. When we use, say, the symbol Δ, we mean to imply that the table normally gives first differences of the function tabulated, but we do not necessarily imply that the differences are given either when they are very small (say a few units of the last place) or when the function is approaching infinity; occasionally we have thought it worth while to specify the point at which differences begin or cease to be given. It is particularly to be noted that "No Δ" is our way of indicating what might be called a plain table, giving simply values of a function corresponding to values of

the argument, without any differences, derivatives, etc. It is not meant to suggest that differences which would have been useful are lacking; sometimes the table is intended to be used without interpolation, or to be interpolated in a special manner. Tables of integral multiples of a quantity do not require differences. In some cases where differences would hardly be expected, we have made no statement about differences unless they are given; see, for example, Art. 7·11, where $\sin x$ and $\cos x$ with radian argument may evidently be used to interpolate one another. Where, however, a list of tables contains information about differences, and such information is lacking in the case of an item, the reason is usually that we have not seen the table mentioned, and that we lack reliable information about means of interpolation; whether or not we have seen the table may be ascertained by consulting Part II, in which, as noted previously, tables that we have not seen are marked with an asterisk. The semicolon convention has occasionally been used in connection with the interpolation column; see, for instance, Art. 7·11, where MT.C. 1931 gives a table of $\sin x$ and $\cos x$ with first differences for $x = 0(\cdot 001)1\cdot 57$, and one without differences for $x = 0(\cdot 1)7\cdot 9$.

It is believed that the list of symbols will make clear the notation adopted, and that any slight extensions which may have been made will cause no difficulty. Thus the reader will not be troubled if a table of natural tangents is described as giving "Δ to 87°" or "PPd (f to 70°)", or a double-entry table with arguments x and y is described as giving "Δ in x, δ^2 in y". In Art. 7·41 we hope that he will understand from:

17 dec.	9′	$v^6(10'')$, 0(18′)30°, 60°(18′)90°	Andoyer 1915

that Andoyer gives sines to 17 decimals at interval 9′ throughout the quadrant, with the first 6 variations for 10″ at alternate arguments from 0 to 30° and from 60° to 90°, but with no variations from 30° to 60°. In any case, such complications are rare. It should be added that brackets have very occasionally been used to indicate some slight qualification; thus (PP) when proportional parts are available, but not at the same opening as the table, and PP(d) when most, but not all, of the proportional parts give an extra decimal.

We now mention a few points concerning the columns giving authorship and date. Information has already been given about the construction of Part II, where the tables are listed in alphabetical order of authors. In many sections of Part I we have given page numbers for particular tables only when it has seemed likely that the information will be specially useful. In other cases it is thought that no difficulty will be experienced in finding the particular table referred to. Page numbers, when given, are enclosed in brackets; Hayashi 1926 (48) means page 48 of Hayashi 1926, and so on. In a few cases, e.g. where the same table occurs in different editions of a work with the same table number but on different pages, it has been more convenient to give the table number, preceded by "T." For brevity, arabic numerals have frequently been used in this connection; thus Norie 1904 (T. 36) means Table XXXVI of Norie 1904. We have already explained that names of actual computers are also occasionally added in brackets; thus Vega 1794 (Wolfram) refers to the well-known table by Wolfram as printed in Vega 1794.

Where a number of tables of one particular function exist, we have frequently printed some names of authors and dates in bold type as an indication of outstanding tables of their kind. The tables so emphasized are those which a computer in this country, possessing only the more important foreign tables, would probably regard as standard. This emphasis has been used particularly in respect of tables of elementary functions, such as logarithms and trigonometrical functions. These tables are so much used that most of them have "reputations" of some kind in

computing circles; in some cases the results of detailed examinations for errors have been published. This has made the selection of tables as outstanding much easier, since we ourselves have naturally been unable to undertake any extensive examination for errors. In the case of tables of transcendental functions we have sometimes felt justified in doing little more than selecting some classic table such as Legendre's large table of elliptic integrals, or one or two tables by computers of recognized eminence, such as Comrie, Peters, the British Association Tables Committee or the New York W.P.A. workers. We have had particular regard to what we believe to be the accuracy of tables selected for emphasis, though naturally a very extensive table, particularly if it is old, may contain a few errors. It is evident that the process of selection is not completely objective, and we have had a little doubt whether bold type should be used at all, but on the whole we think that users of the *Index* will welcome some indication, however imperfect, when they are confronted with a long list of tables. In some cases unemphasized tables may be thoroughly reliable; if we use ordinary type for them, it is occasionally because we have not seen them and know little about them; often because, although we have seen them, we know little or nothing about their accuracy; sometimes we may know them to be accurate, but prefer the arrangement of another equally accurate table; the tables may be of evidently limited scope, or they may be foreign tables possessing no evident superiority over easily-accessible English ones, and so on.

The inaccuracy of certain tables is well known to computers. We may mention particularly the tables of Hayashi, to the errors in which several writers have alluded. We have rarely felt justified in emphasizing by bold type the tables in Hayashi's works, although it may be that some of the individual tables are perfectly accurate. It is only fair to add that a number of Hayashi's tables are unique, and that it is often easier to check Hayashi's values than to make a complete recomputation. A record degree of inaccuracy appears to be reached in the two papers by Kotani, etc., where the 1940 paper gives errata occupying about 20 per cent of the space taken by the 1938 paper, and introduces fresh errors. Other seriously inaccurate tables are mainly old, and need no mention here.

One warning should be given, and that is that our lists should not be used without caution in historical matters. If we refer to two similar tables by different authors, with different dates, reference to Part II may show that we are quoting a later edition of the later dated table, and that the first edition of this preceded the publication of the other. Our information has been arranged for the computer, not for the historian. In a very few cases, where a table is frequently referred to by its year of first publication, we have given this date in brackets in Part I, as well as in Part II; see for example Bremiker 1924 (1852) in Art. 8·41.

Occasionally, where we normally refer to a work by the date of one edition, and have found that we wish to refer to another edition, we have had to use slightly awkward expressions such as "Maxwell 1892 (1881 edition)". We have very occasionally referred to a work by a secondary date when this is unambiguous.

We should perhaps say that one principle which we have followed in regard to including or rejecting items in any article is that some choice should be left when it exists. If similar tables are contained in several works, one version may be superior to the others; but the user of the *Index* may not have convenient access to the work in question, while one of the other versions may be at hand and quite adequate. It is important that lists of tables of such things as common logarithms and trigonometrical functions should not be allowed to become unduly long; on the other hand we have mentioned several 7-figure tables, several 6-figure tables, and so on, when there has been no difficulty in doing so. Further indications about the degree of selectivity of our lists are given in the appropriate places in sections or articles. In the case of

the transcendental functions, we have usually included most of the tables known to us; a few of the commoner tables have been somewhat widely reproduced in abbreviated form, and here some selection has been necessary. We have indexed practically all tables contained in standard collections such as those of Milne-Thomson & Comrie, Jahnke & Emde, and the later part of Dale, because we think that any user of this *Index* will wish to be reminded if a table which he requires is contained in volumes which he probably has at hand. In the case of collections which are less well known we have indexed only those tables which appear to call for mention on their own merits.

We conclude by giving references to works dealing with various aspects of practical mathematics. While our Part II is not exhaustive, and works have usually been included in it because they contain mathematical tables, it appears to be worth while to pick out the books dealing with certain topics. Only a very few papers are mentioned below.

We may perhaps begin by mentioning some collections of mathematical formulæ, synopses of mathematics, etc. Láska 1888–94 (extensive, but to be used with caution), Adams & Hippisley 1922, Silberstein 1923 and Workman 1912 are largely of this type. A number of formulæ may be found in **MT.C. 1931,** Potin 1925, and the various editions of Jahnke & Emde. See also Peters 1921*a* or 1921*b* and Hoüel 1901. Other references are:

Burington 1933	Carr 1886	Percival 1933
Bürklen 1936	Dwight 1934	Schulz 1937
Carmichael & Smith 1931	James & James 1943	

Handbooks for physicists, engineers, etc. containing a certain amount of mathematical information are to be found under Clark, Hodgman, Hütte, Kempe, Kent, Low, Merriman, Oberg & Jones, Peirce & Carver, Templeton, and Trautwine.

A number of the above collections include tables of integrals. Special collections of integrals are **Bierens de Haan 1867** (recently reprinted) and various memoirs by the same author, Petit-Bois 1906 and Peirce 1910. Older collections are Hirsch 1810 and Minding 1849. See also Allen 1941.

We now consider works dealing with numerical methods. The usual textbook in this country is **Whittaker & Robinson 1926.** Other books which may be of use are Freeman 1939, 1936, Scarborough 1930, Steffensen 1927, Runge & König 1924, Lindow 1928 and Sanden 1923. *N.A.* 1937 and several of the *Tracts for Computers* may also be consulted. Mehmke 1902, Mehmke-d'Ocagne 1909, Bennett, Milne & Bateman 1933 and Pearson 1920*a* contain numerous references. Recent developments in special fields are contained in various papers by Comrie, Aitken and Lidstone, and in Southwell 1940. Works of a more abstract character on finite differences are Boole 1860, **Milne-Thomson 1933** and Nörlund 1924, which contains a bibliography. Other references to works dealing wholly or partly with numerical methods or related theory are:

D'Adhémar 1911	C. Jordan 1939	Moors 1905
Andoyer 1906	A. Kowalewski 1917	Nyström 1925
Bauschinger 1901*b*	G. Kowalewski 1932	Oppolzer 1880
Biermann 1905	Lipka 1918	Radau 1891
Boccardi 1902	Lötzbeyer 1934	Rice 1899
Bruns 1903	Lüroth 1900	Runge & Willers 1915
Burkhardt 1904	Maccaferri 1919	Schrutka 1923
Cassina 1928	Mader 1928	T. N. Thiele 1909
Cassinis 1928	Markoff 1896	J. C. Watson 1868
Davis 1933, 1935	Mauderli 1906	Werkmeister 1929
Gibb 1915	Montessus 1911	Willers 1923, 1928

The numerical solution of differential equations is the main topic of **Bennett, Milne & Bateman 1933,** which contains a large number of references. It is also dealt with in Levy & Baggott 1934, in Andoyer 1923 (279), by F. R. Moulton in Adams & Hippisley 1922, in Runge & Willers 1915, and more briefly in a number of the books which treat of numerical methods in general. See also *N.A.* 1937 (1942 issue). The mechanical integration of differential equations (differential analyser, etc.) is described in Hartree 1938, which gives references to a number of other papers. See also Massey, etc. 1938, Rosseland 1939, Lennard-Jones, etc. 1939, Hartree 1940, Wilkes 1940 (with Mallock 1933) and Beard 1942.

We have included few papers on calculating machines and instruments in Part II. Information on the subject may be found in such works as Dyck 1892, 1893, Mehmke 1902, Mehmke-d'Ocagne 1909, Henrici 1910, Jacob 1911, Galle 1912, Horsburgh 1914, Turck 1921, d'Ocagne 1922, 1928, Martin 1925, Baxandall 1926, 1929*a*, 1929*b*, Lenz 1932, Couffignal 1933, 1938, Sabielny 1939; in Comrie 1928*a*, *b*, 1932*b*, *c*, 1933, 1934, 1936, 1942, 1944 and other papers by the same author; and to some extent in literature published by the various makers of calculating machines, and in Office Machines Research 1936–40.

We have not in general included books on nomography and other graphical methods in Part II. See, however, d'Ocagne 1899, Brodetsky 1925 and Allcock & Jones 1941.

As has been mentioned, a number of astronomical tables relating to elliptic and parabolic motion are contained in Part II. The standard collections of tables for dynamical astronomy are Bauschinger 1901 and the later edition, **Bauschinger-Stracke 1934.** Collections of some extent are also given in Crawford 1930, Oppolzer 1870, 1880 and J. C. Watson 1868. Other tables are contained in:

Åstrand 1890	Halley 1705, 1749, 1752	Peters–Stracke 1933
Barker 1757	Innes 1927	Sang (MS. 1848–90)
Bidschof & Vital 1905	Laplace–Bowditch 1829–39	Schlesinger & Udick 1912
Boquet 1920	Láska 1888–94	Stracke 1924, 1928
Burckhardt 1815	Leverrier 1855	Strömgren 1927
Gauss–Davis 1857	Møller 1932, 1940	Subbotin 1929
Gauss–Haase 1865	Olbers 1864 (Luther)	Tietjen 1892
Gauss–Schering 1871	Parkhurst 1889	Wirtz 1918

Astronomical tables of a highly special character (nearly-parabolic motion, solution of equations of Gauss and Lambert, etc.) have not been included in Part II, but a number are contained in Bauschinger–Stracke 1934 and other works mentioned above. Tables of Laplace coefficients are considered in Section 22.

The principal British general nautical tables are those of Burton, Inman, Norie and Raper. A considerable number of nautical tables are included in Part II, and are referred to in Section 9 and in parts of Sections 7 and 8.

In general we have not included physical and chemical tables. Such tables in the English language are Washburn 1926–33, Fowle 1933 and Kaye & Laby 1941. Other tables are Landolt & Börnstein 1923–36 and Marie 1912–. Some of the tables in Smithsonian Institution 1931 may also be found useful.

In general we have not included financial tables, but we have had Glover 1930 and Kent & Kent 1926 available, and have inspected Archer 1936. R. Todhunter 1931 contains short tables and a detailed list of larger financial tables.

Duodecimal tables of most standard mathematical functions are given in Terry 1938.

We conclude by mentioning briefly a number of tables and texts on statistical and related subjects contained in Part II. We have not indexed in Part I the more

technical kinds of statistical table, but some of the more standard kinds have been indexed, mostly in Sections 14 and 15.

We therefore mention a few collections of statistical tables. The two volumes of ***Tables for Statisticians and Biometricians*** **(Pearson 1930, 1931)** are a mine of numerical information. In them are reproduced most of the important tables first published in *Biometrika* up to about 1930. Other collections of statistical tables which are known or believed to be of some generality and extent are:

Carver 1934	Fisher & Yates 1938	Morton 1928
Davenport & Ekas 1936	Holzinger 1925*b*	Rietz 1924
Dunlap & Kurtz 1932		

Some of the textbooks mentioned later also contain useful sets of tables.

Part II also contains various special statistical tables which have not been indexed in Part I. We may instance the tables relating to the correlation coefficient in Soper, etc. 1917, Holzinger 1925*a* and David 1938, and the tables of random numbers in Tippett 1927 and Kendall & Smith 1939.

Accounts of the method of least squares may be found in general works such as Whittaker & Robinson 1926 and Brunt 1931, as well as in books devoted specially to the topic, such as Wright & Hayford 1906, Merriman 1910, Weld 1916, Leland 1921, Helmert 1924 and Deltheil 1930.

Part II contains a considerable number of books on probability and statistics in the wider sense. These range from treatises on fundamentals down to elementary works concerned mainly with applications to education, psychology, medicine and other subjects; some of the latter owe their inclusion to an accurate table of the error integral. Books on probability of an entirely non-mathematical kind have normally been excluded.

In view of the number of books on probability and statistics included in Part II, we limit ourselves to mentioning a few of the more important and recent works. Others may be found by examining Part II (many are referred to in Section 15 of Part I). More are contained in Buros 1938 and 1941, which consist of collections of excerpts from reviews.

An established general textbook is **Yule & Kendall 1937,** which gives a bibliography. Other standard books of their kind are R. A. Fisher 1937, 1941 and Elderton 1938. Among other modern works in English we may mention:

Aitken 1939	D. C. Jones 1924	Sasuly 1934
Bowley 1937	Kendall 1943	Simon 1941
Ezekiel 1930	Kenney 1940	Snedecor 1934, 1937
A. Fisher 1922	Levy & Roth 1936	Tippett 1937
Fry 1928	Peters & Van Voorhis 1940	Tschuprow 1939
Garrett 1937	Plummer 1940	Uspensky 1937
Goulden 1939	Rider 1939	Wilks 1943
Jeffreys 1939	Rietz 1924, 1927	Wolfenden 1942

LIST OF SYMBOLS CONCERNING MEANS OF INTERPOLATION

No Δ	Plain table, giving values of argument and of function only.
Δ	First differences.
Δ^2	First and second differences (second only would be δ^2).
Δ^3	First, second and third differences. Similarly for higher indices.
Δ^n	Differences, as far as needed; usually to different orders in different parts of the range.
Log Δ	Logarithms of first differences.
Δ for 1″	Differences for 1″. If the interval is 1′ throughout, we might also say $\Delta/60$.
IΔ	Indication of first difference (e.g. mean first difference for each line in a "block" table).
Δ, Δ'	First differences for each argument in a double-entry table.
δ^2	Second differences only (first and second would be Δ^2).
δ^4	Second and fourth differences. Similarly for higher even indices.
δ^{2n}	Even differences, as far as needed; usually to different orders in different parts of the range.
Iδ^2	Indication of second difference (e.g. mean second difference for each page or half-page).
δ^2, Iδ^4	Second differences and indication of fourth difference.
δ^2_m	Modified second differences. (See B.A. **1**, 1931 (xi) and *N.A.* 1937 (928), etc.)
δ^4_m	Second and fourth differences, some modified. (See B.A. **1**, 1931 (xi) and *N.A.* 1937 (928), etc.)
δ^{2n}_m	Even differences, as far as needed, some modified. (See B.A. **1**, 1931 (xi) and *N.A.* 1937 (928), etc.)
D	First derivatives.
D^2	First and second derivatives. Similarly for higher indices.
D^n	Derivatives, as far as needed.
v	First variations, i.e. wD = interval times first derivatives.
v^2	First variations wD and second variations $w^2D^2/2!$. Similarly for higher indices.
v^n	Variations $w^nD^n/n!$, as far as needed. Also called reduced derivatives.
Log v	Logarithms of first variations.
v(10″)	First variations for 10″ in table in which interval is different from 10″. For example, equal to $v/6$ if $w = 1'$.
v^2(10″)	First and second variations for 10″ ($\neq w$), e.g. $v/6$ and $v^2/36$ if $w = 1'$. Similarly for higher indices.
v^n(10″)	Variations for 10″, as many as needed, interval not equal to 10″.

PP	Proportional parts giving every tenth of some differences, rounded to nearest integer. (If tenths of all differences occurring in the table are given, they are fPP.)
PPd	Tenths of some differences, exact and therefore given to one decimal place. (The decimal point is sometimes omitted, the quantities given being the first nine multiples of differences.)
PP_{100}	Hundredths of some differences, rounded to nearest integer. Similarly PP_{20}.
fPP	Full proportional parts, all tenths of all differences (except perhaps very small ones) occurring in the table, rounded to nearest integer. (The condition of completeness is sometimes relaxed near an infinity of a function. See, for example, Art. 8·0.)
fPPd	Same, except that tenths are exact and therefore given to one decimal.
fPP_{100}	Hundredths of all differences, rounded to nearest integer.
$fPP_{100}d$	Same, except that hundredths are exact and therefore given to two decimals.
PPMD	Proportional parts of mean differences, tenths rounded to nearest integer. Usually $1(1)9\times MD/10$, sometimes only $1(1)5\times MD/10$. The mean difference is not necessarily integral.
PPdMD	Same, except that one decimal is given.
$PP_{15}MD$	Fifteenths of mean differences (e.g. when interval is $15''$, and interpolation to each $1''$ is provided for). Similarly PP_5MD, $PP_{60}MD$.
σ, τ	Auxiliary functions provided in natural trigonometrical tables. See Art. 7·0.
S, T	Auxiliary functions provided in logarithmic trigonometrical tables. See Art. 8·0.
Sp.	Special means of interpolation.

PART I. INDEX ACCORDING TO FUNCTIONS

SECTION 1

PRIMES, FACTORS, PRODUCTS AND QUOTIENTS

1·0. Tables of primes, factors and products are very numerous, and our lists are highly selective. A full report on tables of primes and factors, with lists of errata in the various tables, is given in D. H. Lehmer 1941, while B.A. 1873 may be consulted on the older tables, some of considerable extent, in all three categories.

The main subdivisions of this section are:

1·1 Lists of Primes
1·2 Factor Tables
1·3 Multiplication Tables, Proportional Parts
1·4 Decimal Values of Vulgar Fractions

1·1. Lists of Primes

D. N. Lehmer 1914 is the standard list. This and Cunningham 1904*a* are arranged in such a way that it is easy to find the *n*th prime for given *n*. Primes are easily picked out from many factor tables, e.g. B.A. 5, 1935. A guide to lists of high primes (beyond ten million) is given in D. H. Lehmer 1941 (38).

To 10,006,721		**D. N. Lehmer 1914**

D. H. Lehmer 1941 gives the following three errata in the above:
Page 11, column 13, for 628151 read 628051
Page 14, column 30, for 854651 read 854647
Page 99, column 20, heading, for 224 read 724

102,001–400,313	See also Art. 1·21	Vega–Hülsse 1849
102,001–400,031	,, ,,	Vega 1797
To 217,219		Rees 1819*c*
To 200,003		Poletti 1920
To 125,683	Below 100,000	**Cunningham 1904*a***
	Above 100,000	**Cunningham 1927**
To 101,977		Lambert 1770, 1798
To 100,103		Barlow 1814
To 30,341		James Glaisher 1879
To 25,409		**Cunningham, etc. 1922**
To 10,000		W. Greve 1911
To 4051		MT.C. 1931
To 3271		Beetle 1933

1·2. Factor Tables

For further references, lists of errors, etc., see **D. H. Lehmer 1941,** B.A. 5, 1935, Dickson 1919, D. N. Lehmer 1909, Cunningham 1904*b*, 1905 and Gram 1893.

1·21. Factor Tables (Complete Decomposition)

Up to 100,000 the standard work is B.A. 5, 1935, which is probably error-free.

To 1,020,000	Multiples of 2, 3, 5 omitted	Chernac 1811
To 256,000	No omissions	**Kaván 1937**

To 102,000	Multiples of 2, 3, 5 omitted	Vega–Hülsse 1849
To 102,000	,, ,, ,,	Vega 1797
To 100,390	,, ,, ,,	Gifford 1931
To 100,000	No omissions	**B.A. 5, 1935**
To 20,000	Multiples of 2, 5 omitted	**G. W. Jones 1893**

Only error, 4629 = 3 . 1543, not 3 . 15 . 43.

To 12,000	No omissions	Stager 1916
To 10,200	,, ,,	Brown & Sharpe 1933
To 10,192	Primes omitted	Peters & Stein 1922
To 10,000	No omissions	**B.A. 8, 1940**
To 6009	,, ,,	Buckingham 1935

Corrections: 5183 = 71 . 73, 5461 = 43 . 127.

Complete decompositions of $p - 1$, p being prime, are given in the following:

$100{,}000 < p < 125{,}683$	Cunningham 1927
$p < 100{,}000$	Cunningham 1904*a*
$p < 25{,}409$	Cunningham, etc. 1922

1·211. Factor Tables (All Divisors, Prime and Compound)

To 10,000	Anjema 1767

1·22. Factor Tables (Least Factor)

To 100,330,201	Multiples of 2, 3, 5 omitted	Kulik (MS. 1866)

D. H. Lehmer 1941 (27) mentions MS. tables by other authors for the 11th, 12th and 16th millions.

To 10,000,000	Multiples of 2, 3, 5, 7 omitted	**D. N. Lehmer 1909**
8,010,001–9,000,000	Multiples of 2, 3, 5 omitted	Dase & Rosenberg 1865
7,002,001–8,010,000	,, ,, ,,	Dase 1863
6,000,001–7,002,000	,, ,, ,,	Dase 1862
5,000,000–6,000,000	,, ,, ,,	James Glaisher 1883
4,000,000–5,000,000	,, ,, ,,	James Glaisher 1880
3,000,000–4,000,000	,, ,, ,,	James Glaisher 1879
2,028,000–3,036,000	,, ,, ,,	Burckhardt 1816, 1909
1,020,000–2,028,000	,, ,, ,,	Burckhardt 1814, 1909
To 1,020,000	,, ,, ,,	Burckhardt 1817, 1909
21,500–1,000,000	Multiples of 2, 3, 5, 11 omitted	Kulik 1825
To 115,500	Multiples of 2, 3, 5, 7 omitted	Hoüel 1864
To 100,000	Multiples of 2, 5 omitted	Brancker 1668
20,000–100,000	,, ,, ,,	Hinkley 1853
To 100,000	,, ,, ,,	Inghirami–Prompt 1919
10,000–100,000	,, ,, ,,	Brown & Sharpe 1933
To 99,000	Multiples of 2, 3, 5 omitted	**Carr 1886**

From Burckhardt. No errors known.

1·3. Multiplication Tables, Proportional Parts

Tables of proportional parts (i.e. with decimal multipliers) giving exact results have been treated as multiplication tables, the decimal point having been disregarded. Upper limits such as 100 have usually been described as 99, in order to show the numbers of digits which may normally be multiplied together.

Sexagesimal tables have been omitted here; see Arts. 9·8 ff.

See also squares, quarter-squares, etc. in Section 2, and triangular numbers in Art. 3·4.

For further references to multiplication tables, see De Morgan 1842*b*, 1861, B.A. 1873, Mehmke 1902 and Mehmke–d'Ocagne 1909. See also the items by Ernst, Leau, Neishuler, Nikitin, R. H. Smith and Zolotarev in Part II.

1·31. Tables giving 9 Multiples

To 9,999,999 × 9	Crelle 1836 (see Art. 1·32 for the usual Crelle)
To 99,999 × 9	Laundy 1865 } See also the authorities mentioned in Art. 1·3
To 9,999 × 9	Mazzotto 1934 }
To 999 × 9	**Peters 1930*a*, *b*,** Zoretti 1917, Burrau 1907
11(1)175 × 1(1)9	MT.C. 1931 (also on convenient loose card supplied with book)
10(1)99 × 1(1)9	Dale 1903

See also tables of proportional parts in many logarithmic and trigonometrical tables.

1·32. Tables giving 99 or more Multiples

To 999 × 999	**Crelle 1923,** Cotsworth 1927, Herwart 1610

Editions of Crelle before those in 1907–08 by O. Seeliger omit multiples of ten.

To 999 × 999	Vriesendorp 1937, Yano 1913

These omit multiples of ten.

To 999 × 999	H. C. Schmidt 1898, Henselin 1897

These give each product only once.

To 999 × 1009	Monlaur 1909

On arrangement see P. V. Neugebauer, *Vierteljahrsschrift d. Astr. Ges.*, **45**, 99, 1910.

To 9999 × 99	**Peters 1924,** L. Zimmermann 1923, Mendizábal y Tamborrel 1903, Riem 1897, Cadet 1802 (which omits multiples of ten after 1000)
To 4000 × 99	Petrick 1875–1879
To 999 × 99	**H. Zimmermann 1929, 1930,** L. Zimmermann 1926, Triebel 1934, C. A. Müller 1897, Kohlmann 1876 (see also Kohlmann 1900 in Part II), Raabe & Söhner 1910
To 99 × 99	Inman 1906, 1940, Unwin 1918
1(1)100 × 1(½)100	Triebel 1934
To 1000 × ·999 }	**H. B. Hedrick 1918**
To 100 × ·99 }	This gives results to the nearest integer
0(·01)1 × 0(10)990	Numerov 1922, which gives results to the nearest integer, with bivariate PPMD in square arrays of 9 rows and 9 columns

1·4. Decimal Values of Vulgar Fractions

Tables of this kind are not very numerous, and we have included most of those that have come to our notice. They have been divided into four groups:

1·41 General tables of N/D, arranged according to the (integral) values of N and D;

1·42 Tables of N/D in which N is rather restricted, e.g. tables of the first nine multiples of reciprocals;

1·43 Tables in which the decimal values of N/D (or of its logarithm) are arranged in order of magnitude in a single series, so that vulgar fractions approximating to a given quantity are easily found, as is useful in connection with gearing;

1·44 Tables of the complete recurring periods that arise when vulgar fractions are expressed as recurring decimals. Tables in this group can be used to continue decimal values found in tables in the other groups to any required number of places.

For reciprocals ($N = 1$) see Art. 2·4. There are numerous engineering and commercial tables for special divisors (e.g. $D = 12$, $D = 64$), but these rank as conversion tables of a kind which we have omitted.

1·41. General Tables of N/D

11 dec. 9 dec.	$\left.\begin{matrix} N < D \\ N > D \end{matrix}\right\} D = 2(1)120,\ N = 1(1)120$	**Page 1942**
8 dec.	$D < 100$, $N < 50$, $N < D$	Pearson 1934 (434)

Columns headed $\bar{x}$ give $p/(p + q)$, where p and q equal $\frac{1}{2}(\frac{1}{2})11(1)50$.

6 dec.	$D = 2(1)100$, $N <$ and prime to D	**Goodwyn 1818**

Means of completing the decimal are also provided (see Art. 1·44).

5 dec.	$D = 2(1)49$, $N <$ and prime to D	Wucherer 1796

Page 1942 also gives separate tables for each value of $N + D$ from 25 to 239. In each of these tables 6-decimal values of N/D (capable of extension by the table indexed above) are arranged in order of increasing N, with the limitations $12 \leq N \leq 120$, $12 \leq D \leq 120$.

See also Rauschelbach 1918 in Part II.

1·42. N/D, N rather restricted

20 dec. 12 dec.	$\left.\begin{matrix} N = 1 \\ N = 2(1)9 \end{matrix}\right\} D = 2(1)99$	Houzeau 1875*b*
13 dec.	$D = 1000(1)9999$, $N = 1(1)9$	**Picarte 1861**
6 dec.	$D = 50(1)999$, $N = 2(1)19$	Woo 1929, Pearson 1931 (16)

Columns headed $\bar{\eta}^2$ give $(n - 1)/(N - 1)$, where $N = 51(1)1000$, $n = 3(1)20$.

5 fig.	$D = 11(1)101$, $N = 10(10)90$	R. W. Burgess 1927
5 dec.	$D = 50(1)999$, $N = 1(1)10$	Wucherer 1796

1·43. Arrangements in Order of Magnitude

We distinguish between tables of decimal equivalents of N/D and tables of $\log_{10} N/D$. Normally N/D itself is given only in its lowest terms, but Page 1942 is noteworthy as giving all equivalent vulgar fractions in which $N \leq 120$, $D \leq 120$.

1·431. N/D in Order of Magnitude

10 dec.	$N < D < 100$	Brocot 1862

German editions give 11 decimals; see Archibald, *M.T.A.C.*, **1**, 132, 1943.

10 dec.	$D < 100$, $N < \frac{1}{2}D$	**Goodwyn 1818**

For $\frac{1}{2} < N/D < 1$, use $1 - N/D$. See also Art. 1·44.

8 dec.	$D \leqslant 1000$, $N < \frac{1}{10}D$	**Goodwyn 1823b**
8 dec.	$N < D \leqslant 120$	Buckingham 1935

On errors see Miller 1942 and Comrie, *M.T.A.C.*, **1**, 92, 1943.

6 dec.	$N \leqslant 20$, $D \leqslant 20$	Sang 1878
5 fig.	$N \leqslant 120$, $D \leqslant 120$	**Page 1942**

Means of extending the decimal are also provided (see Art. 1·41).

5 dec.	$D \leqslant 100$, $N \leqslant \frac{1}{10}D$	Albrecht 1908 (304)
4 dec.	$N < D \leqslant 60$	Oberg & Jones 1936 (630) Brown & Sharpe 1933 (223)

1·432. Log N/D in Order of Magnitude

7 dec.	$N \leqslant 120$, $D \leqslant 120$	**Page 1942**
6 dec.	$D < N \leqslant 120$, $N/D \leqslant 7{\cdot}5$	Oberg & Jones 1936 (948)
6 dec.	$N \leqslant 100$, $D \leqslant 100$, $\frac{24}{100} \leqslant N/D \leqslant \frac{100}{24}$	Brown & Sharpe 1933 (229)

From Wingquist's table in *American Machinist.* N not always prime to D.

5 dec.	$D < N < 100$	Airy 1881

1·44. Complete Recurring Decimal Periods of N/D

The case $N = 1$ has been included here for convenience, although reciprocals really belong to Art. 2·4.

$D \leqslant 1024$, all cycles	**Goodwyn 1823a**
$D = 1$ (primes and powers of primes) 463, all cycles $D = 467$ (primes and powers of primes) 997, $N = 1$	Gauss 1863
$D = 1$ (primes and powers of primes) 347, all cycles	Hoüel 1865

In 1/337, for 63510 read 63501 (Miller).

$D \leqslant 300$, apparently all cycles	Bierstedt 1812
$D = 1(1)100$, all cycles	**Goodwyn 1818**

SECTION 2

POWERS, POSITIVE, NEGATIVE AND FRACTIONAL

2·01. Introduction

Tables of squares, cubes, square roots, cube roots and reciprocals are very numerous and our lists in the corresponding articles are concerned mainly with tables of considerable extent, either in range or in number of figures tabulated. Tables of other powers are less numerous and our selection is correspondingly less restricted.

Compound interest tables are not included although strictly they are tables of powers (with accuracy up to about 12 figures) of rather special numbers. These numbers are usually between 1 and 1·1, and the tables sometimes extend to very high positive or negative index, perhaps ± 100 or ± 200. Antilogarithms, also technically powers, and powers of transcendental numbers such as π, e or M are dealt with in Sections 5, 6 and 10. Tables of various kinds of sums of powers are listed in Arts. 4·4 to 4·7.

We have included in the present section, Art. 2·64, tables involving square roots of various simple rational functions, for example, $\sqrt{1 \pm x^2}$, $\sqrt{x(1-x)}$ or $\sqrt{1+x^2}/x$.

For further details concerning older tables see B.A. 1873. For lists of errata see Cunningham 1905 and B.A. **9**, 1940.

Tables of squares, cubes, square roots, cube roots and reciprocals, and of higher powers, roots and reciprocal powers, expressed in the duodecimal scale of notation are given in Terry 1938. These are not listed in the separate articles.

The subdivision of this section is as follows:

2·02. Collected Tables of Powers

Several collections of tables occur many times in the following articles. In order to help the user to gain an idea of the full scope of the tables of powers contained in them, the more important collections are listed below, with cross-references to the appropriate articles.

Barlow 1814	2·11, 2·2, 2·25, 2·31, (2·32), 2·42, 2·61, 2·71
Barlow 1840	2·11, 2·2, 2·42, 2·61, 2·71
Barlow–Comrie 1930, 1941	2·11, 2·2, 2·31, 2·32, 2·42, 2·61, 2·611, 2·62, 2·71
B.A. **9**, 1940	(2·11, 2·2), 2·31, 2·32
Claessens 1920	2·11, 2·2, 2·61, 2·71

Jahn 1839	2·11, 2·2, (2·31, 2·32), 2·5, 2·61, 2·71, 2·81
MT.C. 1931	2·11, 2·2, 2·31, 2·42, 2·5, 2·61, 2·62, 2·71
New York W.P.A. 1939*a*	(2·11, 2·2), 2·31, 2·32
New York W.P.A. (MS.)	2·5, 2·61, 2·71, 2·72, 2·73, 2·81
Pearson 1931	2·32, 2·42, 2·5, 2·611, 2·62, 2·63, 2·647
Peters & Stein 1922	(2·11, 2·2), 2·31, 2·32, 2·42, 2·5, 2·611
Vega–Hülsse 1849	(2·11, 2·2, 2·31), 2·32, 2·61, 2·71
Weigel 1933	2·32

The bracketed references indicate tables which would be included in the articles listed if they contained more arguments or more digits.

2·11. Squares

The standard table of squares is L. Zimmermann 1938, and Barlow–Comrie 1930 or 1941 is strongly recommended for smaller ranges of the argument. All tables in the following list give exact values.

To 100,009	**L. Zimmermann 1938**
To 100,000	Ludolf 1690
To 100,000	Kulik 1848
To 27,000	Jahn 1839 (97)
To 25,400	Hutton 1781 (54)
To 25,200	Boebert 1812
To 25,000	Babington 1635
To 15,000	Timpenfeld 1937
To 12,500	**Barlow–Comrie 1941**
To 12,000	Büchner 1701
To 10,000	Barlow 1814, 1840
To 10,000	Peters 1871
To 10,000	Claessens 1920
To 10,000	L. Zimmermann 1923

PP for interpolation to the nearest hundred.

To 10,000	**Barlow–Comrie 1930**

Duhamel 1896 gives auxiliary tables, on a single sheet, for the determination, by means of a small number of additions and subtractions, of the squares of all numbers not exceeding 10^9.

Other tables, containing provision for interpolation, are as follows:

4 dec.	1(·001)10·009	fPPd	F. G. Gauss 1870 (112)

Gauss 1870 gives 5 decimals as far as argument 2, but the provision for interpolation is not complete for the extra decimal.

4 dec.	0(·001)10·009	fPPd	F. G. Gauss 1938 (62)
4 dec.	0(·001)3·509	fPPd	Bremiker 1907 (50), 1937 (140)
3½ dec.	1(·01)12	Δ, (PPd)	**MT.C. 1931** (14)

2·12. Quarter-Squares

These are intended primarily for multiplication by the formula

$$ab = \tfrac{1}{4}(a+b)^2 - \tfrac{1}{4}(a-b)^2$$

Glaisher 1878*e* gives a useful account of this and similar methods of multiplication, together with details of early tables of quarter-squares.

To 200,000	Blater 1888
To 100,000	Laundy 1856
To 40,000	Merpaut 1832
To 30,000	Kulik 1851
To 20,009	Plassmann 1933
To 20,000	Bojko 1909

Pryde 1930 (439) gives a table to 5100.

2·2. Cubes

All tables in the following list give exact values. Jahn 1839 is known to contain many errors.

To 100,000	Kulik 1848
To 24,000	Jahn 1839 (161)
To 12,500	**Barlow–Comrie 1941**
To 12,000	Buchner 1701
To 10,000	Babington 1635
To 10,000	Guldinus 1635
To 10,000	Hutton 1781 (54)
To 10,000	Barlow 1814, 1840
To 10,000	Claessens 1920
To 10,000	**Barlow–Comrie 1930**

The following table of curtailed values is also recommended:

$3\frac{1}{2}$ dec.	1(·01)2·5	Δ, (PPd)	**MT.C. 1931** (14)
$2\frac{1}{2}$ dec.	2·5(·01)6		
$1\frac{1}{2}$ dec.	6(·01)12		

2·25. $x^3 \pm x$

These tables are designed to help in the solution of cubic equations. See also Art. 5·64.

6 or 7 dec.	$x^3 \pm x$	0(·0001)3·28		Kulik 1860
8 dec.	$x^3 - x$	1(·0001)1·1549	No Δ	Barlow 1814 (186)
7 dec.	$x^3 - x$	0(·001)1·155		Lambert 1770; Lambert–Felkel 1798

See also Art. 16·41, for $P_{3,\,2}(x) = 15(x - x^3)$.

2·3. Higher Integral Powers

The standard tables for such powers are B.A. **9**, 1940 and New York W.P.A. 1939*a*. All tables listed in Arts. 2·31 and 2·32 give exact values unless otherwise stated.

For sums of positive integral powers, see Sub-section 4·4, and for differences of powers, see Sub-section 4·9, in particular Arts. 4·9242, 4·9332 and 4·952.

2·31. Fourth and Fifth Powers

To 1099		**B.A. 9, 1940** (2)
To 1000	See Cunningham 1905 for errata	Barlow 1814 (170, 176)
To 1000	Fourth powers only	**Barlow–Comrie 1930** (2), **1941** (2)
To 1000		Hayashi 1933 (38)
To 1000		**New York W.P.A. 1939*a*** (1)
To 308		**Peters & Stein 1922** (13)
To 300	Fifth powers to 200	Moore 1650

MT.C. 1931 (14) gives fourth and fifth powers to $4\frac{1}{2}$ figures for arguments 1(·01)10, and to $5\frac{1}{2}$ figures for arguments 10(·01)12, with Δ, (PPd).

Allen 1941 (17) gives 3- to 9-figure values of fifth powers, without Δ, for arguments ·1(·002)·15(·005)·4(·01)1(·02)2(·05)5(·1)10(·2)30(·5)40. Every part of the range can be covered to 6 or more figures by shifting the decimal point. This table is from Peirce & Carver. Trautwine 1937 (67) gives a similar table, with additional arguments 40(·5)50(1)99. Kent & Kent 1923 (51) also gives a similar table for arguments ·1(·05)2·6(·1)10(·2)30(·5)40(1)99, and Ives 1934 (113) gives 4-decimal or 5-figure values, without Δ, for arguments ·1(·1)8(·2)10.

Pearson 1930 (102, Lee & Everitt) gives kn^4 for $k = 1(1)N$:

n	2(1)7	8	9	10, 11	12	13, 14	15(1)19
N	400	350	300	250	200	150	100

2·32. Sixth and Higher Powers, n^p

This article is incomplete for $p \lessdot 10$ and $n < 100$ simultaneously.

To $p = 140$, $n = 11$ To $p = 53$, $n = 31$ To $p = 23$, $n = 50$	See list of errata in B.A. 9	Weigel 1933

Weigel 1933 also gives powers $p = 2(1)17$ for $n = 57, 73, 91, 111, 133, 157, 183, 211, 241, 273, 307, 343, 381$.

To $p = 50$, $n = 120$ To $p = 20$, $n = 299$ To $p = 12$, $n = 1099$	**B.A. 9, 1940** (2)

The values are curtailed to not less than 21 digits for $p = 31(1)39, 41(1)49$ and, when $n > 100$, for $p = 28, 29$ also. The complete values are available in manuscript: Miller, etc. (MS. 1939).

To $p = 10$, $n = 1000$	**New York W.P.A. 1939*a*** (21, 41, 61)
To $p = 10$, $n = 308$	**Peters & Stein 1922** (13)
$p = 11, 12$, to $n = 100$	Hayashi 1933 (60)
To $p = 10$, $n = 100$	Hutton 1781 (101)
To $p = 10$, $n = 100$	Hayashi 1926 (266)
To $p = 10$, $n = 100$ To $p = 20$, $n = 10$	**Barlow–Comrie 1930** (202), **1941** (252)
To $p = 10$, $n = 100$	**Pearson 1931** (258)

Error: In 88^8, for 243 read 248.

To $p = 6$, $n = 200$	Moore 1650
To $p = 12$, $n = 25$	Jäderin 1915 (127)
To $p = 30$, $n = 10$, also $n = 1\frac{1}{2}$ and $2\frac{1}{2}$	Hayashi 1930*b* (170)
To $p = 25$, $n = 2, 3, 5, 7$ only	Salomon 1827

Peters & Stein 1922 gives (33) all powers of primes from 7 to 97 with 32 or fewer digits; also (32) powers to 2^{120}, 3^{70}, 5^{60}.

Powers of 2 have been given in Shanks 1853 for $p = 1(12)721$.

Manuscript tables are available for:

$n = 2$, $p = 1(1)430(10)510$ $n = 3$, $p = 1(1)251$	Miller (MS. 1940)
$n = 2$, $p = 1$(var., about 2)671	Uhler (MS.)
$n = 12$, $p = 5(5)120$	Creak (MS.)

The following tables give curtailed values:

8 dec.	To $p = 11$, $n = 0(\cdot 01)1$	Lambert 1770; Lambert–Felkel 1798; J. C. Schulze 1778; Vega–Hülsse 1849
4 or 5 fig.	To $p = 15$, $n = \cdot 5(\cdot 01)1(\cdot 02)2(\cdot 05)5$	Jahnke–Emde 1933 (2)

See also Emde 1940.

2·4. Reciprocals

Reference should also be made to Art. 1·4, Decimal Values of Vulgar Fractions (in particular Art. 1·44); the present article deals with the particular case (not excluded from Art. 1·4) for which the numerator is unity. See also Arts. 4·612 and 14·48 for sums of reciprocals of integers.

2·41. Complete Recurring Decimal Values

To 1024	**Goodwyn 1823*a***
Primes and powers of primes, to 997	Gauss 1863
Primes and powers of primes, to 347	Hoüel 1865
To 300	Bierstedt 1812
To 100	Goodwyn 1818

2·42. Curtailed Values

The standard tables are Oakes 1865 and Barlow–Comrie 1930, 1941; Picarte 1861 is also a valuable table, giving the first 9 multiples of each reciprocal.

32 dec.	To $n = 100$		**Peters & Stein 1922** (36)
20 dec.	To $n = 100$		Houzeau 1875*b* (107)
20 dec.	To $n = 100$		Pearson 1931 (257)
17–20 dec.	To $n = 1000$		Gifford (MS.), Davis (MS.)
13 dec.	$n = 1000(1)10,000$		**Picarte 1861**
10 dec.	To $n = 1000$		Gelin 1881 (7)
9+ dec.	To $n = 1054$	No Δ	Ives 1929 (118), 1934 (28); Davenport & Ekas 1936 (196)
9 dec.	To $n = 1000$	No Δ	Glover 1930 (438); Trautwine 1937 (48)
8; 9 dec.	To $n = 1000$; 2000	No Δ	Kent & Kent 1923 (55)
10 fig.	$n = 2(1)201(2)301$		Davis 1935 (326, col. A)
9 fig.	To $n = 10,000$		Merpaut 1832
7 fig.	$n = 100,000(1)200,009$	fPPd	**New York W.P.A. 1943*c***

From two independent computations, prepared by C. C. Kiess and others and by the New York W.P.A. workers.

7 fig.	$n = 10,000(1)100,000$	fPP	**Oakes 1865**
7 fig.	$n = 10,000(1)100,000$	PPMD	Cotsworth 1902
7 fig.	To $n = 12,500$	Δ	**Barlow–Comrie 1941**
7 fig.	To $n = 10,000$	No Δ	Barlow 1814, 1840
7 fig.	To $n = 10,000$	No Δ	Burroughs 1918
7 fig.	To $n = 10,000$	No Δ	van Haaften 1926
7 fig.	To $n = 10,000$	Δ	**Barlow–Comrie 1930**
6 fig.	$n = 1(\cdot 01)10$	No Δ	Carmichael & Smith 1931 (174)

6 fig.	To $n = 1000$	No Δ	Fowle 1933 (15)
6 fig.	To $n = 1000$	IΔ	Fisher & Yates 1938 (74), 1943 (82)
5 fig.	To $n = 5000$	Δ	**Lohse 1935**
$5\frac{1}{2}$ fig.	$n = 3(\cdot01)12$	Δ, (fPPd)	**MT.C. 1931** (14)
$4\frac{1}{2}$ fig.	$n = 1(\cdot01)3$		

2·43. Reciprocals of Primes

10–11 dec.	For $p < 10{,}000$	Δ	B.A. **5**, 1935 (285)

2·46. $\dfrac{1+x}{1-x}$

$4\frac{1}{2}$ dec.	$0(\cdot001)\cdot8(\cdot0001)\cdot8999$	No Δ	Foxell 1934 (145)

This table has a number of errors, which may be easily detected by differencing.

2·5. Higher Negative Integral Powers, n^{-p}

32 dec. $n = 1(1)100$, all significant p — **Peters & Stein 1922** (36)

Error: In 42^{-5}, for 85453 21863 read 85452 31863 (C. R. Cosens).

17 d.–18 f. $n = 100(1)1000$, $p = 1(1)5$ — H. T. Davis (MS.)

See *M.T.A.C.*, **1**, 164, 1944.

10 dec.	$n = 1(1)50$, $p = 4(1)9$	Jahn 1839 (242)
7 fig.	$n = 50(1)101(2)199$ and 200, $p = 1(1)3$	Pearson 1931 (244)
5–8 dec.	$n = 2(1)9$, $p = 3(1)6$	MT.C. 1931 (36)
4 fig.	$n = 1(1)100$, $p = 2$	
4–5 fig.	$n = 0(\cdot01)1$, $p = 2$	Kelley 1923 (211)
3–4 dec.	$n = 0(\cdot1)7(\cdot5)9$, $p = 2$	Merriman 1910 (228)

New York W.P.A. has announced the preparation of a "Table of the first 10 powers of the reciprocals of the integers from 1 to 1000".

2·61. Square Roots

The standard tables of square roots are Vega–Hülsse 1849, Barlow–Comrie 1930, 1941 and Milne-Thomson 1929.

50 dec.	$1(1)100$		Miller (MS.)
15 dec.	$1(1)100$		Gelin 1882 (17)
15 dec.	$0(\cdot01)1$; $\cdot1 < \dfrac{p}{1000} < 1$, p prime		New York W.P.A. (MS.)
12 dec.	$1(1)10{,}000$		**Vega–Hülsse 1849**

In Barlow–Comrie 1930, 1941 it is stated that "the differencing of the square roots [in Vega–Hülsse] showed that the error of the twelfth decimal does not exceed a unit".

12 dec.	$0(\cdot001)1(\cdot01)10(\cdot1)20$		Hayashi 1930*a* (72)

This table contains a number of errors.

10 dec.	$1(1)1000$		Maseres 1795, from Hutton 1775
7 dec.	$1(1)10{,}000$	No Δ	Barlow 1814, 1840
7 dec.	$1(1)1054$	No Δ	Ives 1929 (118), 1934 (28); Davenport & Ekas 1936 (196)

7 dec.	1(1)1000	No Δ	Glover 1930 (438)
7 dec.	1(1)1000	No Δ	**Barlow–Comrie 1941**
6 dec.	1000(1)12,500(10)125,000	Δ	
7 dec.	1(1)1000	No Δ	**Barlow–Comrie 1930**
6 dec.	1000(1)10,000(10)100,000	Δ	
6 dec.	100(·1)1000(1)10,000	Δ	**Milne-Thomson 1929**
5 dec.	10,010(1)25,500	No Δ	Jahn 1839
4 dec.	1(1)10,009		

This table contains many errors.

5 dec.	1(·01)10(·1)100	No Δ	Carmichael & Smith 1931 (174)
4 dec.	0(·1)100(1)999	PP	G. W. Jones 1893 (142)
4 dec.	1(·01)10(·1)100	PPMD	Castle 1910 (46)
4 dec.	1(·01)10(·1)100	PPMD	Dale 1903 (10)
5 fig.	100(1)1000(10)10,000	Δ	Lohse 1935 (106)
5 fig.	100(1)1000(10)10,000	PPMD	Fisher & Yates 1938 (70), 1943 (78)
4½ dec.	1(·01)12	Δ, (fPPd)	**MT.C. 1931** (14)
3½ dec.	10(·1)120		
3 dec.	1(1)10,000	No Δ	Claessens 1920

2·611. Many-Figure Square and Cube Roots

In addition to the tables listed in the previous article and in Art. 2·71, there are several which give many-figure roots of selected numbers. See also Art. 5·61.

Egersdörfer & Egersdörfer 1936 (54) gives 28-decimal square roots for the integers from 2 to 42 which do not contain a squared factor; also for 46, 51, 55, 57, 58, 62, 65, 66, 69, 70, 74, 77, 78, 85, 87, 91, 95, 102, 105, 114, 130, 138, 154, 170, 174, 182, 186, 190, 210, 230, 238, 266, 286, 290, 310, 322, 330, 374, 390, 418. In addition, a number of multiples, mainly fractional, of these square roots are given.

Boorman 1882*a* gives square roots of 2, 3, 5, 11, 13, 17 to a number of decimals, of which 124, 91, 106, 62, 44, 54 respectively are known to be correct; revised values are given in Boorman 1884*b*.

The square root of 2 is given in De Morgan 1872 (293) to 110 decimals, calculated by W. H. Colvill and verified by J. Steel.

Peters & Stein 1922 (1) gives $\sqrt{2}$ to 64 decimals and $\sqrt{3}$ to 66 decimals.

Pearson 1931 (262) gives $\sqrt{2}$ and $1/\sqrt{2}$ to 30 decimals.

Gray 1877 gives $\sqrt[3]{2}$ to 56 decimals; this is a verification and extension of a 50-decimal value by Martin 1877. Gray also gives $\sqrt[3]{4}$ to the same accuracy.

Barlow–Comrie 1930 (207), 1941 (257) give 15-decimal values of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{10}$, $\sqrt[3]{2}$, $\sqrt[3]{10}$, $\sqrt[3]{100}$.

2·62. Reciprocal Square Roots

10 dec.	1(1)25		Pearson 1931 (xlvii)
8 fig.	1000(1)12,500(10)125,000	Δ	Comrie (MS. 1942)
8 dec.	1(1)100		Glover 1930 (490)
8 dec.	51(1)101(2)199 and 200		Pearson 1931 (244)
7 dec.	0(·05)1(·1)51·1		K. Pearson 1922 (154)
7 dec.	1(1)1000		**Barlow–Comrie 1930, 1941**
4½ dec.	1(1)100		**MT.C. 1931** (36)

2·63. Powers, $n^{\pm p}$

10 dec.	$\pm p = 3$, $n = 1(1)100(10)1000$	Δ	H. T. Davis (MS.)
8 dec.	$-p = 3$, $n = 2(\cdot 001)7\cdot 5(\cdot 01)20$	Δ	N.A.O. 1933 (122), 1939*b* (116)
7 fig.	$-p = 1(\cdot 1)6$, $n = 50(1)101(2)199$ and 200		Pearson 1931 (244)
6− fig.	$p = 5, \left\{ \begin{array}{l} n = \frac{1}{4}(\frac{1}{4})1(1)16(2)40 \\ \frac{1}{12}n = 3\frac{1}{2}(\frac{1}{4})4(\frac{1}{2})14(1)21 \end{array} \right\}$		Kent & Kent 1923 (933)
5 f. or 6 d.	$p = 5$, $n = 0(\text{var.})8\frac{1}{2}(\frac{1}{4})10(\frac{1}{2})$ $20(1)100$		Trautwine 1937 (69)
5 f. or 4 d.	$p = 5$, $n = \cdot 1(\cdot 1)9(\cdot 2)10$		Ives 1934 (113)
5 fig.	$p = 3$, $n = 1(1)1000$		Kent & Kent 1923 (52)
4 d.; 4 f.	$p = 3$, $n = 0(\cdot 001)1\cdot 499$; $1\cdot 5(\cdot 01)19\cdot 99$		U.S. Bureau of Reclamation 1940
4 dec.	$p = 3$, $n = 0(\cdot 001)1\cdot 49(\cdot 01)12$		Horton 1907 (171)

Horton gives a fifth decimal for most $x < 1\cdot 49$; this was obtained by linear interpolation from 4-decimal values and is almost meaningless.

4 fig.	$p = 3$, $n/1000 = 1(1)25$		Russell 1914 (61)

Some of our information is from *M.T.A.C.*, **1**, 131, 1943, which also refers to 3-decimal tables with $p = 3$.

2·64. Square Roots of Rational Functions

The following lists include a number of cases where one circular or hyperbolic function is tabulated in terms of another as argument, e.g.

$$\sqrt{1-x^2} = \sin(\cos^{-1} x) = \operatorname{sech}(\tanh^{-1} x) \qquad \frac{x}{\sqrt{1-x^2}} = \tan(\sin^{-1} x)$$

H. T. Davis, *M.T.A.C.*, **1**, 164, 1944, announces 14-decimal MS. tables of $x^m(1-x^2)^n$ for $x = 0(\cdot 01)1$, $m = 0, 1, 2$ and $n = \frac{1}{2}(1)4\frac{1}{2}$. See also Art. 17·426.

2·6411. $\sqrt{1-x^2}$

12 d.	$0(\cdot 01)\cdot 75(\cdot 005)1$		Fletcher (MS. 1940)
10 d.	$0(\cdot 01)1$ As $P_{1,1}(x)$, see Art. 16·41	No Δ	Tallqvist 1906 (12)
8 dec.	$0(\cdot 0001)1$	No Δ	Kelley 1938 (14)
8 dec.	$0(\cdot 001)1$ As $P_1^1(x)$, see Art. 16·41		Mursi 1941 (2)
6 dec.	$0(\cdot 0001)1$	No Δ	Miner 1922 (7)
4 dec.	$0(\cdot 001)1$	No Δ	Davenport & Ekas 1936 (187)
4 dec.	$\cdot 7(\cdot 001)\cdot 99(\cdot 0001)1$	IΔ, I$(1/\Delta)$ to ·99	T. Smith 1920
4 dec.	$0(\cdot 01)1$	No Δ	Garrett 1937 (475)

2·6415. $\dfrac{1-\sqrt{1-x^2}}{x}$

4 dec.	$0(\cdot 001)\cdot 8$	No Δ	Card 1943

2·6416. $\dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{x}{\sqrt{1-x^2}}$

10; 9; 8 dec. 7 dec.	$0(\cdot 005)\cdot 1$; $\cdot 1(\cdot 001)\cdot 199$; $\cdot 2(\cdot 001)\cdot 9$ $\cdot 9(\cdot 0^35)\cdot 96(\cdot 0^32)\cdot 99(\cdot 0^31)\cdot 995(\text{var.})\cdot 9^90$	No Δ	New York W.P.A. 1941*f* (31)

The function $x/\sqrt{1-x^2}$ is given with one decimal less up to $x = \cdot 9$.

2·6421. $\sqrt{1+x^2}$

7; 5 dec.	0(·001)·099; ·1(·01)·99		Jahnke–Emde 1933 (17)

Jahnke & Emde 1909 (5) gives fewer decimals.

5 dec.	0(·01)1·02	No Δ	Trautwine 1937 (256)
4 dec.	0(·001)1	No Δ	Miller 1939
		1Δ	Dwight 1941 (180)
3 dec.	0(·01)·1(·005)·3(·002)1	No Δ	Dwight 1934 (184)

See also Emde 1940.

2·6425. $\dfrac{\sqrt{1+x^2}}{x}$ and $\dfrac{\sqrt{1+x^2}-1}{x}$

6 fig., approx.	$\dfrac{\sqrt{1+x^2}-1}{x}$	·010(·001)1		Fergusson 1912
4 dec.	$\dfrac{\sqrt{1+x^2}}{x}$	1(·001)3(·01)10(·5)30(10)50	No Δ	Seeley 1936

2·6431. $\sqrt{x(1-x)}$

8 dec.	0(·0001)1	No Δ	Kelley 1938 (14)
3 dec.	0(·001)·45(·05)·55(·001)1	No Δ	Greiner 1909 (368)

Common logarithms to 5 decimals are also given.

2·6436. $\dfrac{1}{\sqrt{x(x-1)}}$

8 dec.	2(1)100	No Δ	Glover 1930 (490)

2·646. $\dfrac{x-1}{\sqrt{x}}$

7 dec.	0(·05)1(·1)51·1	No Δ	K. Pearson 1922 (154)

2·647. $\dfrac{x-1}{\sqrt{x(x+1)}}$

10 dec.	1(1)24		Pearson 1931 (xlvii)

2·71. Cube Roots

The standard tables of cube roots are Barlow–Comrie 1930, 1941. Barlow 1814, 1840 and Vega–Hülsse 1849 give an extra digit over most of the range; it is stated in Barlow–Comrie 1930, 1941 that the latter contains occasional errors of a unit in the last decimal, and errors in the former are also analysed.

31+ dec.	Primes to 127	Drach 1877
21 dec.	1(1)100	Gwyther (MS.)
15 dec.	10(10)1000(100)10,000	Gwyther (MS.)
13 dec.	100(1)200	Gwyther (MS.)
15 dec.	0(·01)9·99	New York W.P.A. (MS.)
10 dec.	1(1)100	Gelin 1882 (17)

Error: $\sqrt[3]{57}$ should read 3·84850 11313.

7 dec.	1(1)10,000	No Δ	Barlow 1814, 1840
7 dec.	1(1)10,000	No Δ	**Vega–Hülsse 1849**
7 dec.	1(1)1054	No Δ	Ives 1929 (118), 1934 (28) Davenport & Ekas 1936 (196)
7 dec.	1(1)1000	No Δ	Glover 1930 (438)
7 dec.	1(1)1000	No Δ	**Barlow–Comrie 1941**
6 dec.	1000(1)12,500	Δ	**Barlow–Comrie 1941**
7 dec.	1(1)1000	No Δ	**Barlow–Comrie 1930**
6 dec.	1000(1)10,000	Δ	**Barlow–Comrie 1930**
5 dec.	10,010(1)25,500	No Δ	Jahn 1839
4 dec.	1(1)10,009	No Δ	Jahn 1839

This table contains many errors.

5 dec.	1(·01)10(·1)100(1)1000	No Δ	Carmichael & Smith 1931 (174)
4 dec.	0(·01)10(·1)100(1)1000	PP	G. W. Jones 1893 (146)
4 dec.	1(·01)10(·1)100(1)1000	PPMD	Castle 1910 (50)
4 dec.	1(·01)10(·1)100(1)1000	PPMD	Dale 1903 (19)
$4\frac{1}{2}$ dec.	1(·01)12	Δ, (fPPd)	**MT.C. 1931** (14)
$3\frac{1}{2}$ dec.	10(·1)120(1)1200	Δ, (fPPd)	**MT.C. 1931** (14)
3; 4 dec.	1(1)4999; 5000(1)10,000	No Δ	Claessens 1920

See also Art 2·611.

2·72. Cube Roots of Squares, $n^{2/3}$

31+ dec.	Primes to 127		Drach 1877
21 dec.	1(1)100		Gwyther (MS.)
15 dec.	$(10n^2)^{1/3}$, $(100n^2)^{1/3}$; $n = 1(1)100$		Gwyther (MS.)
13 dec.	100(1)200		Gwyther (MS.)
15 dec.	0(·01)9·99		New York W.P.A. (MS.)
5 dec.	0(·01)3(·1)10(1)50	No Δ	Hayashi 1930*b* (106)
3–4 dec.	1(·1)31	Δ	Möller & Rasmusen 1936

2·73. $n^{-1/3}$ and $n^{-2/3}$

15 dec.	0(·01)9·99	New York W.P.A. (MS.)

2·8. General Fractional Powers

2·81. Fractional Powers, n^p

15 dec.	$n = 2(1)9$, $n = 0(·01)1$ and $100 < 1000n$(prime) < 1000; each for $p = 0(·001)·1(·01)1$	New York W P.A. (MS.)
15 dec.	$n = 0(·01)9·99$; $p = \pm\frac{1}{3}, \pm\frac{2}{3}$	New York W P.A. (MS.)
10 dec.	$n = 1(1)400$; $p = 1/12$	Johnston (MS.)
7 dec.	$n = 0(·01)1$; $p = 0(·01)1$	New York W.P.A. (MS.)

With subtabulation near the origin.

6 dec.	n prime, to 199; $1/p = 4(1)10$	Jahn 1839 (243)
5 f. or 4 d.	$n = \cdot 1(\cdot 1)8(\cdot 2)10$; $p = 1/5, 2/5$	Ives 1934 (113)
4–5 fig.	$n = \cdot 5(\cdot 1)1(\cdot 2)3(\cdot 5)5, 10, \frac{1}{10}$; $p = 0(\cdot 05)1$	Jahnke–Emde 1933 (8)
4 fig.	$n/1000 = 1(1)25$; $p = 1\cdot 5(\cdot 05)1\cdot 7$	Russell 1914 (61)
4 fig.	$n = 0(1)999$; $p = 5/4$	Kent & Kent 1923 (790)
5 dec.	$n = 2$; $12p = 1(1)11$	Rayleigh 1894 (11)
4–5 fig.	440×2^p; $12p = -57(1) + 38$	Rayleigh 1894 (11)

See also Emde 1940, and the previous Sub-sections 2·6 and 2·7.

The main 15-decimal New York W.P.A. table is arranged with n^p as a function of p for each n; the 7-decimal table gives it as a function of n for each p.

The values of $n^{1/12}$ by S. Johnston, together with the 50-decimal square roots of Art. 2·61 and the cube roots, by C. E. Gwyther, of Arts. 2·71 and 2·72, form part of a larger project (under the supervision of J. C. P. Miller) for fractional powers with $p = r/s$, $r < s \leqslant 12$, for integral n up to 100 or 200 or more, and for $n = 0(\cdot 01)1$.

2·86. $(1+x^2)^{1/4}$

5 dec.	$0(\cdot 01)1$	Δ	Comrie & Hartley (in press)

2·9. Powers of Complex Numbers

2·91. Reciprocals

Hawelka 1931 (54) and Jahnke & Emde 1933 (13) give u, v to 4 or 5 figures with IΔ for $x = 0(\cdot 01)1$, where

$$\frac{1}{1+ix} = u - iv \qquad \frac{1}{x+i} = v - iu$$

See also Emde 1940.

2·95. Square Roots

Notation: $\sqrt{1+ix} = u+iv \qquad \sqrt{x+i} = U+iV$

5 d.	u, v, U, V	$0(\cdot 01)1$	Δ	Comrie & Hartley (in press)
4–5 f.	u, v, U, V	$0(\cdot 01)1$	IΔ	Hawelka 1931 (54); Jahnke–Emde 1933 (15)
3 d.	u	$0(\cdot 002)\cdot 6(\cdot 02)37(1)250$	No Δ	Pierce 1922 (184)
4; 3 d.	v	$0(\cdot 002)\cdot 6(\cdot 02)1\cdot 18$; $1\cdot 2(\cdot 02)37(1)250$	No Δ	Pierce 1922 (184)

See also Emde 1940.

Comrie & Hartley also give $\sqrt{1+ix}$ in polar form; $(1+x^2)^{1/4}$ is given to 5 decimals and $\frac{1}{2}\tan^{-1} x$ to $0^\circ{\cdot}001$ and to 0·00001 radian, also for $x = 0(\cdot 01)1$; in each case Δ is given.

SECTION 3

FACTORIALS, BINOMIAL COEFFICIENTS, PARTITIONS, ETC.

3·0. Other functions of integers will be found in Section 4. The main subdivisions of the present section are:

3·1 Factorials, etc.
3·2 Permutations
3·3 Combinations
3·4 Triangular and Figurate Numbers
3·5 Double Factorials, etc.
3·6 Partitions

3·1. Factorials, etc.

3·11. $n!$

For non-integral n, see the Gamma Function in Section 14. A table of exact values of the first 120 factorials is announced in Salzer & Hillman 1943. Uhler 1942*a* gives $100!$ exactly.

Exact	To $n = 200$	**Uhler 1944**
,,	To $n = 100$	Joffe (MS. 1921)
,,	To $n = 60$	**Peters & Stein 1922**
,,	To $n = 50$	**Robbins 1917**, Potin 1925

Errors in Potin at n = 18, 38, 45 and 50 (Salzer & Hillman 1943).

,,	To $n = 30$	Beetle 1933, Alliaume 1928
,,	To $n = 25$	Hayashi 1930*b*
,,	To $n = 20$	Brandenburg 1931, 1932; Davenport & Ekas 1936; Hayashi 1926
8 fig.	To $n = 200$	**Fry 1928**
8 fig.	To $n = 100$	**Barlow–Comrie 1930, 1941**
6 fig.	To $n = 300$	Fisher & Yates 1938

Glaisher 1870 tabulates $n.n!$ to 20 figures for $n = 2(1)71$. Note that

$$n.n! = (n+1)! - n!$$

3·111. $n!$ decomposed into Prime Factors

To $n = 1200$ — Alliaume 1928

Each factorial is also expressed as a product of powers of "prime factorials", which are expressions of the form $2.3.5.7.11\ldots p$, where p is prime.

To $n = 60$ — **Peters & Stein 1922**
To $n = 31$ — Davis 1935 (206)

3·12. $1/n!$

475 dec.	To $n = 214$		**Uhler 1937**

Also 70 fig., $n = 215(1)369$, and 815+ dec., $n = 370(1)410$.

108 dec.	To $n = 74$		**Van Orstrand 1921**
85 dec.	To $n = 61$	(At $n = 32$, for final 9 read 8)	**Robbins 1917**
54 dec.	To $n = 43$		**Peters & Stein 1922**

Recurring dec. 28 fig.	$n = 2(1)12$ $n = 13(1)50$	(Never less than 37 dec.)	**Glaisher 1883*a***
30 dec.	To $n = 20$		Brandenburg 1931, 1932
23 dec.	To $n = 23$		Hayashi 1930*b*
21 dec.	To $n = 22$		Jahn 1839
21 dec.	To $n = 20$		Hayashi 1926
12 fig.	To $n = 50$	(From Glaisher)	Pearson 1931 (256)
6 fig.	To $n = 20$		Dale 1903
8 dec.	To $n = 9$		MT.C. 1931 (36)

3·13. $(1/n!)^2$

105+ dec.	To $n = 70$	(Many errors!)	Hayashi 1930*a*
23 dec.	To $n = 14$		Hayashi 1930*b*

3·14. $\text{Log}_{10}\, n!$

33 dec.	To $n = 3000$		Duarte 1927
33 dec.	$n = 3050(50)10{,}000(10{,}000)100{,}000$, also 10^6		Duarte 1933
20 dec.	To $n = 190$		Thoman 1867
18 dec.	To $n = 1200$	(6 errors, see B.A. 1926)	Degen 1824
18 dec.	To $n = 1200$	(Recomputation of Degen)	**Peters & Stein 1922**
18 dec.	To $n = 120$		**Robbins 1917**
12 dec.	To $n = 1000$		Montessus de Ballore & Duarte 1925
10 dec.	$n = 10(10)1300(50)10{,}000$		Boccardi 1932
10 dec.	To $n = 1200$	(From Degen)	Fry 1928
10 dec.	To $n = 1199$		E. S. Pearson 1922
8 dec.	To $n = 99$		Potin 1925 (839) Lupton 1906
7 dec.	To $n = 1000$		Pearson 1930 (98) J. W. Glover 1930
7 dec.	To $n = 300$		Fisher & Yates 1938
7 dec.	To $n = 250$		Dwight 1941
6 dec.	To $n = 1200$		De Morgan 1845
6 dec.	To $n = 100$	(Not in earlier editions)	Allen 1941 (Ballou)
5 dec.	To $n = 100$		Dale 1903 Hayashi 1930*b*
4½ dec.	To $n = 100$		MT.C. 1931

3·141. $\text{Log}_{10}\,(1/n!)$ and $\log_{10}\,(1/n.n!)$

18 dec.	To $n = 1200$	$\text{Log}\,(1/n!)$	Degen 1824
10 dec.	To $n = 71$	$\text{Log}\,(1/n.n!)$	Glaisher 1870

3·145. $2\log_e\{(n-1)!(n-2)!\ldots 2!\} = 2\log_e\{(n-1)(n-2)^2\ldots 2^{n-2}\}$

5–6 fig.	To $n = 30$		Rosa & Grover 1912 (195)
4–5 fig.	To $n = 30$	(Errors)	Rosa & Cohen 1908 (115)

3·15. Derangements (Sub-Factorial n)

The number of derangements of n things, so that none is in its proper place, is given for $n = 1(1)12$ in Whitworth 1878, 1901 (107). The number of derangements is the integer nearest to $n!/e$. It is also $\Delta^n 0!$ or $\nabla^n n!$

$$3\cdot2.\quad {}^nP_r = \frac{n!}{(n-r)!} = n(n-1)\ldots(n-r+1)$$

${}^nP_r$ is the number of permutations of n things, r at a time.

Exact	${}^nP_r$ to $n = 25$, all $r \leqslant n$	Potin 1925 (842)
,,	${}^nP_r$, $n = 26(1)50$, $r = 1(1)25$	
,,	${}^nP_r$ to $n = 15$, all $r \leqslant n$	Hayashi 1930*b*
10 dec.	$\text{Log}_{10}\,({}^nP_{50})$, $n = 50(50)10{,}000$	Boccardi 1932

Several of the functions tabulated on pages 370–385 of Davis 1935 are multiples of reciprocals of ${}^nP_r$ tabulated against n.

$$3\cdot3.\quad {}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$$

Articles 3·31 to 3·33 are concerned only with positive integral n. ${}^nC_r$ is then the number of combinations of n things, r at a time. For negative n, see Art. 3·4. For fractional n, see Arts. 3·53 to 3·57. For binomial interpolation coefficients, see Section 23.

3·31. ${}^nC_r$

Exact	To $n = 60$	(Also given in prime factors)	**Peters & Stein 1922**
,,	To $n = 50$		Glaisher 1917, Potin 1925
,,	To $n = 40$		Gyldén 1877
,,	To $n = 25$		Dwight 1941
,,	To $n = 15$		Hütte 1931, Hayashi 1930*b*
,,	To $n = 12$		**Barlow-Comrie 1930, 1941**; **MT.C. 1931**
,,	$n = 6(1)132$, $r = 2(1)12$ only		Sasuly 1934
,,	$n = 4(1)110$, $r = 2(1)10$ only		C. Jordan 1932*b*
,,	$n = 2(1)50$, $r = 2(1)10$ only		**B.A. 9, 1940**
8 fig.	To $n = 100$		Fry 1928

3·32. $\text{Log}_{10}\,({}^nC_r)$

7 dec.	To $n = 64$	J. W. Glover 1930
7 dec.	To $n = 40$	Gyldén 1877

3·33. ${}^nC_r/n$

Exact	To $n = 53$, all $r < n$	Weigel 1933

3·4. Triangular and Figurate Numbers

The triangular numbers are the numbers $\frac{1}{2}n(n+1) = {}^{n+1}C_2 = T_n$, which have been tabulated to much higher values of n than the binomial coefficients in general. Glaisher has pointed out that the triangular numbers provide a useful variant of the quarter-squares method of multiplication, by means of the formula

$$mn = T_m + T_{n-1} - T_{m-n}$$

in which no suffix greater than the greater of m and n occurs. Thus multiplication of any two 5-figure numbers would require a table of T_n up to n = 100,000, compared with a table of quarter-squares up to 200,000.

Exact	T_n to $n = 20{,}000$	de Joncourt 1762
,,	T_n to $n = 5000$	Barbette 1910
,,	T_{n-1} to $n = 50$	B.A. **9**, 1940

On Arnaudeau 1896, see Mehmke 1902 (949) and Mehmke–d'Ocagne 1909 (220).

The figurate numbers ${}^nH_r$ of the $(r + 1)$th order are the numbers

$$\frac{n(n+1)\dots(n+r-1)}{r!} = {}^{n+r-1}C_r = (-)^r({}^{-n}C_r)$$

Triangular numbers are thus figurate numbers of the third order. The figurate numbers ${}^nH_r$ for any r up to 10 and all n up to $51 - r$ stand in a vertical column in the table of binomial coefficients in **B.A. 9, 1940**. A similar remark applies to the more extensive tables in Sasuly 1934 and C. Jordan 1932*b* (see Art. 3·31). Values of ${}^nH_r$ are also given for $n = 1(1)30$, $r = 1(1)12$ in Lambert 1798 (184), and for $n = 5(1)20$, $r = 1(1)7$ in Goulden 1939 (242). See also Salzer in Part II.

For figurate numbers ${}^nH_r$ with half-integral n, see Arts. 3·53 ff.

3·5. Double Factorials, etc.

3·51. $(2n-1)!!$ and $(2n)!!$

$(2n - 1)!!$ stands for $1.3.5\dots(2n-1)$, and $(2n)!!$ for $2.4.6\dots 2n$, the differences between consecutive factors being 2, instead of 1 as in ordinary factorials. We have $(2n)!! = 2^n.n!$

Exact	Both functions, $n = 1(1)25$	Potin 1925 (840) Hayashi 1930*b*
,,	$(2n-1)!!$, $n = 1(1)20$	Egersdörfer 1936

3·515. $\{(2n-1)!!\}^2$, etc.

C. E. Gwyther (MS. 1943) gives $\{(2n-1)!!\}^2$ and $\{(2n-5)!!\}^2(2n-3)(2n-1)$ to 16–21 figures for $n = 1(1)56$.

3·52. $\text{Log}_{10}(2n-1)!!$ and $\log_{10}(2n)!!$

20 dec.	Log $(2n-1)!!$, $n = 1(1)50$	Thoman 1867
18 dec.	Both functions, $n = 1(1)60$	**Robbins 1917**
8 dec.	Both functions, $n = 1(1)50$ Quantities tabulated are cologarithms	Elderton 1902 Pearson 1930 (30)
8 dec.	Log $(2n-1)!!$, $n = 1(1)24$	Hoüel 1921 (105)

3·53. $(2n-1)!!/(2n)!!$, etc.

Notation

$$A = (2n-1)!!/(2n)!! = {}^{\frac{1}{2}}H_n = (-)^n({}^{-\frac{1}{2}}C_n)$$
$$B = (2n-3)!!/(2n)!! = -({}^{-\frac{1}{2}}H_n) = (-)^{n-1}({}^{\frac{1}{2}}C_n)$$
$$C = (2n-1)!!/(2n)!!(2n+1) = A/(2n+1)$$
$$D = (2n-3)!!/(2n)!!(2n+1) = B/(2n+1)$$

Apart from sign, A and B are thus coefficients in the expansions of $(1 + x)^{-\frac{1}{2}}$ and $(1 + x)^{\frac{1}{2}}$ respectively, while C and D are those in the expansions of

$$\int(1 - x^2)^{-\frac{1}{2}}dx = \sin^{-1} x \quad \text{and} \quad \int(1 - x^2)^{\frac{1}{2}}dx$$

Tables

A, $2A$	Exact, as decimals 18 dec.	To $n = 20$ $n = 21(1)32$	Fletcher (MS. 1941)
A, B	Exact, as decimals	To $n = 15$	Lambert 1798
$2A$	12 dec.	To $n = 11$	Leverrier 1856 ([1])

First coefficient in expansion of $b^{(n)}_{\frac{1}{2}}$. At $n = 9$, for 110 read 109.

A, B, C, D	10 dec., also as vulgar fractions	To $n = 10$	**Hayashi 1926** **Hayashi 1930*b*** (164)
A, B, C, D	10 dec.	To $n = 11$ or 12	Vega–Hülsse 1849

In D at $n = 6$, in this and next three items, for 75251 read 75240.

A, B, C, D	10 dec.	To $n = 10$	Vega 1797 Barlow 1814 (256) Köhler 1847

$2^{2n-1}B$, which is integral, is given (from Euler) exactly for $n = 2(1)24$ in Grunert 1841. See also Art. 3·55.

3·54. $\text{Log}_{10}\{(2n-1)!!/(2n)!!\}$, etc.

The following tables of the common logarithms of the quantities defined in Art. 3·53 may be mentioned:

A	8 dec.	$n = 1(1)25$	Hayashi 1930*b* (59) Potin 1925 (859) Hoüel 1901 (xxvii)
A	7 dec.	$n = 1(1)30$	**Hill 1884**
A, B, C, D	7 dec.	$n = 1(1)11$ or 12	Vega–Hülsse 1849
A, B, C, D	7 dec.	$n = 1(1)10$	Vega 1797 Barlow 1814 (256) Köhler 1847
A, B, C, D	7 dec.	$n = 1(1)9$	Láska 1888–94 (99)

3·55. ${}^nC_r$, n half-integral

Thiele 1909 (164) gives the coefficients, all integral, of $x^{0(1)17}$ in the expansions of $(1 - 4x)^n$ for $n = -\frac{1}{2}(1)+8\frac{1}{2}$. See also Art. 3·57.

3·555. $\text{Log}_{10}({}^nH_r)$, n half-integral

${}^nH_r$ stands for $\dfrac{n(n + 1)\ldots(n + r - 1)}{r!}$, as in Art. 3·4.

7 dec.	$n = \frac{1}{2}(1)\frac{9}{2}$, $r = 1(1)30$	Hill 1884

3·56. $\{(2n-1)!!/(2n)!!\}^2 = ({}^{\frac{1}{2}}H_n)^2 = ({}^{-\frac{1}{2}}C_n)^2$

Leverrier 1856 gives, for $n = 0(1)11$, 12-decimal values of $2({}^{\frac{1}{2}}H_n)^2$, which is the coefficient of α^{2n} in the expansion of $b^{(0)}$ on page [1]. The common logarithm of $2({}^{\frac{1}{2}}H_n)^2$ is also tabulated on page [2], to 10 decimals for $n = 0(1)15$ and to 7 decimals for $n = 16(1)25$.

3·561. $\{(2n-3)!!/(2n)!!\}^2 = ({}^{-\frac{1}{2}}H_n)^2 = ({}^{\frac{1}{2}}C_n)^2$

Exact	$n = 1(1)8$	Hayashi 1933

3·562. $2({}^{\frac{1}{2}}H_n)({}^{\frac{1}{2}}H_{n+r})$

Leverrier 1856 gives, for $n = 0(1)11$, $r = 0(1)11$, 12-decimal values of this expression, which is the coefficient of α^{2n} in the expansion of $b^{(r)}$ on page [1]. The common logarithm of the same quantity is also tabulated on pages [2]–[4] to 7 or 10 decimals for $r = 0(1)15$, $n = 0(1)N$, where N depends on r and is never less than 15.

3·57. ${}^nC_r$, n fractional

Tables of binomial coefficients for values of n which include a number of simple vulgar fractions (denominators 2, 3 and 4) are given in Madelung 1936 (363); also, differently, in earlier editions.

3·6. Partitions

We have not sought specially for tables of partitions, but the following references may be found of use:

Notation

${}^n\Pi_r$ = number of partitions of n into exactly r parts
$p(n)$ = number of unrestricted partitions of $n = {}^n\Pi_1 + {}^n\Pi_2 + \ldots + {}^n\Pi_n$
$q(n)$ = number of odd partitions of n (i.e. partitions into odd integers)
= number of unrestricted partitions of n, without repetitions
$q_0(n)$ = number of odd partitions of n, without repetitions
(n, m) = number of partitions of n in which m is the smallest part, so that $p(n) = (n+1, 1)$

Tables

Actual partitions	$n = 1(1)18$	Cayley 1881
${}^n\Pi_r$	$n = 1(1)27$, $r \leqslant 20$	Whitworth 1901 (99)
$p(n)$	$n = 1(1)300$; $301(1)600$	Gupta 1935; 1937
$p(n)$	$n = 1(1)200$	MacMahon 1918
$q(n)$, $q_0(n)$	$n = 1(1)400$	Watson 1937
$q(n)$	$n = 1(1)100$	Darling 1918
(n, m)	To $n = 300$, $m = 50$	Gupta 1939

SECTION 4

BERNOULLI AND EULER NUMBERS AND POLYNOMIALS. SUMS OF POWERS AND OF INVERSE POWERS. DIFFERENCES AND DERIVATIVES OF ZERO, ETC.

4·01. Introduction

The constants and polynomials considered in this section are in the main closely connected with the Bernoulli and allied numbers; almost all of them occur as coefficients in well-known series.

The notations and definitions used are very diverse; we shall follow closely those of Glaisher, one of the largest single contributors to the theory, and to the numerical output. It should be noted that we choose the neutral $U(x)$, $V(x)$ notation of Glaisher 1898*a* (115) as a foundation for the Euler and Bernoulli polynomials respectively.

The subdivision of this section is as follows:

4·02. Definitions and Interconnections

4·021. Bernoulli, Euler and Other Numbers

These are largely extracted, with modifications, from Glaisher 1898*a*, *q.v.*

$$\frac{te^{xt}}{e^t-1} = 1 + tV_1(x) + \frac{t^2}{2!}V_2(x) + \frac{t^3}{3!}V_3(x) + \dots$$

$$\frac{te^{xt}}{e^t+1} = tU_1(x) + \frac{t^2}{2!}U_2(x) + \frac{t^3}{3!}U_3(x) + \dots$$

$$U_n(x) = V_n(x) - 2^n V_n(\tfrac{1}{2}x)$$

$$\frac{d}{dx}V_n(x) = nV_{n-1}(x) \qquad \frac{d}{dx}U_n(x) = nU_{n-1}(x)$$

$V_n(x)$, $\dfrac{2}{n+1}U_{n+1}(x)$ are called in Davis 1935 (181, 275) the nth Bernoulli and Euler polynomials respectively.

Special numbers may be defined as follows:

$$\tfrac{1}{2}t\cot\tfrac{1}{2}t = 1 - \frac{B_1}{2!}t^2 - \frac{B_2}{4!}t^4 - \frac{B_3}{6!}t^6 - \dots$$

$$\tan\tfrac{1}{2}t = \frac{\alpha_1}{2!}t + \frac{\alpha_2}{4!}t^3 + \frac{\alpha_3}{6!}t^5 + \dots \qquad \alpha_n = 2(2^{2n}-1)B_n$$

$$t\operatorname{cosec} t = 1 + \frac{\beta_1}{2!}t^2 + \frac{\beta_2}{4!}t^4 + \frac{\beta_3}{6!}t^6 + \dots \qquad \beta_n = (2^{2n}-2)B_n$$

$$\sec t = 1 + \frac{E_1}{2!}t^2 + \frac{E_2}{4!}t^4 + \frac{E_3}{6!}t^6 + \dots$$

$$\frac{3t\cos\frac{1}{2}t}{\sin\frac{3}{2}t} = 2 + \frac{\delta_1}{2!}t^2 + \frac{\delta_2}{4!}t^4 + \frac{\delta_3}{6!}t^6 + \dots \qquad \delta_n = (3^{2n}-3)B_n$$

$$\frac{\sin\frac{1}{2}t}{\cos\frac{3}{2}t} = \frac{\zeta_1}{2!}t + \frac{\zeta_2}{4!}t^3 + \frac{\zeta_3}{6!}t^5 + \dots \qquad \zeta_n = \tfrac{1}{3}(2^{2n}-1)(3^{2n}-3)B_n$$

$$\frac{3\sin\frac{1}{2}t}{2\sin\frac{3}{2}t} = I_0 + \frac{I_1}{2!}t^2 + \frac{I_2}{4!}t^4 + \frac{I_3}{6!}t^6 + \dots$$

$$\frac{3\cos\frac{1}{2}t}{2\cos\frac{3}{2}t} = H_0 + \frac{H_1}{2!}t^2 + \frac{H_2}{4!}t^4 + \frac{H_3}{6!}t^6 + \dots \qquad H_n = (2^{2n+1}+1)I_n$$

$$\frac{3\sin\frac{1}{2}t\sin t}{2\sin\frac{3}{2}t} = \frac{\gamma_1}{2!}t + \frac{\gamma_2}{4!}t^3 + \frac{\gamma_3}{6!}t^5 + \dots \qquad \gamma_n = \tfrac{3}{4}(3^{2n}-1)B_n$$

$$\frac{\sin t\cos\frac{1}{2}t}{2\cos\frac{3}{2}t} = \frac{\theta_1}{2!}t + \frac{\theta_2}{4!}t^3 + \frac{\theta_3}{6!}t^5 + \dots \qquad \theta_n = \tfrac{1}{4}(2^{2n}-1)(3^{2n}-1)B_n$$

$$\frac{3t\cos 2t}{\sin 3t} = 1 - \frac{\epsilon_1}{2!}t^2 - \frac{\epsilon_2}{4!}t^4 - \frac{\epsilon_3}{6!}t^6 - \dots \qquad \epsilon_n = \tfrac{1}{2}(2^{2n}-2)(3^{2n}-3)B_n$$

$$\frac{\sin 2t}{2\cos 3t} = T_1 t + \frac{T_2}{3!}t^3 + \frac{T_3}{5!}t^5 + \dots$$

$$\frac{3\sin 2t}{\sin 3t} = J_0 + \frac{J_1}{2!}t^2 + \frac{J_2}{4!}t^4 + \frac{J_3}{6!}t^6 + \dots \qquad J_n = 2(2^{2n}+1)I_n$$

$$\frac{\cos 2t}{\cos 3t} = S_0 + \frac{S_1}{2!}t^2 + \frac{S_2}{4!}t^4 + \frac{S_3}{6!}t^6 + \dots \qquad S_n = \tfrac{1}{2}(3^{2n}+1)E_n$$

$$\frac{3\sin^2 t}{2\sin 3t} = \frac{\eta_1}{2!}t + \frac{\eta_2}{4!}t^3 + \frac{\eta_3}{6!}t^5 + \dots \qquad \eta_n = \tfrac{3}{8}(2^{2n}-2)(3^{2n}-1)B_n$$

$$\frac{\cos^2 t}{\cos 3t} = R_0 + \frac{R_1}{2!}t^2 + \frac{R_2}{4!}t^4 + \frac{R_3}{6!}t^6 + \dots \qquad R_n = \tfrac{1}{4}(3^{2n+1}+1)E_n$$

$$\frac{\cos t}{\cos 2t} = P_0 + \frac{P_1}{2!}t^2 + \frac{P_2}{4!}t^4 + \frac{P_3}{6!}t^6 + \dots$$

$$\frac{\sin t}{\cos 2t} = Q_1 t + \frac{Q_2}{2!}t^3 + \frac{Q_3}{4!}t^5 + \dots$$

$$J_n = H_n + I_n \qquad 3S_n = 2R_n + E_n$$

Of these numbets, the Bernoulli numbers B_n, the Euler numbers E_n, and the numbers I_n, T_n, P_n and Q_n are fundamental, not being simply expressible in terms of others. For the expression of the various numbers in terms of $V_n(x)$ or $U_n(x)$ for the particular values $x = 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{12}$, see Glaisher 1898*a* (131–133).

We give a table of the first six values of the various numbers; this may be helpful for identification, since notation is somewhat diverse in many cases. (The S_n of this Art. 4·021 is not used elsewhere in the *Index*; other references are to the sum S_n of Arts. 4·022 and 4·6111.) The table is mainly from Glaisher 1898*c* (51, 76).

n	0	1	2	3	4	5
B_n	−1	1/6	1/30	1/42	1/30	5/66
α_n	0	1	1	3	17	155
β_n	1	1/3	7/15	31/21	127/15	2555/33
γ_n	0	1	2	13	164	3355
δ_n	2	1	13/5	121/7	1093/5	49205/11
ϵ_n	−1	1	91/5	3751/7	1 38811/5	251 43755/11
ζ_n	0	1	13	363	18581	15 25355
η_n	0	1	14	403	20828	17 14405
θ_n	0	1	10	273	13940	11 44055
E_n	1	1	5	61	1385	50521
R_n	1	7	305	33367	68 15585	22374 23527
S_n	1	5	205	22265	45 44185	14916 32525
I_n	1/2	1/3	1	7	809/9	1847
H_n	3/2	3	33	903	46113	37 84503
J_n	2	10/3	34	910	4 15826/9	37 86350
T_n	0	1	23	1681	2 57543	676 37281
P_n	1	3	57	2763	2 50737	365 81523
Q_n	0	1	11	361	24611	28 73041

For the theory of Bernoulli polynomials, etc., including very many practically useful recurrence formulæ, see Glaisher 1898*a*, *c*, 1911*c* and Lehmer 1935, 1936.

4·022. Sums of Inverse Powers

Series of inverse powers which may be summed by means of Bernoulli and Euler numbers, etc., include, with Glaisher's notation:

$$S_n = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\dots \qquad S_{2n} = \tfrac{1}{2}(2\pi)^{2n}B_n/(2n)! \qquad (4{\cdot}611)$$

$$s_n = 1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\dots \qquad s_{2n} = \tfrac{1}{2}\pi^{2n}\beta_n/(2n)! \qquad (4{\cdot}616)$$

$$U_n = 1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{9^n}+\frac{1}{11^n}+\dots \qquad U_{2n} = \tfrac{1}{4}\pi^{2n}\alpha_n/(2n)! \qquad (4{\cdot}621)$$

$$u_n = 1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\frac{1}{9^n}-\frac{1}{11^n}+\dots \qquad u_{2n+1} = \tfrac{1}{2}(\tfrac{1}{2}\pi)^{2n+1}E_n/(2n)! \qquad (4{\cdot}626)$$

$$G_n = 1+\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{8^n}+\dots \qquad G_{2n} = \tfrac{2}{3}(\tfrac{2}{3}\pi)^{2n}\gamma_n/(2n)! \qquad (4{\cdot}631)$$

$$g_n = 1-\frac{1}{2^n}+\frac{1}{4^n}-\frac{1}{5^n}+\frac{1}{7^n}-\frac{1}{8^n}+\dots \qquad \sqrt{3}\cdot g_{2n+1} = (\tfrac{2}{3}\pi)^{2n+1}I_n/(2n)! \qquad (4{\cdot}632)$$

$$j_n = 1+\frac{1}{2^n}-\frac{1}{4^n}-\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{8^n}-\dots \qquad \sqrt{3}\cdot j_{2n+1} = (\tfrac{1}{3}\pi)^{2n+1}J_n/(2n)! \qquad (4{\cdot}633)$$

$$v_n = 1 - \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} - \dots \qquad v_{2n} = \tfrac{1}{3}(\tfrac{1}{3}\pi)^{2n}\eta_n/(2n)! \qquad (4{\cdot}634)$$

$$W_n = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \dots \qquad W_{2n} = 2(\tfrac{1}{3}\pi)^{2n}\theta_n/(2n)! \qquad (4{\cdot}641)$$

$$h_n = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \dots \qquad \sqrt{3}\cdot h_{2n+1} = (\tfrac{1}{3}\pi)^{2n+1}H_n/(2n)! \qquad (4{\cdot}642)$$

$$r_n = 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \dots \qquad r_{2n+1} = 2(\tfrac{1}{6}\pi)^{2n+1}R_n/(2n)! \qquad (4{\cdot}643)$$

$$t_n = 1 - \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} - \dots \qquad \sqrt{3}\cdot t_{2n} = 6(\tfrac{1}{6}\pi)^{2n}T_n/(2n-1)! \qquad (4{\cdot}644)$$

$$p_n = 1 + \frac{1}{3^n} - \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} - \dots \qquad p_{2n+1} = \sqrt{2}(\tfrac{1}{4}\pi)^{2n+1}P_n/(2n)! \qquad (4{\cdot}651)$$

$$q_n = 1 - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} - \dots \qquad q_{2n} = \sqrt{2}(\tfrac{1}{4}\pi)^{2n}Q_n/(2n-1)! \quad (4{\cdot}656)$$

It will be noted that, in each case, only even suffix or odd suffix is dealt with in this way, never both for the same series. The values thus omitted are almost equally important, but are much more troublesome to compute.

The series may all be derived from particular cases of the series

$$\sin 2\pi x + \frac{\sin 4\pi x}{2^{2n+1}} + \frac{\sin 6\pi x}{3^{2n+1}} + \dots = (-1)^{n+1}.\tfrac{1}{2}\frac{(2\pi)^{2n+1}}{(2n+1)!}V_{2n+1}(x)$$

$$\cos 2\pi x + \frac{\cos 4\pi x}{2^{2n}} + \frac{\cos 6\pi x}{3^{2n}} + \dots = (-1)^{n+1}.\tfrac{1}{2}\frac{(2\pi)^{2n}}{(2n)!}V_{2n}(x)$$

from which similar formulæ involving U_n are readily derived.

Glaisher 1913*b* gives many useful formulæ for checking these series.

Series of inverse powers of primes considered by Glaisher and others include

$$\Sigma_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \dots + \frac{1}{p^n} + \dots \qquad (4{\cdot}71)$$

$$\Upsilon_n = \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} - \frac{1}{13^n} - \frac{1}{17^n} + \dots + (-1)^{\alpha}\frac{1}{p^n} + \dots \qquad \alpha = \tfrac{1}{2}(p+1) \qquad (4{\cdot}72)$$

$$\Gamma_n = \frac{1}{2^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{17^n} + \dots + (-1)^{\beta}\frac{1}{p^n} + \dots \qquad \beta = p - 3[\tfrac{1}{3}p] \qquad (4{\cdot}73)$$

4·023. Generalized Bernoulli and Euler Numbers. Differences of Zero, etc.

Bernoulli and Euler polynomials of order n are defined, Milne-Thomson 1933 (127, 143), by

$$\frac{t^n e^{xt}}{(e^t-1)^n} = \sum_{p=0}^{\infty} B_p^{(n)}(x)\frac{t^p}{p!}$$

$$\frac{2^n e^{xt}}{(e^t+1)^n} = \sum_{p=0}^{\infty} E_p^{(n)}(x)\frac{t^p}{p!}$$

See Milne-Thomson for further information concerning Euler polynomials.

Most formulæ for interpolation, numerical differentiation and integration are directly expressible in terms of generalized Bernoulli polynomials, through the relation

$$B_p^{(n+1)}(x) = \frac{p!}{n!}\left(\frac{d}{dx}\right)^{n-p}(x-1)^{(n)}$$

where the factorial notation $x^{(n)} = x(x-1)\dots(x-n+1)$ is used. Tables with general x are considered incidentally under other notations in Section 23; here, in Subsection 4·9, we deal with $B_p^{(n)}(x)$ mainly for $x = 0$ and $\frac{1}{2}n$, and to a lesser extent for $x = \frac{1}{2}$ and 1, and for $x = \frac{1}{2}m$, where m is an integer from 0 to $2n$. For further details concerning these polynomials see Milne-Thomson 1933, mainly Chapters VI and VII, Nörlund 1924, and Steffensen 1927, Sections 6, 7, 13 and 18, in which the differences and derivatives of zero are used in place of the B notation.

$B_p^{(n)} = B_p^{(n)}(0)$ are called Bernoulli *numbers* of order n; the order may be negative. We have

$$B_0^{(0)} = 1 \qquad B_p^{(0)} = 0, p \neq 0 \qquad B_{2m}^{(1)} = (-1)^m B_m \qquad B_{2m+1}^{(1)} = 0$$

$$\left\{\frac{1}{t}\log_e(1+t)\right\}^n = \sum_{p=0}^{\infty}\frac{n}{n+p}B_p^{(n+p)}\frac{t^p}{p!}$$

$$\frac{t}{(1+t)\log_e(1+t)} = \sum_{p=0}^{\infty}B_p^{(p)}\frac{t^p}{p!}$$

We may also note that

$$B_p^{(n)}(x) = B_0^{(n)}x^p + pB_1^{(n)}x^{p-1} + p_2B_2^{(n)}x^{p-2} + \dots + B_p^{(n)}$$

where p_s is used to denote the binomial coefficient ${}^pC_s$.

$$\frac{d}{dx}B_p^{(n)}(x) = pB_{p-1}^{(n)}(x) \qquad \Delta B_p^{(n)}(x) = pB_{p-1}^{(n-1)}(x)$$

$$nB_p^{(n+1)}(1) = (n-p)B_p^{(n)} \qquad nB_{2q}^{(n+1)}(\tfrac{1}{2}n) = nB_{2q}^{(n+1)}(\tfrac{1}{2}n+1) = (n-2q)B_{2q}^{(n)}(\tfrac{1}{2}n)$$

Formulæ giving connections with differences and derivatives of zero are

$$x^n = \sum_{p=0}^{n} n_p x^{(p)} B_{n-p}^{(-p)} \qquad x^{(n)} = \sum_{p=0}^{\infty}(n-1)_{p-1}x^p B_{n-p}^{(n)}$$

$$\frac{\Delta^p 0^n}{p!} = n_p B_{n-p}^{(-p)} \qquad \frac{D^p 0^{(n)}}{p!} = (n-1)_{p-1}B_{n-p}^{(n)}$$

Again, writing $x^{[n]} = x(x+\frac{1}{2}n-1)(x+\frac{1}{2}n-2)\dots(x-\frac{1}{2}n+1)$ for the central factorial,

$$x^n = \sum_{p=1}^{n} n_p x^{[p]} B_{n-p}^{(-p)}(-\tfrac{1}{2}p) \qquad x^{[n]} = \sum_{p=1}^{n}(n-1)_{p-1}x^p B_{n-p}^{(n)}(\tfrac{1}{2}n)$$

$$\frac{\delta^p 0^n}{p!} = n_p B_{n-p}^{(-p)}(-\tfrac{1}{2}p) \qquad \frac{D^p 0^{[n]}}{p!} = (n-1)_{p-1}B_{n-p}^{(n)}(\tfrac{1}{2}n)$$

In these latter formulæ n and p are both even or both odd, since $B_{2q+1}^{(n)}(\frac{1}{2}n) = 0$ for all positive integral q.

Formulæ for numerical differentiation (including Markoff's) are, in symbolic form,

$$D^p = \sum_{n=p}^{\infty} \frac{D^p o^{(n)}}{n!} \Delta^n \qquad = \sum_{n=p}^{\infty} \frac{p}{n} B_{n-p}^{(n)} \frac{\Delta^n}{(n-p)!}$$

$$= \sum_{n=p}^{\infty} \frac{D^p o^{[n]}}{n!} \delta^n \qquad = \sum_{n=p}^{\infty} \frac{p}{n} B_{n-p}^{(n)} (\tfrac{1}{2}n) \frac{\delta^n}{(n-p)!}$$

$$= \sum_{n=p}^{\infty} \frac{D^{p+1} o^{[n+1]}}{n!(p+1)} \mu\delta^n = \sum_{n=p}^{\infty} B_{n-p}^{(n+1)} (\tfrac{1}{2}n + \tfrac{1}{2}) \frac{\mu\delta^n}{(n-p)!}$$

Both sides are supposed to act on y_0, or both on $y_{1/2}$, whichever is appropriate to the type of difference on the right. Reverse formulæ of similar type give differences in terms of derivatives.

More generally

$$D^p y_r = \sum_{n=p}^{\infty} B_{n-p}^{(n+1)}(r+1) \frac{\Delta^n}{(n-p)!} y_0$$

In a similar way integration involves the coefficients $B_n^{(n)}/n!$ and $B_n^{(n)}(1)/n!$ and others.

Lists of formulæ to a few terms involving connections between differential and difference operators are common; we have not attempted to mention all of them.

4·1. Bernoulli Numbers and Rational Multiples

There is considerable diversity of notation for these numbers, the most important variations being B_{2n} or B_{2n+1}, or $(-1)^{n-1} B_{2n}$, in each case for our B_n.

4·111. Exact Values of B_n

Unless otherwise stated, the numbers are given in the form N/D, where N and D are co-prime integers. The order is chronological in the first list, which gives what appear to be in part original calculations.

$n = 1(1)5$	Bernoulli 1713 (97)
$n = 1(1)15$	Euler 1755 (420), 1913 (321)
Error: D_{13} should be 6, not 2.	
$n = 1(1)18$	Rothe 1800 (336)
$n = 1(1)25$	Rothe 1817
$n = 1(1)31$	Rothe-Ohm 1840 (11)
$n = 1(1)62$	Adams 1877*a* (10), 1878*a* (270), 1890
$n = 1(1)62$	Adams 1877*a* (12)
These values are given as recurring decimals.	
$n = 1(1)90$	Serebrennikov 1905 (3)
$n = 91, 92$	Serebrennikov 1906 (6)
$n = 98$	D. H. Lehmer 1935 (648)
$n = 91(1)110$	D. H. Lehmer 1936 (462)

Euler 1770 (130) and 1785 (266) give, in effect, B_n to $n = 17$, see Art. 4·18.
Other tabulations include

$n = 1(1)90$	Peters & Stein 1922 (83)
$n = 1(1)90$, $n = 1(1)62$	Davis 1935 (230, 236)
The second table gives the values as recurring decimals.	

$n = 1(1)50$	Potin 1925 (860)
$n = 1(1)32$	Saalschütz 1893 (208)
$n = 1(1)25$	*Penny Cyclopædia*, **16**, 366, 1840
Error: D_{13} should be 6, not 2.	
$n = 1(1)25$	*English Cyclopædia, Arts & Sciences*, 5, 999, 1860
D_{13} correct.	
$n = 1(1)22$	Hayashi 1930*b* (176)
$n = 1(1)21$	Hayashi 1926 (256)
$n = 1(1)20$	Markoff 1896 (125)
$n = 1(1)20$	Ramanujan 1911 (*Coll. Papers*, 5)
$n = 1(1)15$	Lacroix 1800 (126)
$n = 1(1)15$	Nielsen 1923 (176)
$n = 1(1)15$	Nörlund 1924 (457)
$n = 1(1)15$	Duarte 1927 (xii)
$n = 1(1)13$	Jones 1893 (38)
$n = 1(1)13$	Joffe 1915 (38)
$n = 1(1)11$	Dwight 1934 (9)
$n = 1(1)11, 13$	Dwight 1941 (206)

For $n = 1(1)10$, values are given in Hall 1845 (264), Boole 1860 (84; 1872 edition, 90), Chrystal 1889 (207), Dale 1903 (31), Adams & Hippisley 1922 (140), Rietz 1924 (10), Jahnke & Emde 1933 (322), 1938 (272) and Sasuly 1934 (27).

4·112. Approximate Values of B_n

13 dec.	$n = 1(1)10$	Hall 1845 (264)
13 dec.	$n = 1(1)8$	Lacroix 1800 (135)
9 fig.	$n = 1(1)250$	Glaisher 1873*c* (390); Davis 1935 (240)
9 fig.	$n = 1(1)15$	Jones 1893 (38)
7 fig. *N*; *D* exact	$n = 1(1)35$	Dwight 1941 (206)
5 fig.	$n = 1(1)20$	Dale 1903 (31)

4·12. $\text{Log}_{10} B_n$

20 dec.	$n = 1(1)40$	Thoman 1860 (906)
10 dec.	$n = 1(1)250$	Glaisher 1873*c* (386); Peters & Stein 1922 (86); Davis 1935 (240)
10 dec.	$n = 1(1)18$	Grunert 1833 (80); *Penny Cyclopædia*, **16**, 366, 1840; *English Cyclopædia, Arts & Sciences*, 5, 999, 1860

There are some errors in the two final digits (see Glaisher 1873*c*); this table appeared originally in Eytelwein 1824.

10 dec.	$n = 1(1)15$	Jones 1893 (38)
8 dec.	$n = 1(1)31$	Hoüel 1901 (60)
8 dec.	$n = 1(1)30$	Potin 1925 (861)
7 dec.	$n = 1(1)200$	Glover 1930 (433)
7 dec.	$n = 1(1)35$	Dwight 1941 (206)

4·13. $\dfrac{(-1)^{n-1}B_n}{(2n-1)2n}$

Uhler 1942*a* gives values for $n = 1(1)71$ to at least $(128-2n)$ figures, rising to $(138-2n)$ when n is small or large; some early values are given as recurring decimals. Duarte 1927 (xii) gives the values as fractions to $n = 14$. These numbers are the coefficients in Stirling's series for $\log_e \Gamma(x)$.

4·14. $\dfrac{B_n}{B_{n-1}}$, etc.

4 fig.	$n = 2(1)250$	Peters & Stein 1922 (86)

Hutchinson 1925 (50) gives $b_n = \dfrac{B_{n+1}}{(2n+1)(2n+2)B_n}$ for $n = 1(1)13$; the first four are given as exact fractions, the rest to 9 decimals. b_{13} is indistinguishable, to 9 decimals, from $b_\infty = 1/4\pi^2$.

4·15. Tangent Numbers

$$T_n \text{ (Peters)} = (-1)^n C_{2n-1} \text{ (Davis)} = 2^{2n}\frac{a_n}{4n} = \frac{2^{2n}(2^{2n}-1)}{2n}B_n$$

These numbers are integers and all tables below give exact values.

$n = 1(1)30$	Peters & Stein 1922 (88)
$n = 1(1)15$	Saalschütz 1893 (23); Nielsen 1923 (177)
$n = 1(1)15$	Davis 1935 (187)
$n = 1(1)10$	Nörlund 1924 (458)
$n = 1(1)8$	Sheppard 1899 (21)
$n = 1(1)7$	Euler 1748 (132)

The numbers are given as numerators.

$n = 2(1)7$	Gudermann 1830*b* (172)

10 dec.	$\log_{10} T_n$ (Peters)	$n = 1(1)50$	Peters & Stein 1922 (88)
4 fig.	T_n/T_{n-1}	$n = 2(1)50$	

4·16. Cosecant Numbers, β_n

$$2(-1)^n R_{2n} \text{ (Lucas)} = (-1)^n D_{2n} \text{ (Davis)} = P_n \text{ (Sheppard)} = \beta_n = (2^{2n}-2)B_n$$

These are given as exact fractions N/D, where N and D are co-prime integers.

$n = 1(1)15$	Davis 1935 (187)
$n = 1(1)13$	Joffe 1915 (38)
$n = 1(1)10$	Sheppard 1912 (334)
$n = 1(1)10$	Nörlund 1924 (458)
$n = 1(1)5$	Glaisher 1898*c* (51)
$n = 1(1)5$	Lehmer 1935 (640)

The numbers $\frac{1}{2}\beta_n$ are tabulated, without sign.

4·171. Genocchi's Numbers

$$(-1)^n G_{2n} \text{ (Lehmer)} = \lambda_n \text{ (Sheppard)} = a_n = 2(2^{2n}-1)B_n$$

These are the simplest integer set of multiples of B_n.

$n = 1(1)110$ Lehmer (MS. 1936)

Deposited in the Library of the American Mathematical Society.

$n = 1(1)10$ Sheppard 1912 (334)

$n = 1(1)5$ Glaisher 1898c (76)

$n = 1(1)5$ Lehmer 1935 (640)

4·172. $\dfrac{\alpha_n}{2n}$ and $\dfrac{\alpha_n}{4n}$

$n = 1(1)17$ $M_{n+1} = a_n/4n$ Oettinger 1856 (5)

Errors in M_8, M_{13} and M_{16}.

$n = 1(1)13$ $a'_n \doteq a_n/2n$ Joffe 1915 (38)

4·18. Further Multiples of B_n

Euler 1748 (131) gives, as detachable parts of expressions for S_{2n}, the values of $2(2n+1)B_n$ for $n = 1(1)13$; Euler 1770 (130) and 1785 (266) give the values for $n = 1(1)17$. Euler 1755 (419) gives $(2n+1)B_n$ and $(n+1)!\,(2n+1)B_n$ for $n = 1(1)15$.

As listed in Art. 4·02, Glaisher 1898c (51, 76) gives, for $n = 1(1)5$, the values of

$$\gamma_n = \tfrac{3}{4}(3^{2n}-1)B_n \qquad \delta_n = (3^{2n}-3)B_n$$
$$\epsilon_n = \tfrac{1}{2}(2^{2n}-2)(3^{2n}-3)B_n \qquad \zeta_n = \tfrac{1}{3}(2^{2n}-1)(3^{2n}-3)B_n$$
$$\eta_n = \tfrac{3}{8}(2^{2n}-2)(3^{2n}-1)B_n \qquad \theta_n = \tfrac{1}{4}(2^{2n}-1)(3^{2n}-1)B_n$$

Rothe 1800 (329–332, 337–338) gives $2^{2n}(2^{2n}-1)B_n/(2n)!$ and $2^{2n}B_n/(2n)!$ for $n = 1(1)18$, and $2^{2n}(2^{2n}-2)B_n/(2n)!$ for $n = 1(1)11$. He denotes these (with obvious slight changes of notation) respectively by

$$\frac{(-1)^{n-1}}{2n-1}\,p_{-(2n-1)}(n) \qquad \frac{(-1)^n}{2n-1}\,p_{-(2n-1)}(n+1) \qquad (-1)^n p_{-2n}(n+1)$$

Crout 1929 (123, 127) gives the two first for $n = 1(1)4$, and Sang 1884b (616) gives the same quantities divided by $2n$ for $n = 1(1)10$; Sang also gives 20-decimal values of his constants, which are coefficients in expansions for $\log_e \cos x$ and $\log_e \sin x$. Legendre 1816 (185) gives $\log_{10} H_n$ to 15 decimals for $n = 1(1)15$, where $H_n = 2^{2n-1}B_n/(2n)!$, and Newman 1892 (12) gives exact values of H_n for $n = 1(1)5$ and 16-decimal values for $n = 6(1)16$; note that $H_n \pi^{2n} = S_{2n}$ (see Art. 4·6111).

The coefficients $B_n/(2n)!$ in the Euler–Maclaurin formula connecting sums and integrals are given for $n = 1(1)4$ by Sheppard 1900b (477, eqn. 120), by Andoyer 1906 (116) and 1923 (265), and by Scarborough 1930 (319). These numbers to $n = 5$ occur as numerical coefficients in Gram 1925 (316) and in Jahnke & Emde 1933 (319), 1938 (269).

See also Sub-sections 4·6 and 4·8 for further coefficients which are irrational multiples of B_n.

4·19. $B_x = \dfrac{2.(2x)!}{(2\pi)^{2x}} S_{2x}$

7 dec. $x = 2(\cdot 1)4$ Glaisher 1872c (18)

Glaisher also gives the minimum value 0·02377 ... at $x = 2\cdot 93$....

See also Sub-section 22·1 on the Riemann ζ-function, since $S_x = \zeta(x)$.

4·2. Euler Numbers and Rational Multiples

Again some diversity of notation exists, of type similar to that referred to in Art. 4·1.

4·21. Exact Values of E_n

Chronological order is adopted in the first list below, which contains what appear to be in part original calculations.

$n = 1(1)8$	Euler 1755 (542), 1785 (270)
A ninth number was incorrectly given.	
$n = 1(1)14$	Scherk 1825 (7)
$n = 1(1)9$	Gudermann 1830*b* (171)
$n = 9$	Rothe–Ohm 1840 (12)
$n = 10$	Stern 1875 (90)
$n = 1(1)27$	Glaisher 1914*a* (16)
$n = 1(1)32$	Joffe 1916 (110)
$n = 1(1)50$	Joffe 1920 (269)
$n = 1(1)30$	Peters & Stein 1922 (89)

Other tabulations include

$n = 1(1)50$	Davis 1935 (294)
$n = 1(1)14$	Saalschütz 1893 (23)
$n = 1(1)14$	Nielsen 1923 (178)
$n = 1(1)10$	Nörlund 1924 (458)
$n = 1(1)10$	Rietz 1924 (10)
$n = 1(1)9$	Chrystal 1889 (318)
$n = 1(1)9$	Dale 1903 (91)
$n = 1(1)8$	Jones 1893 (38)
$n = 1(1)8$	Sheppard 1899 (21)
$n = 1(1)8$	Andoyer 1915 (xi)
$n = 1(1)8$	MT.C. 1931 (227)

4·22. Approximate Values of E_n

10 fig.	$n = 1(1)250$	Davis 1935 (298)
7 fig.	$n = 1(1)35$	Dwight 1941 (207)

4·23. $\text{Log}_{10}\, E_n$

12 dec.	$n = 1(1)250$	Davis 1935 (298)
10 dec.	$n = 1(1)50$	Peters & Stein 1922 (89)
7 dec.	$n = 1(1)35$	Dwight 1941 (207)

4·24. $\dfrac{E_n}{E_{n-1}}$

4 fig.	$n = 2(1)50$	Peters & Stein 1922 (89)

4·25. $2^{2n}E_n$

Exact	$n = 1(1)7$	Called E_n'	Glaisher 1913*a* (110)

4·26. $R_n = \frac{1}{4}(3^{2n+1}+1)E_n$

Exact	$n = 1(1)5$	Glaisher 1897*a* (166), 1898*a* (71), 1898*c* (51)

4·27. $S_n = \frac{1}{2}(3^{2n}+1)E_n$

Exact	$n = 1(1)5$	Glaisher 1898*c* (51)

4·28. $\mathrm{Log}_{10} \dfrac{E_n}{(2n)!}$

14 dec.	$n = 1(1)15$	Called $\log_{10} K_n$	Legendre 1816 (185)

4·3. Glaisher's *I*, *T*, *P* and *Q* Numbers and Rational Multiples. Other Special Numbers

4·311. Glaisher's *I* Numbers, I_n

These numbers are integers, except for $I_0 = 1/2$, and for I_n when $2n+1$ is a multiple of 3^a, in which case $3^a I_n$ is an integer.

Exact	$n = 0(1)5$	Glaisher 1896 (157), 1897*a* (165), 1898*a* (36), 1898*c* (51)
	$n = 1(1)13$	Glaisher 1899*b* (224)
	$n = 0(1)16$	Miller (MS.)
	$n = 0(1)5$	Glaisher 1893*b* (53)

This reference gives $\frac{1}{3}I_n$, denoted by H_n.

4·312. Glaisher's $G_n = (2n+1)I_n$

Exact	$n = 1(1)13$	Glaisher 1899*b* (224)
	$n = 0(1)16$	Miller (MS.)

4·313. Glaisher's $K_n = (2n+1)2^{2n+1}I_n$

Exact	$n = 1(1)5$	Glaisher 1893*b* (55)

4·314. Glaisher's $H_n = (2^{2n+1}+1)I_n$

These are integers, except for $H_0 = \frac{3}{2}$.

Exact	$n = 0(1)5$	Glaisher 1897*a* (165), 1898*a* (49), 1898*c* (51)
	$n = 1(1)13$	Glaisher 1899*b* (229)
	$n = 1(1)13$	Glaisher 1899*b* (232)

This reference gives $\frac{1}{3}H_n$, denoted by H_n'.

4·315. Glaisher's $J_n = 2(2^{2n}+1)I_n$

These are integers, except for denominators 3^a as mentioned in Art. 4·311.

Exact	$n = 0(1)3$	Glaisher 1896 (162), 1898*a* (44)
	$n = 0(1)5$	Glaisher 1897*a* (165), 1898*c* (51)
	$n = 0(1)13$	Glaisher 1899*b* (229)

4·32. Glaisher's T Numbers, T_n

Exact (integers)	$n = 1(1)5$	Glaisher 1897*a* (166), 1898*a* (76), 1898*c* (51)
	$n = 1(1)7$	Miller (MS.)

4·33. Glaisher's P and Q Numbers, P_n and Q_n

These are two "parallel" but independent sets of integers.

Exact	$n = 0(1)4$ P_4 and Q_4 in error.	Gudermann 1830*c* (315)
	$n = 1(1)5$	Glaisher 1897*a* (166), 1898*a* (63, 66), 1898*c* (51)
	$n = 1(1)7$	Glaisher 1913*a* (111)
	$n = 1(1)20$	Glaisher 1914*c* (201, 202)

4·34. Other Special Numbers

D. H. Lehmer 1935 (649) gives coefficients in $3(e^z+e^{\omega z}+e^{\omega^2 z})^{-1} = \sum_0^\infty W_n z^n/n!$, where ω is a complex cube root of unity. $W_n = 0$ except when n is a multiple of 3, and Lehmer gives

n	0	3	6	9	12
W_n	1	-1	19	-1513	315523

4·4. Bernoulli Polynomials; Sums of Integral Powers of Integers

4·41. Coefficients

We consider first tables of coefficients of these polynomials, expanded in various ways. Firstly

$$V_n(x) = x^n - \tfrac{1}{2}nx^{n-1} + n_2B_1x^{n-2} - n_4B_2x^{n-4} + \ldots + V_n(0) \qquad (4·4111)$$

where $V_{2n}(0) = (-1)^{n-1}B_n$; $V_{2n+1}(0) = 0$, $n > 0$; $V_1(0) = -\frac{1}{2}$, and n_r is the binomial coefficient $\dfrac{n!}{r!(n-r)!}$. Again, with $A_n(x) = \dfrac{1}{n}V_n(x)$,

$$S_{n-1}(x-1) = \sum_{r=1}^{x-1} r^{n-1} = \frac{1}{n}\{V_n(x) - V_n(0)\} = B_n(x)\,\text{(Glaisher)} = A_n(x) - A_n(0)\,\text{(Glaisher)}$$

and the following expansions have been considered:

$S_n(x)$ in powers of x	(4·4112)
$S_{2n+1}(x)$ and $S_{2n}(x)/(2x+1)$ in powers of x	(4·4113)
$S_n(x)$ in powers of $2x+1$	(4·412)
$S_{2n+1}(x)$ and $S_{2n}(x)/(2x+1)$ in powers of $x(x+1)$	(4·413)
$S_n(x)$ in terms of factorials	(4·414)

Other series of positive powers are considered in Arts. 4·416 and 4·417, and series of powers in which the terms alternate in sign in Arts. 4·514 and 4·515. For yet other possibilities see Glaisher (*Quart. J. Math.*, **29**, 303–328, 1898 and **43**, 101–122, 1912).

$S_n(x) = \sum_{r=1}^{x} r^n$ should not be confused with $S_n = \sum_{r=1}^{\infty} r^{-n}$.

4·4111. $V_n(x)$ in Powers of x

The coefficients are n times those in the following Arts. 4·4112 and 4·4113.

$n = 1(1)8$	Glaisher 1898*a* (121)
$n = 0(1)10$	Steffensen 1927 (120)
$n = 0(1)6$	Milne-Thomson 1933 (136)
$n = 0(1)15$	Davis 1935 (181)

See also Renfer 1900 (29).

4·4112. $S_{n-1}(x)$ in Powers of x

$n-1 = 1(1)25$	Joffe 1915 (43); Davis 1935 (190)
$n-1 = 1(1)50$	B.A. 9, 1940 (128)

4·4113. $S_{n-1}(x-1) + \frac{1}{n}V_n(0) = \frac{1}{n}V_n(x) = A_n(x)$, in Powers of x

$n = 1(1)8$	Glaisher 1896 (110)

Denoted by $(-1)^n f(-x, n-1)$.

$n = 1(1)8$	Glaisher 1897*b* (21)
$n = 1(1)8$	Renfer 1900 (73)

4·4114. $\dfrac{S_{2m}(x)}{x(x+1)(2x+1)}$ and $\dfrac{S_{2m+1}(x)}{x^2(x+1)^2}$ in Powers of x

Suffix $= 2(1)15$	Davis 1935 (188)

4·4121. $2^{n+1}S_n(x)$ in Powers of $2x+1$

$n = 1(1)25$	Joffe 1915 (46); Davis 1935 (192)
$n = 1(1)11$	Sheppard 1912 (333)

4·4122. $A_n(x)$ in Powers of $x-\frac{1}{2}$

$a_n(x) = A_n(x+\frac{1}{2})$ in powers of x $\quad n = 1(1)4 \quad$ Glaisher 1897*b* (36)

4·4131. $V_{2n}(x) - V_{2n}(0)$ and $\dfrac{V_{2n+1}(x) - V_{2n+1}(0)}{2x-1}$ in Powers of $x-x^2$

Suffix $= 3(1)18$	Steffensen 1927 (126, 127)

4·4132. $S_{2n+1}(x)$ and $\dfrac{S_{2n}(x)}{2x+1}$ in Powers of $x+x^2$

Suffix $= 1(1)11$	Glaisher 1899*a* (170)
Suffix $= 1(1)25$	Joffe 1915 (49); Davis 1935 (194)

4·414. $S_{n-1}(x)$ and $S_n(x-1)$ in Terms of Factorials $x^{(r)}$

$n = 1(1)10$ Kramp 1800 (357, 358)

Error: p. 359. The coefficient of y_3 in Σy^{10} should be 511/3 not 551/3.

4·416. The Series $x^n + (x-q)^n + (x-2q)^n + \ldots + r^n$

Glaisher 1900*b* sums a number of these series. These include expansions in powers of x for the cases

$q = 2(1)4$	$r = 1(1)q$	$n = 1(1)4$	p. 197

with expansions in powers of $x + \frac{1}{2}q$ on page 211. Also, writing $x = q(m-1)+r$, expansions are given in powers of m for the cases

$q = 2, 3, 4, 6$	r prime to and less than q	$n = 1(1)5$	p. 222

Sasuly 1934 (27) gives expansions in powers of r for $q = 2$, $x = -r$ (i.e. twice sums of powers of odd numbers) in the cases $n = 2(2)12$ (omit his factor $(\frac{1}{2}h)^n$).

4·417. $S_{n,n}(x) = 1^n(x-1)^n + \ldots + r^n(x-r)^n + \ldots + (x-1)^n 1^n$

Glaisher 1900*c* sums a number of these series. Expansions are given in powers of x for the cases

$n = 1(1)8$	r proceeding by units		p. 242
$n = 1(1)4$	r proceeding by twos	x even	p. 245

At the top of the page: for Σ read S.

4·42. Special Values of $V_n(x)$, etc.

Glaisher 1898*a* gives the values of $V_n(x)$ and related polynomials in terms of the numbers B_n, E_n, I_n, etc. for various special values of x; in particular see pages 131–132. Alternatively see Glaisher 1911*c* (95, 100). Glaisher 1898*c* (48–50) gives a larger collection of values of $V_n(x)$ only, for $x = p/q$, $0 < p < q$, $q = 2, 3, 4, 6, 8, 12$. In addition (1898*a*) he gives exact numerical values (in fractional form) as follows:

$2^n A_n(\frac{1}{2})$ and $2^n\{A_n(\frac{1}{2}) - A_n(0)\}$	$n = 2(2)8$	p. 26
$3^n A_n(\frac{1}{3})$ and $3^n\{A_n(\frac{1}{3}) - A_n(0)\}$	$n = 2(2)8$	p. 36
$4^n A_n(\frac{1}{4})$ and $4^n\{A_n(\frac{1}{4}) - A_n(0)\}$	$n = 2(2)8$	p. 31

Lehmer 1940 gives a 10-place table of zeros of $V_{2n}(x)$ and of corresponding extreme values of $V_{2n+1}(x)$.

4·431. Tables of $V_n(x) - V_n(0)$ for $0 < x < 1$

Exact	$n = 2(1)5$	$x = 0(\cdot 01)1$	Davis 1935 (224, McClain)
10 dec.	$n = 6(1)8$		

4·432. Small Tables of $V_n(x) - V_n(0)$, etc.

Renfer 1900 (92) gives 6-decimal values, for

$n = 1(1)6$ $\quad\quad x = -4(1)-1(\frac{1}{2})+2(1)5$

of $B^{n+1}(x) = A_n(x) - A_n(0)$, $\phi(x, n) = V_n(x) - V_n(0)$, $\chi_n(x) = V_n(x)/n!$ and $A_n(x) = V_n(x)/n$.

4·44. Sums of Integral Powers of Integers, $S_n(x)$

All the tables listed below give exact values. See also Art. 3·4 for $n = 1$, Triangular Numbers.

$n = 1(1)7$	$x = 1(1)20$	Pearson 1902*b* (10)
$n = 1(1)7$	$x = 1(1)100$	Elderton 1903 (479); Pearson 1930 (40)
$n = 1(1)10$	$x = 1(1)100$	Davis 1935 (258, Terkhorn)
$n = 1(1)3$	$x = 1(1)1000$	Davis 1935 (258, Terkhorn)

The three following items tabulate $2S_n(x)$:

$n = 2(2)16$	$x = 1(1)10$	Whittaker & Robinson 1926 (294)
$n = 2(2)10$	$x = 1(1)25$	Jäderin 1915 (129)
$n = 2(2)6$	$x = 1(1)25$	Bailey 1931 (357)

In addition, Bailey gives $2(2x+1)S_4 - 4S_2^2$, $4(S_6S_2 - S_2^4)$ and S_4/S_2, and Jäderin also tabulates many such quantities, to 15 decimals, as exact fractions, and as 7-decimal logarithmic values.

During the preparation of the Table of Powers, B.A. **9**, 1940 (see Section 2), a number of sums Σx^n, $50k < x < 50k+q$, were formed; these are available in manuscript, as follows:

Sums of 50 consecutive powers, $q = 49$

$n = 2(1)12$	$k = 0(1)21$
$n = 13(1)20$	$k = 0(1)5$
$n = 21(1)50$	$k = 0, 1$

Sums of 21 consecutive powers, $q = 20$

$n = 21(1)50$	$k = 2$

These sums have all been fully checked. Besides these, material is available for the calculation (already carried out in some cases) for the same n and k, and for $q = 4(5)44$ or, sometimes, $q = 9(10)39$ or, when $n = 2(1)50$ and $k = 100$, for $q = 4$, 9 and 14 only.

Campbell 1930 (756) gives all leading forward differences, at $k = 0$, for the evaluation of the sums

$$\sum_{x=mk+1}^{m(k+1)} x^n \qquad n = 0(1)6 \qquad m = 2(1)10$$

4·45. Sums of Powers of Odd Integers

Bailey 1931 (358) gives $2\sum_{r=1}^{x}(2r-1)^n$ for $n = 2(2)6$, $x = 1(1)25$. He also gives, with $R = 2r-1$, $4\{x\Sigma R^4 - (\Sigma R^2)^2\}$, $4\{\Sigma R^6 \Sigma R^2 - (\Sigma R^2)^4\}$ and $\Sigma R^4/\Sigma R^2$; all values are exact. Jäderin 1915 (130) gives $2\sum_{r=1}^{x}(2r-1)^n$ for $n = 2(2)10$, $x = 1(1)10$; he also gives many dependent quantities similar to those given by Bailey.

4·5. Euler Polynomials

The arrangement is similar to that of Art. 4·4.

4·51. Coefficients

Notations are as follows:

$$U_n(x) = V_n(x) - 2^n V_n(\tfrac{1}{2}x)$$
$$= \tfrac{1}{2}nx^{n-1} - n_2(2^2-1)B_1x^{n-2} + n_4(2^4-1)B_2x^{n-4} - \ldots + (-1)^x U_n(0) \quad (4{\cdot}5111)$$

where $U_{2n}(0) = (-1)^n(2^{2n}-1)B_n$; $U_{2n+1}(0) = 0$, $n > 0$; $U_1(0) = \frac{1}{2}$, and n_r is the binomial coefficient ${}^nC_r$. Also, writing $nA_n'(x) = U_n(x) = \frac{1}{2}nE_{n-1}(x)$ (Davis), we have

$$(-1)^{x-1}\Sigma_{n-1}(x-1) = \sum_{r=1}^{x-1}(-1)^{x-r+1}r^{n-1} = \frac{1}{n}\{U_n(x) - (-1)^x U_n(0)\}$$
$$= A_n'(x) - (-1)^x A_n'(0) \text{ (Glaisher)}$$

The following polynomial expansions have been considered:

$E_n(x)$ in powers of x (4·5112)

$\dfrac{E_{2n+1}(x)}{x-\frac{1}{2}}$ and $\dfrac{E_{2n}(x)}{x(x-1)}$ in powers of x (4·5113)

$E_n(x) = (x-\frac{1}{2})^n - n_2 2^{-2}E_1.(x-\frac{1}{2})^{n-2} + n_4 2^{-4}E_2.(x-\frac{1}{2})^{n-4} - \ldots$ (4·512)

This corresponds to the expansion of $\Sigma_n(x)$ in powers of $x+\frac{1}{2}$.

$\dfrac{E_{2n}(x)}{2x+1}$ and $E_{2n+1}(x)$ in powers of x^2+x (4·513)

For other series of powers with alternating signs, see Arts. 4·514 and 4·515.

$\Sigma_n(x) = \sum_{r=1}^{x}(-1)^r r^n$ should not be confused with $\Sigma_n = \sum_{p=1}^{\infty} p^{-n}$, p prime.

4·5111. $U_n(x)$ in Powers of x

$n = 1(1)8$ — Glaisher 1898*a* (121)

4·5112. $\frac{1}{2}E_{n-1}(x) = A_n'(x)$ in Powers of x

$n = 1(1)8$ — Glaisher 1897*b* (20), 1898*a* (94)

4·5113. $\dfrac{E_{2n+1}(x)}{x-\frac{1}{2}}$ and $\dfrac{E_{2n}(x)}{x(x-1)}$ in Powers of x

$n = 1(1)10$ — Davis 1935 (275)
$n = 1(1)6$ — Nörlund 1924 (24)
$n = 1(1)6$ — Milne-Thomson 1933 (146)

4·512. $A_n'(x)$ in Powers of $x-\frac{1}{2}$

$a_n'(x) = A_n'(x+\frac{1}{2}) = \frac{1}{2}E_{n-1}(x+\frac{1}{2})$ in powers of x $n = 1(1)8$ Glaisher 1897*b* (35)

4·513. $\dfrac{\Sigma'_{2m-1}(x)}{2x+1}$ and $\Sigma_{2m}(x)$ in Powers of x^2+x

$$\Sigma'_{2m-1}(x) = \Sigma_{2m-1}(x) + \frac{(-1)^m(2^{2m}-1)}{2m}B_m$$

Suffix = 1(1)13 Glaisher 1899*a* (176)

4·514. The Series $x^n - (x-q)^n + (x-2q)^n - \ldots \pm r^n$

Glaisher 1900*b* sums a number of these series. These include expansions in powers of x for the cases

$q = 2(1)4$	$r = 1(1)q$	$n = 1(1)4$	p. 202

with expansions in powers of $x+\frac{1}{2}q$ on page 213. Also, writing $x = q(m-1)+r$, expansions are given in powers of m for the cases

$q = 2, 3, 4, 6$	r prime to and less than q	$n = 1(1)5$	p. 226

4·515. $\Sigma_{n,n}(x) = 1^n(x-1)^n - \ldots \pm r^n(x-r)^n \mp \ldots \pm (x-1)^n 1^n$

Glaisher 1900*c* sums a number of these series. Expansions are given in powers of x for the cases

$n = 1(1)6$	r proceeding by units	x even	p. 243
$n = 1(1)5$	r proceeding by twos	$\frac{1}{2}x$ even	p. 245

4·52. Special Values of $U_n(x)$ and $E_n(x)$

Glaisher 1898*a* gives values of $U_n(x)$ and related polynomials in terms of the numbers B_n, E_n, I_n, etc. for various special values of x; in particular see pages 107 and 131–132. Alternatively see Glaisher 1911*c* (106).

Davis 1935 also gives a few special values of $E_n(x)$ on page 277.

4·53. Tables of $E_n(x)$ for $0 < x < 1$

Exact	$n = 2(1)5$	$x = 0(\cdot 01)1$	Davis 1935 (288, Kantz)
10 dec.	$n = 6(1)8$		

4·6. Sums of Inverse Powers of Integers

The notation we use has been given in Art. 4·022. We note here, however, connections with the psi function, Section 14, and with the Riemann zeta function, Sub-section 22·1. In fact, $\zeta(s) = 1 + \dfrac{1}{2^s} + \dfrac{1}{3^s} + \ldots$ when $s > 1$; we confine ourselves here to integral s, except in Arts. 4·6112, 4·6162 and 4·6212.

Again, using the notation

$$\psi(x) = \frac{d}{dx}\log_e x! = \frac{d}{dx}\log_e \Gamma(x+1)$$

the reduced derivatives of $\psi(x)$ are

$$\frac{(-1)^n\psi^{(n-1)}(x)}{(n-1)!} = \frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \frac{1}{(x+3)^n} + \ldots$$

so that all the series considered in this sub-section can be expressed as sums or differences of these reduced derivatives, mainly for fractional x in the interval (0, 1) but on occasion for larger x, e.g. $x = 2$, 3, etc.

Natural values of various sums are considered in Arts. 4·6111 to 4·66. Logarithmic values are considered in Art. 4·68 and certain irrational multiples of the various series are deferred to Sub-section 4·8. Art. 4·67 contains references to certain sums of series of inverse factorials, etc. Many of the sums are multiples of B_n, E_n, etc., as indicated in Art. 4·022.

4·6111. $S_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \ldots$ and $S'_n = S_n - 1$

The constant γ (see Art. 5·41) is often given for S_1, otherwise undefined.

16 dec.	$n = 2(1)16$	Euler 1755, 1913 (349)

Final digits (3 to 6) in error for n = 5, 7, 11 and 13 in Euler 1755, corrected in Euler 1913.

16 dec.	$n = 2(1)10$	Lacroix 1800 (139)

From Euler, with his errors at n = 5 and 7.

16 dec.	$n = 2(1)35$	Legendre 1814 (65), 1826 (432)

Also given in De Morgan 1842*a* (554), Merrifield 1881 (8), Gram 1884 (269), Brunel 1886 (51), Newman 1892 (38) and Edwards 1922 (144).

22 dec.	$n = 3$	Clausen 1830*a* (381)
16; 17; 21 d.	$n = 2(1)17;\ 18(1)23;\ 24(1)40$	Oettinger 1856 (7)

Errors:	n	Decimals	For	Read
	6	2	1·077	1·017
	22	16, 17	502 56	502 72
	26	17, 18	839	828
	32	16, 17	203	183

22–24 dec.	$n = 2(2)12$	Glaisher 1877*b* (145)
32 dec.	$n = 2(1)70$	Stieltjes 1887 (300)
24 dec.	$n = 24(2)38$	Glaisher 1891*a* (352)
20 dec.	$n = 2(2)12$	Jones 1893 (38)
32 dec.	$n = 2(1)107$	Glaisher 1914*b* (148); Davis 1935 (244, 218)
32 dec.	$n = 2(1)100$	Peters & Stein 1922 (90)
16 dec.	$n = 2(1)35$	B.A. **1**, 1931 (xxii, Lodge)
16 dec.	$n = 2(1)20$	Davis 1933 (280)

In general the accuracy of these tables is high, only the final digit containing errors of a unit or so. Euler 1755, Lacroix and Oettinger have larger errors, as indicated, but there are none exceeding a final unit in the tables of Legendre, Stieltjes, Glaisher and Peters & Stein, all of which are from original calculations.

Other tables include

10 dec.	$n = 2(1)20$	Dale 1903 (92)
10 dec.	$n = 2(1)11$	Hayashi 1930*b* (171)
10 dec.	$n = 2(1)11$	Adams & Hippisley 1922 (140)

4·6112. S_n for Fractional n

6 dec.	$n = 1 + \frac{1}{m},\ m = 1(1)20$	Glaisher 1872*a* (55)

$$4{\cdot}6113.\quad \frac{S'_n}{n} = \frac{S_n - 1}{n}$$

32 dec.	$n = 1(1)51$	Johnston (MS.)
10 dec.	$n = 1(2)1{\cdot}1$	Láska 1888–94 (272)

Error: $\frac{1}{3}S'_3$ should read 0·06735 23010 5.

$$4{\cdot}6114.\quad \frac{S_n}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \ldots \text{ and } 1 + \frac{S_n}{2^n}$$

23 dec.	$n = 2(2)48$	Euler 1748 (153)

Errors: $n = 16$, for 69977 read 71502; $n = 22$, for 9154 read 9255.

16; 17; 21 d.	$n = 2(1)17;\ 18(1)23;\ 24(1)40$	Oettinger 1856 (19)

One error: $n = 22$, 20th and 21st decimals, for 79 read 95.

32 dec.	$n = 2(1)100$	Peters & Stein 1922 (91)

$$4{\cdot}6115.\quad \frac{S_n - 1}{2^n} = \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \ldots$$

32 dec.	$n = 2(1)50$	Peters & Stein 1922 (92)

$$4{\cdot}6116.\quad \frac{2}{n}\,\frac{S_n - 1}{2^n},\ \text{etc.}$$

20 dec.	$n = 2(2)30$	Euler 1748 (155)
20 dec.	$n = 2(2)30$	Callet 1795 (52)

Callet (50) also gives 20-decimal values of $2S_n/n.2^n$ for $n = 2(2)24$.

$$4{\cdot}6117.\quad 2(n-1)\frac{S_n - 1}{2^{2n}}$$

Glaisher 1912*b* gives incidentally the following quantities in which the notation $n_r = n(n-1)\ldots(n-r+1)/r! = {}^nC_r$ is used.

	33 dec.	$2(n-1)(S_n - 1)/2^{2n}$	$n = 3(2)37$	p. 51
	33 dec.	$2(n-1)(S_n - 1 - 1/2^n)/2^{2n}$	$n = 3(2)31$	p. 52
	31 dec.	$2(n-1)_3(S_n - 1)/2^{2n}$	$n = 5(2)39$	p. 54
	31 dec.	$2(n-1)_3(S_n - 1 - 1/2^n)/2^{2n}$	$n = 5(2)31$	p. 55
	29 dec.	$2(n-1)_5(S_n - 1)/2^{2n}$	$n = 7(2)39$	p. 56
	29 dec.	$2(n-1)_5(S_n - 1 - 1/2^n)/2^{2n}$	$n = 7(2)31$	p. 57
Also	18 dec.	$2(n-1)_7(S_n - 1 - 1/2^n)/2^{2n}$	$n = 9(2)21$	Glaisher 1912*a* (40)

$$4{\cdot}612.\quad \phi(x) = \sum_{r=1}^{\infty} \frac{x}{r(r+x)} = \sum_{r=1}^{x} \frac{1}{r}$$

Certain of these sums are given as *recurring* decimals; these are not specially indicated. The order is chronological.

42 dec.	$x = 10, 20, 25, 50, 100$	Oettinger 1862 (376)
264+ dec.	$x = 1000$	Adams 1877*b* (15)
264+ dec.	$x = 500, 1000$	Adams 1878*b* (93, 94)

24 dec.	$x = 6(1)25$	Aldis 1899 (223)
18 dec.	$x = 26(1)28$	
10 dec.	$x = 2(1)450$	Glover 1930 (458)

Shanks 1867 to 1872*b* give a number of values; these are unreliable—see Glaisher 1871*b* and Adams 1878*b*.

See also Art. 14·48.

4·613. $\left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \ldots + \frac{1}{x^n}\right)(x!)^n$

Glaisher (*Quart. J. Math.*, **31**, 338, 1900) gives these quantities for

$$x = p - 1 = 4, \quad n = 1(1)20 \qquad x = p - 1 = 6, \quad n = 1(1)16$$

4·614. $S_n - 1 - \frac{1}{2^n} - \ldots - \frac{1}{x^n} = \frac{1}{(x+1)^n} + \frac{1}{(x+2)^n} + \ldots = \frac{(-1)^n}{(n-1)!}\psi^{(n-1)}(x)$

32 dec.	$x = 2(1)9$	$n = 2(1)41$	Miller (MS.)

Some cases are included, with $x > 6$, which have 32 zeros.

4·6161. $s_n = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \ldots$

12 dec.	$n = 2(2)10$	Spence 1809 (50)
16; 17; 21 d.	$n = 2(1)17;\ 18(1)23;\ 24(1)40$	Oettinger 1856 (11)

Errors affecting the first 16 decimals are as follows; there are also others.

n	Decimals	For	Read
10	5, 6	039	093
10	14, 15	0225	0715
24	9	941	940
37	12	993	992

16 dec.	$n = 1(1)35$		Glaisher 1872*a* (56)
22–24 dec.	$n = 2(2)12$		Glaisher 1877*b* (145)
10 dec.	$n = 1(1)20$		Dale 1903 (92)
32 dec.	$n = 1(1)107$		Glaisher 1914*b* (153)
32 dec.	$n = 1(1)100$	Called t_n	Davis 1935 (247)

4·6162. s_n for Fractional n

6 dec.	$n = \frac{1}{m}$	$m = 1(1)20$	Glaisher 1872*a* (55)
5 dec.	$n = 1 + \frac{1}{m}$		

4·6163. $\frac{2}{n}s_n$, etc.

20 dec.	$n = 2(2)18$	Errors!	Gudermann 1830*b* (194)
5 dec.	$n = 2(2)16$		Knight 1935 (279)

Knight also gives 6-figure values of $2(2q+2p-1)_{2q}s_{2q+2p}$ for $p = 1(1)6$, $q = 0(1)8$.

$$4\cdot 6171.\quad s'_n = 1 - s_n = \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \dots$$

16 dec.	$n = 2(1)34$	Newman 1892 (39)

Error: $n = 11$, for 8324 7126 read 8285 6501.

20 dec.	$n = 2(2)18$	Newman 1892 (39)

Error: $n = 4$, for 7640 read 7540.

32 dec.	$n = 1(1)107$	Glaisher 1914*b* (156)
32 dec.	$n = 1(1)100$	Peters & Stein 1922 (93)

$$4\cdot 6172.\quad \frac{s'_n}{2^{2n-2}}$$

18 dec.	$n = 1(2)19$	Glaisher 1912*c* (101)

$$4\cdot 6173.\quad \frac{s'_n}{2^n(n+1)}$$

22– dec.	$n = 2(2)34$	Glaisher 1877*c* (202)

$$4\cdot 618.\quad s_n - 1 + \frac{1}{2^n} = \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \dots$$

32 dec.	$n = 1(1)53$	Peters & Stein 1922 (94)

$$4\cdot 6211.\quad U_n = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \dots \text{ and } U_n - 1$$

23 dec.	$n = 2(2)44$	Euler 1748 (150)

Error: $n = 10$, 19th and 20th decimals, for 50 read 48.

16; 17; 21 d.	$n = 2(1)17;\ 18(1)23;\ 24(1)40$	Oettinger 1856 (16)

Errors:

n	Decimals	For	Read
5	8	782	762
25	17	228	238
26	18	413	412
34	19, 20	62	96

16 dec.	$n = 2(1)35$		Glaisher 1872*a* (56)
22–24 dec.	$n = 2(2)12$	Called σ_n	Glaisher 1877*b* (145)
16 dec.	$n = 2(1)23$		Newman 1892 (38)

Error: $n = 13$, for 6218 read 4218.

12 dec.	$n = 2(2)24$		Sheppard 1899 (43)
10 dec.	$n = 2(1)20$	Called T_n	Dale 1903 (92)
32 dec.	$n = 2(1)67$		Glaisher 1914*b* (151)
32 dec.	$n = 2(1)50$		Peters & Stein 1922 (92)
32 dec.	$n = 2(1)67$	Called s_n	Davis 1935 (245)

$$4\cdot 6212.\quad U_n \text{ for Fractional } n$$

5; 4 dec.	$n = 1 + \frac{1}{m},\quad m = 1(1)18;\ 19, 20$	Glaisher 1872*a* (55)

4·6213. $\frac{2}{n}(U_n - 1)$, etc.

20 dec.	$n = 2(2)34$	Euler 1748 (156)
20 dec.	$n = 2(2)38$	Callet 1795 (51)

Callet 1795 (50) also gives 20-decimal values of $2U_n/n$ for $n = 2(2)24$.

4·622. $\frac{1}{n}\left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right)$

Exact fractions	$n = 1(1)14$	Rothe 1800 (313)

4·6261. $u_n = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \ldots$

The sum u_2 is known as Catalan's constant.

7 dec.	$n = 3(2)13$	Euler 1785 (251)

The final one or two digits are unreliable.

14 dec.	$n = 2$		Catalan 1864 (620)
25 dec.	$n = 2$		Bresse 1867 (1140)
9 dec.	$n = 2$		Glaisher 1876 (76)
20 dec.	$n = 2$		Glaisher 1877*c* (203), 1877*d* (190), 1893*a* (34)
12 dec.	$n = 3(2)25$		Sheppard 1899 (43)
16; 15 dec.	$n = 4;\ 6$		Glaisher 1903 (21, 27)
23; 21 dec.	$n = 3;\ 5(2)11$		Glaisher 1912*a* (45)
18 dec.	$n = 1(1)38$		Glaisher 1912*a* (49)
32; 31; 29 d.	$n = 2;\ 4;\ 6$		Glaisher 1912*b* (53, 55, 58)
7; 8 dec.	$n = 2;\ 4, 6$		Adams & Hippisley 1922 (140)
13 dec.	$n = 2$		Silberstein 1923 (87)
18 dec.	$n = 1(1)38$	Called T_n	Davis 1935 (304)
32; 30; 28 d.	$n = 2;\ 4;\ 6$		

4·6262. $\frac{2}{n}u_n$

28–16 dec.	$n = 1(2)19$	Gudermann 1830*b* (193)

Errors: $n = 13$, 15th decimal, for 74429 read 74426; $n = 19$, the value is 0·10526 31578 04174 8; the last two digits are unreliable throughout.

30–18 dec.	$n = 1(1)37$	Miller (MS.)

4·6271. $u_n' = 1 - u_n = \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} - \ldots$

13 dec.	$n = 2(1)20$	Newman 1892 (45)

Error: $n = 4$, for 5488 read 5448.

32 dec.	$n = 1(1)53$	Peters & Stein 1922 (94)
23; 21 dec.	$n = 1, 3;\ 5(2)11$	Glaisher 1912*a* (45)
18 dec.	$n = 1(1)38$	Glaisher 1912*a* (48)
33; 31; 29 d.	$n = 2;\ 4;\ 6$	Glaisher 1912*b* (52, 54, 57)

$$4{\cdot}6272.\quad \frac{2}{n}u_n' = \frac{2}{n}(1-u_n)$$

25–18 dec.	$n = 1(2)17$	Errors!	Gudermann 1830*b* (194) Newman 1892 (45)

Gudermann also gives a wrong value for $n = 19$.

18 dec.	$n = 2(2)34$	Glaisher 1912*a* (43)

$$4{\cdot}6273.\quad \frac{u_n'}{2^{n-2}}$$

21 dec.	$n = 2(2)28$	Glaisher 1912*c* (96)

$$4{\cdot}6274.\quad (n-1)\frac{u_n'}{2^{n-2}}$$

20 dec.	$n = 2(2)28$	Glaisher 1912*c* (98)

$$4{\cdot}6281.\quad u_n'' = 1 - \frac{1}{3^n} + \frac{1}{5^n} - u_n = \frac{1}{7^n} - \frac{1}{9^n} + \frac{1}{11^n} - \dots$$

33; 31; 29 d.	$n = 2;\ 4;\ 6$	Glaisher 1912*b* (51, 54, 56)

$$4{\cdot}6282.\quad u_n''' = \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{15^n} - \dots$$

33; 31; 29 d.	$n = 2;\ 4;\ 6$	Glaisher 1912*b* (52, 55, 57)

$$4{\cdot}6291.\quad \frac{(u_n)'}{2^{n-2}} = \left(\frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \dots\right)\Big/ 2^{n-2}$$

21 dec.	$n = 2(2)20$	Glaisher 1912*c* (97)

$$4{\cdot}6292.\quad (n-1)\frac{(u_n)'}{2^{n-2}}$$

20 dec.	$n = 2(2)22$	Glaisher 1912*c* (99)

4·63–4·66. Other Series

Very few have been evaluated numerically, though several have been summed in terms of known constants and of the numbers considered in Sub-section 4·3. Note that some of the equation numbers of Art. 4·022 are not represented as article numbers; this indicates that we have not found any published numerical values.

Soddy 1943 (126) tabulates sums of the "harmonic" series

$${}_dS_S^a = \frac{1}{a} - \frac{1}{a+d} + \frac{1}{a+2d} - \dots = \frac{1}{2d}\left[\psi\left(\frac{a}{2d} - \frac{1}{2}\right) - \psi\left(\frac{a}{2d} - 1\right)\right]$$

for all integers $a < d < 16$ with a prime to d. He gives 12 to 15 decimals when $a < \frac{1}{2}d$, and 7 decimals when $a > \frac{1}{2}d$.

$$4{\cdot}651.\quad p_n = 1 + \frac{1}{3^n} - \frac{1}{5^n} - \frac{1}{7^n} + + - - \ldots$$

31 dec.	$p_1 = \pi/2\sqrt{2}$	Glaisher 1912*c* (97–99)
19 dec.	p_2	Davis 1935 (285)

Glaisher also gives what seems to be a fairly reliable 20th decimal for p_2, and the values of $1 + \frac{1}{3^2} - p_2$ and $p_2 - 1 - \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}$ to the same accuracy.

Templeton-Maynard 1858 (171) gives $\pi\sqrt{2}$ and $\pi/\sqrt{2}$ to 31 decimals, and their common logarithms to 30 decimals.

$$4{\cdot}656.\quad q_1 = \frac{1}{\sqrt{2}} \log_e (\sqrt{2}+1) = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + - - + \ldots$$

21– dec.	Glaisher 1912*c* (96, 97)
20 dec.	Davis 1935 (285)

Glaisher also gives the values of $1 - \frac{1}{3} - q_1$ and $q_1 - 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ to 21 decimals, and (page 101) the value of $q_1 - 1 + \frac{1}{3} + \frac{1}{5}$ to 18 decimals.

$$4{\cdot}66.\quad 1 + \frac{1}{5^2} + \frac{1}{9^2} + \ldots \quad \text{and} \quad \frac{1}{3^2} + \frac{1}{7^2} + \frac{1}{11^2} + \ldots$$

9 dec.	Glaisher 1876 (75, 76); Davis 1935 (284)
20 dec.	Glaisher 1877*c* (203), 1877*d* (190)

Glaisher 1876 also gives 9-decimal values of various partial sums.

4·67. Sums involving Factorials, etc.

$$4{\cdot}671.\quad M_n = \frac{0!}{n!} + \frac{1!}{(n+1)!} + \frac{2!}{(n+2)!} + \ldots = \frac{1}{(n-1).(n-1)!}$$

24–41 dec.	$n = 2(1)21$	Oettinger 1856 (33)

$$4{\cdot}672.\quad K_n = \frac{0!}{n!} - \frac{1!}{(n+1)!} + \frac{2!}{(n+2)!} - \frac{3!}{(n+3)!} + \ldots$$

48– dec.	$n = 1(1)12$	Miller (MS.)
18–20 dec.	$n = 2(1)10$	Oettinger 1856 (31)

K_6, K_7 and K_8 are in error. Correct by using $K_n = nN_{n+1}$; see the next article.

$$4{\cdot}673.\quad N_n = \frac{0!}{n!} + \frac{2!}{(n+2)!} + \frac{4!}{(n+4)!} + \ldots = \frac{K_{n-1}}{n-1}$$

18 dec.	$n = 2(1)11$	Oettinger 1856 (35)

N_4 and N_6 are in error. Correct by using $nN_{n+1} = K_n$; see the preceding article.

$$4{\cdot}674. \quad Q_n = \frac{1!}{(n+1)!} + \frac{3!}{(n+3)!} + \frac{5!}{(n+5)!} + \ldots$$

48 dec.	$n = 2(1)12$	Miller (MS.)
18 dec.	$n = 2(1)11$	Oettinger 1856 (39)

Errors: Q_6, for 815 729 read 814 229.
Q_7 should read 0·000 026 791 975 113 542.
Q_8 should start 0·000 002 930 ...

Note that $M_n = K_n + 2Q_n = \dfrac{1}{(n-1)\,.\,(n-1)!}$ is correct throughout.

$$4{\cdot}676. \quad K_n = \frac{1}{e}\sum_{t=0}^{\infty} \frac{t^n}{t!} \quad \text{and} \quad p_m = \sum_{t=1}^{\infty} \frac{(\log_e t)^m}{t!}$$

It may be noted that $K_n = \sum_{p=0}^{n} \Delta^p 0^n/p!$ when n is a positive integer (see Art. 4·9241), and that it is the coefficient of x^n in the expansion of $\exp(e^x - 1)$; see Art. 5·216. Early coefficients are given in a number of references; for $n = 1(1)9$ the values are 1, 2, 5, 15, 52, 203, 877, 4140 and 21147. Touchard 1933 gives two further values, and Epstein 1939 (155) gives them for $n = 1(1)20$, together with a number of earlier references. Epstein also gives the values to 10 decimals for $-n = 1(1)20$.

Epstein 1939 (159) gives p_m and $p_m/m!$ to 10 figures or 15 decimals and $\sum_{k=1}^{m} p_k/k!$ to 10 decimals, all for $m = 1(1)15$.

4·68. Logarithmic Values

4·6811. $\log_e S_n$

15 dec.	$n = 2(1)34$	Merrifield 1881 (9); Gram 1884 (269)
24– dec.	$n = 2(2)80$	Glaisher 1891*a* (350)
24 dec.	$n = 2(1)80$	Davis 1935 (249, Glaisher, Kantz & Scotten)

4·6812. $\log_{10} S_n$

15 dec.	$n = 2(1)35$	Gram 1884 (269)

4·6821. $\log_e U_n$

12 dec.	$n = 2(4)22$	Glaisher 1891*d* (380)
14 dec.	$n = 6$	Glaisher 1893*a* (39)
14 dec.	$n = 4, 12, 20$	Glaisher 1893*a* (41)

4·6822. $-\log_e u_n$

12 dec.	$n = 1(2)25$	Glaisher 1891*d* (379)
20 dec.	$n = 2$	Glaisher 1893*a* (36)
12 dec.	$n = 6$	Glaisher 1893*a* (38)
14 dec.	$n = 10, 14, 22, 26$	Glaisher 1893*a* (40, 41)

4·6823. $\mathrm{Log}_{10}\, u_n$

18 dec.	$n = 1(1)38$		Davis 1935 (304, Kantz & Scotten)

4·6831. $\mathrm{Log}_e\, G_n$

12 dec.	$n = 2(4)14$		Glaisher 1893*b* (57)

4·6832. $-\mathrm{Log}_e\, g_n$

12 dec.	$n = 1(2)11$		Glaisher 1893*b* (56)

4·6841. $\mathrm{Log}_e\, W_n$

12 dec.	$n = 2(4)14$	W_n is called G'_n	Glaisher 1893*b* (57)

4·6842. $-\mathrm{Log}_e\, h_n$

12 dec.	$n = 1(2)17$	h_n is called g'_n	Glaisher 1893*b* (57)

4·7. Sums and Products involving Primes only

4·711. $\Sigma_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \dots$

15 dec.	$n = 2(2)36$	Euler 1748 (237)

Errors in 5 final digits corrected in *Opera Omnia*, (1) **8**, 1922.

15 dec.	$n = 2(2)36$	Glaisher 1877*f* (175)
24 dec.	$n = 2$	Glaisher 1878*d* (192)
15 dec.	$n = 2(1)35$	Merrifield 1881 (10); Gram 1884 (269)

Merrifield and Gram have Σ_3 in error; the first 5 digits should read 17476. The value opposite Σ_1 refers to the constant $G - g$ (Art. 4·79); see Glaisher 1891*c* (373).

15 dec.	$n = 3$	Glaisher 1891*c* (373)
24 dec.	$n = 2(2)80$	Glaisher 1891*a* (353)
24 dec.	$n = 2(1)80$	Davis 1935 (249, Glaisher, Kantz & Scotten)

4·712. $\frac{2}{n}\Sigma_n$

24 dec.	$n = 2(2)74$	Glaisher 1891*a* (355)

The table is incorrectly headed Σ_n/n.

4·72. $Y_n = \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} - \frac{1}{13^n} - \frac{1}{17^n} + \dots$

12 dec.	$n = 1(2)25$	Glaisher 1891*d* (381)
14– dec.	$n = 2$	Glaisher 1893*a* (42)

Euler 1785 (255) gives values to 7 decimals (5 or 6 correct) for $n = 1(2)13$.

$$4{\cdot}731. \quad \Gamma_n = \frac{1}{2^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{11^n} - \frac{1}{13^n} + \dots$$

12 dec.	$n = 1(2)39$	Glaisher 1893*b* (59)

$$4{\cdot}732. \quad \Gamma_n' = \Gamma_n - \frac{1}{2^n}$$

12 dec.	$n = 1(2)17$	Glaisher 1893*b* (59)

4·75. Other Series of Primes

Glaisher 1893*b* (62, 63) gives 12-decimal values of $Y_1 - \frac{1}{3}$, $\Gamma_1 - \frac{1}{2}$,

$$\tfrac{1}{2}(Y_1 + \Gamma_1 - \tfrac{5}{6}) = \tfrac{1}{11} - \tfrac{1}{13} + \tfrac{1}{23} - \tfrac{1}{37} + \tfrac{1}{47} + \tfrac{1}{59} - \tfrac{1}{61} - \dots$$

and

$$\tfrac{1}{2}(\Gamma_1 - Y_1 - \tfrac{1}{6}) = \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{17} - \tfrac{1}{19} + \tfrac{1}{29} - \tfrac{1}{31} + \tfrac{1}{41} + \dots$$

$$4{\cdot}78. \quad P(p) = (1 - \tfrac{1}{2})(1 - \tfrac{1}{3})(1 - \tfrac{1}{5})\dots\left(1 - \frac{1}{p}\right)$$

6 dec.	$2P(p)$	$p < 1229$	Legendre 1808 (T. 9)
7 dec.	$P(p)$	$p < 10{,}000$	Glaisher 1898*b* (2)
7 dec.	$\text{Log}_{10} P(p)$		

4·79. The Constant $G - g$

Glaisher writes

$$\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{5} + \dots + \frac{1}{p} \sim g + \log_e \log_e p$$

and

$$\log_e \left(\tfrac{2}{1} \cdot \tfrac{3}{2} \cdot \tfrac{5}{4} \cdot \dots \cdot \frac{p}{p-1} \right) \sim G + \log_e \log_e p$$

Then

$$G - g = 0{\cdot}31571\ 84520\ 73890$$

according to Merrifield 1881 (7), who incorrectly identifies it with g; see Glaisher 1891*c* (373) and 1893*c* (105). The latter also gives $\log_e 2 - G + g$ to 15 decimals.

Rosser 1938 (43) finds G to be Euler's constant γ and gives

$$G - g = 0{\cdot}31571\ 84520\ 53890\ 07685\ 1085 \quad \text{(last digit doubtful)}$$

He also gives g to 15 decimals. Note the discrepancy in the 11th decimal of $G - g$. Rosser uses the notations B_∞ and C_∞ for g and $e^{-\gamma}$ respectively.

Rosser also finds (p. 44) $D_\infty = 0{\cdot}83242\ 90656\ 62$ and gives $\log_e D_\infty$ and $1/D_\infty$ to 13 figures, where

$$\log_e \left(\tfrac{3}{1} \cdot \tfrac{5}{3} \cdot \dots \cdot \frac{p}{p-2} \right) \sim -\log_e D_\infty + 2 \log_e \log_e p$$

Sutton 1937 (33) evaluates, to about 4 decimals,

$$2 \prod_{p=3}^{\infty} \left(1 - \frac{1}{(p-1)^2} \right) = \tfrac{1}{2} D_\infty / C_\infty^2 = \tfrac{1}{2} e^{2\gamma} D_\infty = 1{\cdot}32032\ 36317$$

4·8. Irrational Multiples of Bernoulli Numbers, etc., or of Sums of Inverse Powers of Integers

$$4{\cdot}811.\quad \frac{4}{\pi}\frac{S_{2n}-1}{2^{2n}}$$

These are coefficients in an expansion for cot $\frac{1}{2}\pi x$.

24 dec.	$n = 1(1)20$	Andoyer 1915 (2)
13 dec.	$n = 1(1)10$	Euler 1748 (102); Hobson 1891 (341), 1921 (364)

$$4{\cdot}812.\quad \frac{8}{\pi}\frac{S_{2n}-1}{2^{4n}}$$

24 dec.	$n = 1(1)13$	Peters & Stein 1922 (96)

4·813. $\frac{2S_3}{\pi^3} = 0{\cdot}07753\ 63592\ 05833\ 59787$ Watson 1923 (579)

$\frac{7S_3}{8\pi^3} = 0{\cdot}03392\ 21571\ 52552\ 19907$ Schott & Watson 1916 (332)

$$4{\cdot}814.\quad \frac{M}{n}\frac{2^{2n-1}(2^{2n}-1)B_n}{(2n)!}\quad \text{and}\quad \frac{M}{n}\frac{2^{2n-1}B_n}{(2n)!}$$

20 dec.	$n = 1(1)10$	Sang 1884*b* (618)

$$4{\cdot}815.\quad (-1)^{n-1}\frac{MB_n}{(2n-1)2n}$$

40–18 dec.	$n = 1(1)14$	Duarte 1927 (xiii)

$$4{\cdot}816.\quad \frac{M}{n}\frac{S_{2n}-1}{2^{2n}},\ \text{etc.}$$

These are coefficients in expansions for $\log_{10}$ sin $\frac{1}{2}\pi x$.

21 dec.	$n = 1(1)16$	Andoyer 1911 (7)
20 dec.	$n = 1(1)15$	Callet 1795 (53)
15 dec.	$n = 1(1)10$	Euler 1748 (157); Hobson 1891 (344), 1921 (367)

Callet 1795 also gives other 20-decimal values, as follows:

$\frac{M}{n}\frac{S_{2n}}{2^{2n}}$	$n = 1(1)12$	p. 51
$\frac{M}{n}\frac{1}{2^{2n}}\left(S_{2n}-1-\frac{1}{2^{2n}}-\dots-\frac{1}{r^{2n}}\right)$	$r = 2(1)5,\quad n = 1(1)13-r$	pp. 54–56

$$4{\cdot}817.\quad \frac{M}{n}\frac{S_{2n}-1}{2^{4n}}$$

24 dec.	$n = 1(1)12$	Peters & Stein 1922 (97)

$$4{\cdot}818.\quad \frac{M}{n}(S_n - 1)$$

14–12 dec.	$n = 1(2)15$	Schlömilch 1848 (171)
10 dec.	$n = 1(2)15$	Láska 1888–94 (289)

Common logarithms to 10 decimals are also given in each case.

$$4{\cdot}821.\quad \frac{4}{\pi}\frac{1 - s_{2n}}{2^{2n}}$$

These are coefficients in an expansion for cosec $\frac{1}{2}\pi x$.

24 dec.	$n = 1(1)20$	Andoyer 1915 (3)

$$4{\cdot}822.\quad \frac{8}{\pi}\frac{1}{2^{4n}}\left(s_{2n} - 1 + \frac{1}{2^{2n}}\right)$$

24 dec.	$n = 1(1)11$	Peters & Stein 1922 (96)

$$4{\cdot}826.\quad \frac{M}{n}\frac{1}{2^{2n}}\left(s_{2n} - 1 + \frac{1}{2^{2n}}\right)$$

These are coefficients in an expansion for $\log_{10}$ tan $\frac{1}{4}\pi x$.

24 dec.	$n = 1(1)14$	Peters & Stein 1922 (97)

$$4{\cdot}831.\quad \frac{4}{\pi}(U_{2n} - 1)$$

These are coefficients in an expansion for tan $\frac{1}{2}\pi x$.

24 dec.	$n = 1(1)25$	Andoyer 1915 (2)
13 dec.	$n = 1(1)13$	Euler 1748 (102)
13 dec.	$n = 1(1)13$	Hobson 1891 (341), 1921 (364)

$$4{\cdot}832.\quad \frac{8}{\pi}\frac{1}{2^{2n}}(U_{2n} - 1)$$

24 dec.	$n = 1(1)15$	Peters & Stein 1922 (95)

$$4{\cdot}835.\quad \frac{360}{\pi}\frac{U_{2n}}{90^{2n}}$$

33 dec.	$n = 1(1)8$	Herrmann 1848*b* (474)

$$4{\cdot}836.\quad \frac{M}{n}(U_{2n} - 1),\ \text{etc.}$$

These are coefficients in expansions for $\log_{10}$ cos $\frac{1}{2}\pi x$.

21 dec.	$n = 1(1)20$	Andoyer 1911 (7)
20 dec.	$n = 1(1)18$	Callet 1795 (52)
13 dec.	$n = 1(1)13$	Euler 1748 (158); Hobson 1891 (344), 1921 (367)

Callet 1795 also gives other 20-decimal values, as follows:

$\dfrac{M}{n} U_{2n}$	$n = 1(1)12$	p. 50
$\dfrac{M}{n}\left(U_{2n} - 1 - \dfrac{1}{3^{2n}} - \dots - \dfrac{1}{r^{2n}}\right)$	$r = 3(2)9,\ n = 1(1)N,\ N = 13, 10, 9, 8$	pp. 54-56

4·837. $\dfrac{M}{n}\dfrac{U_{2n}-1}{2^{2n}}$

24 dec.	$n = 1(1)14$	Peters & Stein 1922 (97)

4·841. $\dfrac{4}{\pi}(1 - u_{2n+1})$

These are coefficients in an expansion for sec $\frac{1}{2}\pi x$.

24 dec.	$n = 1(1)25$	Andoyer 1915 (3)

4·842. $\dfrac{4}{\pi}\dfrac{1}{2^{2n}}(1 - u_{2n+1})$

24 dec.	$n = 1(1)15$	Peters & Stein 1922 (96)

4·843. $\dfrac{2}{\pi}u_2$

22– dec.	Glaisher 1877*c* (203)
20 dec.	Glaisher 1877*d* (190), 1893*a* (34)

4·844. Exp $\left(-\dfrac{1}{2} + \dfrac{2}{\pi}u_2\right)$

8 dec.	Glaisher 1876 (73)

4·846. $\dfrac{2M}{2n+1}(1 - u_{2n+1})$

These are coefficients in an expansion for $\log_{10}$ tan $\frac{1}{4}\pi(1-x)$.

18 dec.	$n = 0(1)16$	Andoyer 1911 (7)

4·9. Generalized Bernoulli and Euler Numbers and Polynomials. Differences and Derivatives of Zero, etc. Coefficients in Formulæ connecting Derivatives and Integrals with Differences

See Art. 4·023 for definitions, relationships, etc. All values listed are exact unless otherwise stated. Throughout this sub-section n, m, p, q and r denote integers.

4·911. $B_p^{(n)}$ as a Polynomial in n

$B_0^{(n)} = 1 \qquad B_1^{(n)} = -\frac{1}{2}n \qquad B_2^{(n)} = \frac{1}{12}n(3n-1) \qquad B_3^{(n)} = -\frac{1}{8}n^2(n-1)$

$p = 0(1)12$	Nörlund 1924 (459); Davis 1935 (209)
$p = 0(1)6$	Milne-Thomson 1933 (127)

G. Scott 1867 (27) gives, for $p = 0(1)11$, expressions of which the part in curled brackets is $a \,.\, 2^p B_p^{(1-z)}/(z-1)$ where a is the coefficient of the highest power of z in each bracket. For $p = 9$ the polynomial should read

$$15z^8 + 60z^7 - 210z^6 - 392z^5 + 1295z^4 - 100z^3 - 1676z^2 + 1008z$$

and for $p = 10$ the coefficient of z^5 should be 1281, not 128.

Nielsen 1906 (76) gives $\psi_{p-1}(n) = B_p^{(n)}/n \,.\, p!$ for $p = 1(1)7$ as polynomials in n.

Cayley 1860 (367) gives $rB_r^{(r+p)}/(r+p) \,.\, p! = B_r^{(r+p+1)}(1)/p!$ for $p = 0(1)6$ as polynomials in r.

Glaisher 1900*a* (24) gives, for $p = 1(1)7$, expressions for $(-1)^p (n-1)_p B_p^{(n)}$, and (p. 25) for $(-1)^p n_p B_p^{(n+1)}$, in powers of n, factorized as far as possible.

4·912. $B_p^{(n)}(\frac{1}{2}n)$ as a Polynomial in n

$B_{2q+1}^{(n)}(\frac{1}{2}n) = 0$ $\quad B_0^{(n)}(\frac{1}{2}n) = 1$ $\quad B_2^{(n)}(\frac{1}{2}n) = -\frac{1}{12}n$ $\quad B_4^{(n)}(\frac{1}{2}n) = \frac{1}{240}n(5n+2)$

$p = 0(2)12$	$2^p B_p^{(n)}(\frac{1}{2}n) = D_p^{(n)}$	Nörlund 1924 (460)
$p = 0(2)8$	$B_p^{(n)}(\frac{1}{2}n)$	Milne-Thomson 1933 (160)

4·916. $E_p^{(n)}(0)$ as a Polynomial in n

$E_0^{(n)}(0) = 1$ $\quad E_1^{(n)}(0) = -\frac{1}{2}n$ $\quad E_2^{(n)}(0) = \frac{1}{4}n(n-1)$ $\quad E_3^{(n)}(0) = -\frac{1}{8}n^2(n-3)$

$E_4^{(n)}(0) = \frac{1}{16}n(n^3 - 6n^2 + 3n + 2)$ $\quad E_5^{(n)}(0) = -\frac{1}{32}n^2(n^3 - 10n^2 + 15n + 10)$

4·917. $E_p^{(n)}(\frac{1}{2}n)$ as a Polynomial in n

$E_{2q+1}^{(n)}(\frac{1}{2}n) = 0$ $\quad E_0^{(n)}(\frac{1}{2}n) = 1$ $\quad E_2^{(n)}(\frac{1}{2}n) = -\frac{1}{4}n$ $\quad E_4^{(n)}(\frac{1}{2}n) = \frac{1}{16}n(3n+2)$

4·921. $B_p^{(n)}$

The following short table may be used for identification and for the determination of signs.

n	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
$B_0^{(n)}$	1	1	1	1	1	1	1	1	1	1	1	1	1
$B_1^{(n)}$	3	$\frac{5}{2}$	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2	$-\frac{5}{2}$	-3
$B_2^{(n)}$	$\frac{19}{2}$	$\frac{20}{3}$	$\frac{13}{3}$	$\frac{5}{2}$	$\frac{7}{6}$	$\frac{1}{3}$	0	$\frac{1}{6}$	$\frac{5}{6}$	2	$\frac{11}{3}$	$\frac{35}{6}$	$\frac{17}{2}$
$B_3^{(n)}$	$\frac{63}{2}$	$\frac{75}{4}$	10	$\frac{9}{2}$	$\frac{3}{2}$	$\frac{1}{4}$	0	0	$-\frac{1}{2}$	$-\frac{9}{4}$	-6	$-\frac{25}{2}$	$-\frac{45}{2}$
$B_4^{(n)}$	$\frac{1087}{10}$	$\frac{331}{6}$	$\frac{243}{10}$	$\frac{43}{5}$	$\frac{31}{15}$	$\frac{1}{5}$	0	$-\frac{1}{30}$	$\frac{1}{10}$	$\frac{19}{10}$	$\frac{251}{30}$	24	$\frac{274}{5}$
$B_5^{(n)}$	$\frac{777}{2}$	$\frac{675}{4}$	$\frac{185}{3}$	$\frac{69}{4}$	3	$\frac{1}{6}$	0	0	$\frac{1}{6}$	$-\frac{3}{4}$	-9	$-\frac{475}{12}$	-120

$n = -9(1)+10, \quad p = 1(1)10$		Miller (MS.)
$n = -27(1)-1, \quad p = 1(1)7$		Gordon 1939 (179)
$n = 1(1)10, \quad p = 1(1)10$	Called $\dfrac{D^n}{\Delta^n} 0^p$	Jeffery 1862 (94)

Error: $n = 9, p = 9$. The denominator should be 20.

$n = p = 0(1)12$ and $n = p - 1 = 1(1)11$	Nörlund 1924 (461, T. 7, 8)
$n = p = 1(1)10$	Milne-Thomson 1933 (136)

4·9221. $\dfrac{B_p^{(n)}}{p!}$

When $p = n$ these are coefficients in the formula for $\int^{(0)} f(x)dx$ in terms of $\Delta^n f_1$. For $p = n = 0(1)4$ the values are 1, $-\frac{1}{2}$, $\frac{5}{12}$, $-\frac{3}{8}$ and $\frac{251}{720}$.

$n = -7(1)+2$, $p = 0(1)7+n$	Called $D_j^i = D_{p-n}^{-n}$	Andoyer 1906 (105)
$n = -10(1)-1$, $p = 0(1)5$	Signs omitted	Glover 1930 (489)
$n = -4(1)+2$, $p = 0(1)5$	Called β_p, with $s = -n$	Ser 1933 (7)
$n = 2$, $p = 0(1)9$	Called G_{-2}^{p-2}	Andoyer 1923 (265)
$n = p = 1(1)20$		Lowan & Salzer 1943 (49)
$n = p = 0(1)12$	12-decimal values	Nörlund 1924 (462, T. 9)
$n = p = 0(1)7$	Signs omitted, eqn. (i)	Bennett, Milne & Bateman 1933 (28)
$n = p = 0(1)5$	Signs omitted, eqn. (7)	Sheppard 1900*a* (425)
$n = p = 0(1)5$	Signs omitted, eqn. (36)	Nyström 1925 (32)
$n+1 = p = 2(1)6$	Signs omitted, called S_n	Nyström 1925 (35)

See also Art. 4·612, since $\sum^{n} \frac{1}{r} = (-1)^{n-1} B_{n-1}^{(n+1)}/(n-1)!$

4·9222. $\dfrac{B_n^{(n)}}{n\,.\,n!}$

$n = 1(1)6$ — Cayley 1860 (366)

4·9231. $\dfrac{D^p 0^{(n)}}{p!} = (n-1)_{p-1} B_{n-p}^{(n)} = n_p B_{n-p}^{(n+1)}(1)$

These numbers are integers, known as Stirling's numbers (of the first kind), to which the sign $(-1)^{n-p}$ is attached; signs are omitted in most of the following tables. We tabulate a few values:

	$n = 1$	2	3	4	5
$p = 1$	$+1$	-1	$+2$	-6	$+24$
2		$+1$	-3	$+11$	-50
3			$+1$	-6	$+35$
4				$+1$	-10

$n = 1(1)22$	$p = 1(1)n$	Glaisher 1900*a* (26)
$n = 1(1)16$	$p = 1(1)n$	Thiele 1909 (31)

Due to N. P. Bertelsen, who carried the calculations to $n = 29$.

$n = 1(1)12$	$p = 1(1)n$	Jordan 1933 (255); Davis 1935 (215)
$n = 1(1)10$	$p = 1(1)n$	Markoff 1896 (27); Steffensen 1927 (57); Sasuly 1934 (329)
$n = 1(1)9$	$p = 1(1)n$	Adams & Hippisley 1922 (159)

The table is arranged with arguments n (including also negative arguments to $n = -4$) and $n-p$, which runs from 0 to 8.

$n = 1(1)9$	$p = 1(1)n$	Jeffery 1861 (364)

Jordan 1933 (259) gives coefficients, to $r = 7$, in an expansion of $D^{n-r}0^{(n)}/(n-r)!$ as a series of binomial coefficients in n. Ward 1934 (92) gives coefficients to $r = 10$ in a similar expansion.

$$4\cdot9232. \quad \frac{D^p 0^{(n)}}{n!} = \frac{p}{n}\frac{B^{(n)}_{n-p}}{(n-p)!} = \frac{B^{(n+1)}_{n-p}(1)}{(n-p)!}$$

These are coefficients in the formulæ (Markoff's) for $D^p f_0$ in terms of $\Delta^n f_0$. A few values follow:

	$n = 1$	2	3	4	5
$p = 1$	$+1$	$-\frac{1}{2}$	$+\frac{1}{3}$	$-\frac{1}{4}$	$+\frac{1}{5}$
2		$+1$	-1	$+\frac{11}{12}$	$-\frac{5}{6}$

$p = -5(1)+5$	$n = p(1)p+7$		Thiele 1909 (83)
$p = -4(1)+4$	$n = p(1)p+5$		Ser 1933 (7)

Denoted by $(-1)^{n-p}a_{n-p}$, with $p = s$.

$p = -2(1)+7$	$n = p(1)7$	Called E^n_p	Andoyer 1906 (105)
$p = 1(1)20$	$n = p(1)20$	Signs omitted	Lowan, Salzer & Hillman 1942 (922)
$p = 1(1)12$	$n = p(1)12$	Signs omitted	Bickley & Miller 1942 (11)
$p = 1(1)8$	$n = p(1)8$		Sheppard 1911 (164); Davis 1933 (73)
$p = 2$	$n = 4(1)8$	Called $(-1)^n B^n_2$	Lindow 1928 (165)

$$4\cdot9241. \quad \frac{\Delta^p 0^n}{p!} = n_p B^{(-p)}_{n-p} = (n-1)_{p-1} B^{(1-p)}_{n-p}(1)$$

These numbers are positive integers, also known as Stirling's numbers (of the second kind). We tabulate a few values:

	$p = 1$	2	3	4	5
$n = 1$	1	1	1	1	1
2		1	3	7	15
3			1	6	25
4				1	10

$n = 2(1)25$	$p = 2(1)n$	Fisher & Yates 1938 (52), 1943 (60)
$n = 1(1)20$	$p = 1(1)n$	Cayley 1881 (2)
$n = 1(1)16$	$p = 1(1)n$	Thiele 1909 (32)

Due to N. P. Bertelsen, who carried the calculations to $n = 29$.

$n = 1(1)12$	$p = 1(1)n$	Grunert 1843 (279); Jordan 1933 (263); Davis 1935 (212)
$n = p(1)p+7$	$p = 1(1)4$	Adams & Hippisley 1922 (159)

This table is an extension for negative n (replaced by $-p$ in this article) of a table of Stirling's numbers of the first kind; see Art. 4·9231.

$n = 1(1)10$	$p = 1(1)n$	Kramp 1800 (359)

Error: p. 360, in X^{10}, the second term should be $511x_2$.

$n = 1(1)10$	$p = 1(1)n$	Markoff 1896 (26); Steffensen 1927 (55); Sasuly 1934 (330)

$n = 1(1)10$	$p = 1(1)n-1$	De Morgan 1842*a* (253)
$n = 2(1)9$	$p = 1(1)n$	Fisher & Wishart 1931 (197)

Cayley 1881 (3) gives coefficients, to $r = 6$, in the expansion of $\Delta^{n-r}0^{(n)}/(n-r)!$ as a series of binomial coefficients in n. Coefficients in similar expansions are given by Jordan 1933 (265) to $r = 7$, and by Ward 1934 (92) to $r = 10$.

See Art. 4·676 for $K_n = \sum_{p=0}^{n} \frac{\Delta^p 0^n}{p!}$.

4·9242. $\Delta^p 0^n = \dfrac{n!}{(n-p)!} B_{n-p}^{(-p)}$, Differences of Zero

$n = 1(1)12$	$p = 1(1)n$	Burgess 1898 (267); Davis 1935 (212)
$n = 1(1)10$	$p = 1(1)n$	Herschel 1820 (9); De Morgan 1842*a* (253); Hall 1845 (238)
$n = 1(1)10$	$p = 1(1)n$	Boole 1860 (20)

In the 1860 edition only.

$n = 1(1)10$	$p = 1(1)n$	King 1902 (424)
$n = 1(1)10$	$p = 1(1)$ min. $(6, n)$	Whittaker & Robinson 1926 (7)

4·9243. $\dfrac{\Delta^p 0^{n+p}}{(n+p)!} = \dfrac{B_n^{(-p)}}{n!}$

12 dec.	$n = 0(1)20$	$p = 1(1)20$	E. M. Elderton & Moul 1931 (307)
12 dec.	$n = 0(1)20$	$p = 1(1)20$	Pearson 1931 (246)
12 dec.	$n = 0(1)10$	$p = 1(1)10$	Pearson & Elderton 1925 (200)
Exact	$n = 0(1)6$	$p = 7-n$	Andoyer 1923 (265)

4·9251. $\dfrac{p}{n-p}\,\dfrac{B_n^{(n-p)}}{n!} = -\dfrac{B_n^{(n-p+1)}(1)}{n!}$

When $p = 1$ these are coefficients in the formula (Gregory's) for $\int^{(0)} f(x)\,dx$ in terms of $\Delta^n f_0$. Signs are frequently omitted or reversed. For $p = 1$, $n = 0(1)7$, the values are -1, $-\frac{1}{2}$, $\frac{1}{12}$, $-\frac{1}{24}$, $\frac{19}{720}$, $-\frac{3}{160}$, $\frac{863}{60480}$ and $-\frac{275}{24192}$; more are given as follows:

$n = 1(1)20$		Lowan & Salzer 1943 (49)
$n = 1(1)17$	10^+ to 5-decimal values	Fisher & Yates 1938 (61), 1943 (69)
$n = 1(1)14$	7-decimal values	Pearson 1902 (274)
$n = 2(1)13$		Clausen 1830*b* (288)
$n = 1(1)13$		Ser 1933 (15)

Error: $n = 10$, the denominator should read 479001600.

$n = 2(1)12$	12-decimal values	Nörlund 1924 (462, T. 10)
$n = 2(1)9$		B.A. 1880 (330)
$n = 2(1)9$	Called J_1^{n-1}	Lindow 1928 (172)
$n = 0(1)8$	Called H_{-1}^{n-1}	Andoyer 1923 (270)

For $p = 2$, $n = 0(1)5$ the values are -1, -1, $-\frac{1}{12}$, 0, $\frac{1}{240}$ and $-\frac{1}{240}$; more are given as follows:

$n = 4(1)10$	Called J_2^{n-2}	Lindow 1928 (172)
$n = 0(1)9$	Called H_{-2}^{n-2}	Andoyer 1923 (270)
$n = 0(1)7$	Equation (298)	Rice 1899 (163)

$$4\cdot9252. \quad \frac{B_n^{(n-1)}}{n(n-1)\,.\,n!} = -\frac{B_n^{(n)}(1)}{n\,.\,n!}$$

$n = 1(1)14$	Signs omitted	Bretschneider 1837 (258)

Bretschneider also gives 37-decimal values for the same n.

$4\cdot931.\ B_p^{(n)}(\frac{1}{2}n)$

The following short table may be used for identification and for the determination of signs. $B_{2q+1}^{(n)}(\frac{1}{2}n)$ is zero.

n	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
$B_0^{(n)}(\frac{1}{2}n)$	1	1	1	1	1	1	1	1	1	1	1	1	1
$B_2^{(n)}(\frac{1}{2}n)$	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	0	$-\frac{1}{12}$	$-\frac{1}{6}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{5}{12}$	$-\frac{1}{2}$
$B_4^{(n)}(\frac{1}{2}n)$	$\frac{7}{10}$	$\frac{23}{48}$	$\frac{3}{10}$	$\frac{13}{80}$	$\frac{1}{15}$	$\frac{1}{80}$	0	$\frac{7}{240}$	$\frac{1}{10}$	$\frac{17}{80}$	$\frac{11}{30}$	$\frac{9}{16}$	$\frac{4}{5}$
$B_6^{(n)}(\frac{1}{2}n)$	$\frac{32}{21}$	$\frac{1135}{1344}$	$\frac{17}{42}$	$\frac{205}{1344}$	$\frac{1}{28}$	$\frac{1}{448}$	0	$-\frac{31}{1344}$	$-\frac{5}{42}$	$-\frac{457}{1344}$	$-\frac{31}{42}$	$-\frac{1835}{1344}$	$-\frac{191}{84}$

$n = -10(1)+10$	$p = 0(2)8$	$B_p^{(n)}(\frac{1}{2}n)$	Miller (MS.)
$n = p$ or $p-1$	$p = 0(2)12$	$2^p B_p^{(n)}(\frac{1}{2}n) = D_p^{(n)}$	Nörlund 1924 (463, T. 11, 12)

$$4\cdot9321. \quad \frac{Dp\,0^{[n]}}{p!} = (n-1)_{p-1} B_{n-p}^{(n)}(\tfrac{1}{2}n)$$

These are integers when n and p are both even; they vanish if $n-p$ is odd. They are usually given in two tables, with n and p both even or both odd; a few values follow:

	$n = 1$	3	5	7
$p = 1$	$+1$	$-\frac{1}{4}$	$+\frac{9}{16}$	$-\frac{225}{64}$
3		$+1$	$-\frac{5}{2}$	$+\frac{259}{16}$
5			$+1$	$-\frac{35}{4}$

	$n = 2$	4	6	8
$p = 2$	$+1$	-1	$+4$	-36
4		$+1$	-5	$+49$
6			$+1$	-14

$n = 2(2)16$	$p = 2(2)n$	Signs omitted	Thiele 1909 (35)
$n = 1(1)12$	$p = 1$ or $2(2)n$		Davis 1935 (215)
$n = 1(1)12$	$p = 1$ or $2(2)n-2$		*N.A.* 1937 (803)

$$(-1)^s C_{r-1}^{(s)} = D^{2(r-s)} 0^{[2r]}/(2r-2s)! \qquad (-1)^s S_r^{(s)} = D^{2r-2s+1} 0^{[2r+1]}/(2r-2s+1)!$$

Error in original edition: C_3'' should be 49.

$n = 1(1)10$	$p = 1$ or $2(2)n$	Steffensen 1927 (59)

The same quantities, multiplied by 2^{n-p} and without signs, are given as follows:

$n = 3(1)20$	$p = 1$ or $2(2)n$	Oppolzer 1880 (7)
$n = 1(2)15$	$p = 1(2)n$	Thiele 1909 (36)

$$4\cdot9322. \quad \frac{D^p 0^{[n]}}{n!} = \frac{p}{n} \frac{B_{n-p}^{(n)}(\tfrac{1}{2}n)}{(n-p)!}$$

These are coefficients in the formulæ for $D^p f_r$ in terms of $\delta^n f_r$. Take $r = 0$ when p and n are even and $r = \frac{1}{2}$ when p and n are odd. A few values follow:

	$n = 1$	3	5	7		$n = 2$	4	6	8
$p = 1$	$+1$	$-\frac{1}{24}$	$+\frac{3}{640}$	$-\frac{5}{7168}$	$p = 2$	$+1$	$-\frac{1}{12}$	$+\frac{1}{90}$	$-\frac{1}{560}$
3		$+1$	$-\frac{1}{8}$	$+\frac{37}{1920}$	4		$+1$	$-\frac{1}{6}$	$+\frac{7}{240}$
5			$+1$	$-\frac{5}{24}$	6			$+1$	$-\frac{1}{4}$

$p = 1(1)19$	$n = p(2)19$ or 20	Called M_1^n, N_2^n, M_3^n, etc.	Oppolzer 1880 (21)
$p = 2(2)52$	$n = p(2)N$		Salzer 1943*a* (120)

Exact values, $N = 32$ (at least), $p+10$ or 52 (at most); 18-decimal extension to $N = 42$.

$p = 1(1)5$	$n = p(2)p+10$		Thiele 1909 (95)
$p = 1(1)12$	$n = p(2)11$ or 12		Jäderin 1915 (29)

Given as exact fractions, to 15 decimals and as 7-decimal logarithms.

$p = 1(1)12$	$n = p(2)11$ or 12		*N.A.* 1937 (802)
$p = 0(1)12$	$n = p(2)11$ or 12	Signs omitted	Bickley & Miller 1942 (12, 14)
$p = 1, 2$	$n = p+2(2)11$ or 12	Called M_1^n and N_2^n	Lindow 1928 (164)
$p = 0(1)8$	$n = p(2)7$ or 8		Sheppard 1900*b* (465)
$p = 0(1)6$	$n = p(2)7$ or 8		Sheppard 1906 (327), 1911 (165)
$p = 0(1)7$	$n = p(2)6$ or 7	Called G_1^n, G_2^n, etc.	Andoyer 1906 (117)

See Art. 4·9343 for negative p.

$$4\cdot9323. \quad \frac{D^p 0^{[n]}}{p\,.\,(n-1)!} = \frac{B_{n-p}^{(n)}(\tfrac{1}{2}n)}{(n-p)!}$$

These are coefficients in the formulæ for $D^{p-1} f_r$ in terms of $\mu\delta^{n-1} f_r$. Take $r = 0$ when p and n are even and $r = \frac{1}{2}$ when p and n are odd. A few values follow:

	$n = 1$	3	5	7		$n = 2$	4	6	8
$p = 1$	$+1$	$-\frac{1}{8}$	$+\frac{3}{128}$	$-\frac{5}{1024}$	$p = 2$	$+1$	$-\frac{1}{6}$	$+\frac{1}{30}$	$-\frac{1}{140}$
2		$+1$	$-\frac{5}{24}$	$+\frac{259}{5760}$	4		$+1$	$-\frac{1}{4}$	$+\frac{7}{120}$
3			$+1$	$-\frac{7}{24}$	6			$+1$	$-\frac{1}{3}$

$p = 1(1)19$	$n = p(2)20$ or 21	Called M_0^{n-1}, N_1^{n-1}, M_2^{n-1}, etc.	Oppolzer 1880 (21)
$p = 2(2)52$	$n = p(2)N$		Salzer 1943*a* (120)

Exact values, $N = 32$ (at least), $p+10$ or 52 (at most); 18-decimal extension to $N = 42$.

$p = 1(1)6$	$n = p(2)p+10$		Thiele 1909 (96)
$p = 1(1)13$	$n = p(2)12$ or 13		Jäderin 1915 (29)

Given as exact fractions, to 15 decimals and as 7-decimal logarithms.

$p = 1(1)13$	$n = p(2)12$ or 13		*N.A.* 1937 (802)
$p = 1(1)13$	$n = p(2)12$ or 13	Signs omitted	Bickley & Miller 1942 (12, 14)

$p = 1, 3$	$n = 3(2)13$	Called M_0^{n-1} and M_2^{n-1}	Lindow 1928 (164)
$p = 2$	$n = 4(2)12$	Called N_1^{n-1}	
$p = 1(1)9$	$n = p(2)8$ or 9		Sheppard 1900*b* (465)
$p = 1(1)7$	$n = p(2)8$ or 9		Sheppard 1906 (327), 1911 (165)
$p = 1(1)8$	$n = p(2)7$ or 8	Called $G_0'^{n-1}$, $G_1'^{n-1}$, etc.	Andoyer 1906 (117)

See Art. 4·9341 for p zero or negative.

4·9331. $\dfrac{\delta^p 0^n}{p!} = n_p B_{n-p}^{(-p)}(-\tfrac{1}{2}p)$, etc.

These numbers, like $D^p 0^{[n]}/p!$, are integers when n and p are both even, and vanish if $n-p$ is odd; as with $D^p 0^{[n]}/p!$ they are usually split into two tables. A few values follow:

	$n = 1$	3	5	7		$n = 2$	4	6	8
$p = 1$	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$p = 2$	1	1	1	1
3		1	$\frac{5}{2}$	$\frac{91}{16}$	4		1	5	21
5			1	$\frac{35}{4}$	6			1	14

$n = 2(2)16$	$p = 2(2)n$		Thiele 1909 (35)
$n = 1(2)17$	$p = 1(2)n$	Multiplied by 2^{n-p}	Thiele 1909 (36)
$n = 1(1)13$	$p = 1$ or $2(2)n$		Sheppard 1913 (95)
$n = 1(1)12$	$p = 1$ or $2(2)n$		Davis 1935 (213)
$n = 1(1)10$	$p = 1$ or $2(2)n$		Steffensen 1927 (57)

4·9332. $\delta^p 0^n = \dfrac{n!}{(n-p)!} B_{n-p}^{(-p)}(-\tfrac{1}{2}p)$, Central Differences of Zero

$n = 1(1)12$	$p = 1(1)n$		Davis 1935 (213)
$n = 2(2)12$	$p = 2(2)n$		Jäderin 1915 (27, 28)
$n = 1(2)11$	$p = 1(2)n$	Multiplied by 2^p	

Jäderin also gives $\mu\delta^{p-1}0^n$ and $\mu(2\delta)^{p-1}0^{n-1}$, which are the same quantities divided by $2p$, to $n = 13$; in addition he gives logarithmic values to 7 decimals.

4·9333. $\dfrac{\delta^{2q}0^{2m}}{2^q} = \dfrac{(2m)!}{2^m \cdot (2m-2q)!} B_{2(m-q)}^{(-2q)}(-q)$

$m = 1(1)32$	$q = 1(1)m$	Called $e_{q,m}$	Joffe 1916 (110)
$m = 33(1)50$	$q = 1(1)m$		Joffe 1920 (203)
$m = 1(1)12$	$q = 1(1)m$		Davis 1935 (283)

4·9334. $\dfrac{\delta^p 0^n}{n!} = \dfrac{1}{(n-p)!} B_{n-p}^{(-p)}(-\tfrac{1}{2}p)$

These are coefficients in the formulæ for $\delta^p f_r$ in terms of $D^n f_r$. Take $r = 0$ when p and n are even and $r = \frac{1}{2}$ when p and n are odd.

$\pm p = 1(1)5$	$n = p(2)p+12$		Thiele 1909 (102)
$p = 2(2)12$	$n = p(2)12$	12-decimal values	Hayashi 1933 (8)
$p = 2(2)8$	$n = p(2)10$		*N.A.* 1937 (806)
$p = 1(1)8$	$n = p(2)7$ or 8		Sheppard 1900*b* (464)
$p = -2(1)+7$	$n = p(2)6$ or 7		Andoyer 1906 (116)
$p = 1$	$n = 1(2)9$	Equation (112)	Sheppard 1900*b* (476)

4·9335. $\dfrac{n}{p\,.\,(n-p)!}B_{n-p}^{(-p)}(-\tfrac{1}{2}p)$

These are coefficients in the formulæ for $\mu\delta^{p-1}f_r$ in terms of $D^{n-1}f_r$. Take $r = 0$ when p and n are even and $r = \frac{1}{2}$ when p and n are odd.

$p = -4(1)+6$	$n = p(2)p+6$		Thiele 1909 (103)
$p = 2(2)12$	$n = p(2)12$	12-decimal values	Hayashi 1933 (6)
$p = 2(2)8$	$n = p(2)10$		*N.A.* 1937 (806)
$p = 1(1)9$	$n = p(2)8$ or 9		Sheppard 1900*b* (464)
$p = -1(1)+8$	$n = p(2)7$ or 8		Andoyer 1906 (116)

4·9336. $\dfrac{2^n\,.\,n!}{2p\,.\,(n-p)!}B_{n-p}^{(-p)}(-\tfrac{1}{2}p)$

$p = 1(2)13$	$n = p(2)13$		Jäderin 1915 (28)

Jäderin also gives 7-decimal logarithms of these constants.

4·9341. $\dfrac{1}{(2m)!}B_{2m}^{(2m-p)}(m-\tfrac{1}{2}p) = -\dfrac{2m-p}{p\,.\,(2m)!}B_{2m}^{(2m-p+1)}(m-\tfrac{1}{2}p)$

When $p = 0$, 1 and 2 these are coefficients in the formulæ for $\int^{(r)} f(x)\,dx$, $\iint^{(r)} f(x)(dx)^2$ and $\iiint^{(r)} f(x)(dx)^3$ respectively, in terms of $\mu\delta^{2m-p-1}f_r$. Take $r = 0$ when p is even and $r = \frac{1}{2}$ when p is odd. A few values follow, for $m = 0(1)3$:

$p = 0$	1	$-\frac{1}{12}$	$\frac{11}{720}$	$-\frac{191}{60480}$
$p = 1$	1	$-\frac{1}{24}$	$\frac{17}{1920}$	$-\frac{367}{193536}$

$p = 0, 1$	$m = 1(1)10$	Q_1^{2m-1} and P_2^{2m-2}	Oppolzer 1880 (545)

Oppolzer also gives 20-decimal values of $\log Q_1^{2m-1}$ and $\log P_2^{2m-2}$.

$p = 0$	$n = 1(1)28$	M_{2m}	Salzer 1944*a* (263)

Values are exact for $m \leqslant 10$, 18-decimal for $m > 10$.

$p = 0$	$m = 1(1)6$	Q_1^{2m-1}	Lindow 1928 (170, 171)
$p = 1$	$m = 1(1)7$	P_2^{2m-2}	
$p = 2$	$m = 2(1)4$	Q_3^{2m-3}	
$p = 0$	$m = 0(1)6$	12-decimal values	Nörlund 1924 (463, T. 13)
$p = 0(1)4$	$m = 0(1)5$		Thiele 1909 (97)
$p = 0$; 1	$m = 0(1)5$; 0(1)4		Sasuly 1934 (126, Hill)
$p = 0, 1$	$m = 0(1)4$	Equations (11) and (38)	Sheppard 1900*a* (429, 446)
$p = 0, 1$	$m = 0(1)4$	$G_{-1}^{\prime 2m-1}$ and $G_{-2}^{\prime 2m-2}$	Andoyer 1906 (117)
$p = 0, 1$	$m = 0(1)4$	$N_{-p-1}^{\prime 2m-p-1}$	Andoyer 1923 (277)
$p = 0, 1$	$m = 1(1)4$	M_{2m} and $-J_{2m}$	Steffensen 1927 (110, 117)
$p = 0, 1$	$m = 0(1)4$		*N.A.* 1937 (809)
$p = 0$	$m = 1(1)4$	M_{2m}	Nyström 1925 (34)

The third column gives the notation used by the author concerned.

B.A. 1880 (357) gives a formula with halved coefficients for $p = 0$, $m = 0(1)4$, but that for $m = 2$ has not been halved, and that for $m = 4$ is incorrect.

See Art. 4·9323 for negative p.

$$4{\cdot}9342.\quad \frac{2^{2m}}{(2m)!}B_{2m}^{(2m)}(m)$$

$m = 0(1)4$ Equation (49) Sheppard 1900*b* (457)

$$4{\cdot}9343.\quad \frac{-p}{(2m-p)\,.\,(2m)!}B_{2m}^{(2m-p)}(m-\tfrac{1}{2}p) = \frac{1}{(2m)!}B_{2m}^{(2m-p+1)}(m-\tfrac{1}{2}p)$$

When $p = 1$, 2 and 3 these are coefficients in the formulæ for $\int^{(r)} f(x)\,dx$, $\iint^{(r)} f(x)(dx)^2$ and $\iiint^{(r)} f(x)(dx)^3$ respectively, in terms of $\delta^{2m-p}f_r$. Take $r = 0$ when p is even and $r = \frac{1}{2}$ when p is odd. A few values follow, for $m = 0(1)3$:

$p = 1$	1	$+\frac{1}{24}$	$-\frac{17}{5760}$	$+\frac{367}{967680}$
$p = 2$	1	$+\frac{1}{12}$	$-\frac{1}{240}$	$+\frac{31}{60480}$

$p = 1, 2$	$m = 1(1)10$	P_1^{2m-1} and Q_2^{2m-2}	Oppolzer 1880 (545)
	Oppolzer also gives 20-decimal values of $\log P_1^{2m-1}$ and $\log Q_2^{2m-2}$.		
$p = 1$	$m = 1(1)6$	P_1^{2m-1}	Lindow 1928 (170, 171)
$p = 2$	$m = 1(1)7$	Q_2^{2m-2}	
$p = 3$	$m = 1(1)4$	P_3^{2m-3}	
$p = 1$	$m = 1(1)6$	12-decimal values	Nörlund 1924 (463, T. 14)
$p = 1$	$m = 0(1)6$	Errors, $m = 4$, $m = 6$	B.A. 1880 (358)
$p = 1(1)5$	$m = 0(1)5$		Thiele 1909 (96)
$p = 1$; 2	$m = 0(1)4$; 0(1)5		Sasuly 1934 (126, Hill)
$p = 1, 2$	$m = 0(1)4$	G_{-p}^{2m-p}	Andoyer 1906 (117)
$p = 1, 2$	$m = 0(1)4$	N_{-p}^{2m-p}	Andoyer 1923 (277)
$p = 1, 2$	$m = 1(1)4$	K_{2m}, $-N_{2m}$	Steffensen 1927 (108, 118)
$p = 1, 2$	$m = 0(1)4$		*N.A.* 1937 (809)
$p = 1$	$m = 0(1)4$	Equation (12)	Sheppard 1900*a* (430)
$p = 2$	$m = 0(1)4$	Equation (62)′	Nyström 1925 (37)

The third column gives the notation used by the author concerned.
See Art. 4·9322 for p negative or zero.

4·9344. Sheppard 1900*b* (456, 457) gives

$$\frac{2^{2n}}{(2n-1)\,.\,(2n)!}B_{2n}^{(2n-1)}(n-\tfrac{1}{2}) \qquad \frac{-2^{2n}}{(n-1)\,.\,(2n)!}B_{2n}^{(2n-2)}(n-1) \qquad \frac{2^{2n}}{(2n)!}B_{2n}^{(2n-1)}(n-\tfrac{1}{2})$$

in equations (47), (48) and (50) respectively, for $n = 0(1)4$.

$$4{\cdot}941.\quad B_p^{(n)}(\tfrac{1}{2})$$

$n = -6(1)+6$ $p = 0(1)5$. Miller (MS.)

See also Art. 4·99.

4·942. $-\dfrac{B_n^{(n)}(\frac{1}{2})}{n!}$

These are coefficients in the formula for $\int^{(\frac{1}{2})} f(x)\,dx$ in terms of $\Delta^n f_1$. The values for $n = 0(1)5$ are 1, 0, $\frac{1}{24}$, $-\frac{1}{24}$, $\frac{223}{5760}$ and $-\frac{103}{2880}$.

$n = 2(1)13$		Clausen 1830*b* (289)
$n = 2(1)9$	Signs omitted, called H_1^{n-1}	Lindow 1928 (172)

4·943. $\dfrac{B_{n-q}^{(n+1)}(\frac{1}{2})}{(n-q)!}$

These are coefficients in the formulæ for $D^q y_{1/2}$ in terms of $\Delta^n y_1$.

$q = 0(1)12$	$n = q(1)12$	Signs omitted	Bickley & Miller 1942 (13)

4·951. $\dfrac{B_{n-p}^{(n)}(x)}{(n-p)!}$

These are the coefficients of the non-central differences in formulæ for derivatives or integrals which involve partly central differences and partly advancing or retreating differences.

$p = 2(1)5$	$n = p(1)13$	$x = 1(1)n-1$		Bickley & Miller 1942 (T. III. $p-1$)
$p = 1(1)4$	$n = p(1)13$	$x = \frac{1}{2}(1)n-\frac{1}{2}$		Bickley & Miller 1942 (T. VI. $p-1$)
$p = 2, 3$	$n-1 = p(1)7$	$x = 3/2$	Called A_1^n, A_2^n	Lindow 1928 (165, 172, signs omitted)
$p = -1$	$n = 3(1)9$	$x = 3/2$	Called H_2^{n-2}	
$p = 2$	$n = 2(1)9$	$x = \frac{3}{2}$	Equation (94)	Thiele 1909 (84)
$p = 3$	$n = 3(1)10$	$x = 2$	Equation (96)	

Tables III. 1–4 and VI. 0–3 in Bickley & Miller 1942 have all been "spread" a line as compared with the "natural" table—two samples of the latter are given on page 10 of the paper. The result of this is that the coefficient $B_{n-p}^{(n)}(x)/(n-p)!$ accompanies a difference δ_s^{n-1} where $s+x = \frac{1}{2}(n-1)$, $\frac{1}{2}(n+1)$ respectively in the upper and lower halves of Tables III. 1–4 and $s+x = \frac{1}{2}n$, $\frac{1}{2}n+1$ respectively in the upper and lower halves of Tables VI. 0–3. MS. tables give further values for $x = \frac{1}{2}(\frac{1}{2})n-\frac{1}{2}$, $n = p(1)13$ to $p = 13$.

4·952. $\Delta^p x^n = \dfrac{n!}{(n-p)!} B_{n-p}^{(-p)}(x)$, etc.

Hayashi 1933 (6) gives exact values, in each case with $n = p(1)12$, and with unit interval in x, as follows:

p		1	2	3	4	5	6	7	8	9	10	11	12
x	from	−54	−55	−56	−58	−58	−58	−58	−57	−59	−58	−60	−60
	to	+53	+53	+53	+54	+53	+52	+51	+49	+50	+48	+49	+48

Hayashi 1933 (4, etc.) also gives decimal values of $\delta^p x^n/n! = B_{n-p}^{(-p)}(-\frac{1}{2}p+x)/(n-p)!$ to varying accuracy, mainly about 10 figures, for $n = p(1)12$, $p = 1(1)12$ and x or $x-\frac{1}{2} = 0(10)50$, $x-\frac{1}{2}p$ being integral.

These tables are all connected with the evaluation of differences in terms of derivatives.

4·961. $E_p^{(n)}(0)$

The following short table may be used for identification.

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$E_0^{(n)}(0)$	1	1	1	1	1	1	1	1	1	1	1	1	1
$E_1^{(n)}(0)$	3	$\frac{5}{2}$	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2	$-\frac{5}{2}$	-3
$E_2^{(n)}(0)$	$\frac{21}{2}$	$\frac{15}{2}$	5	3	$\frac{3}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{3}{2}$	3	5	$\frac{15}{2}$
$E_3^{(n)}(0)$	$\frac{81}{2}$	25	14	$\frac{27}{4}$	$\frac{5}{2}$	$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{2}$	0	-2	$-\frac{25}{4}$	$-\frac{27}{2}$
$E_4^{(n)}(0)$	168	90	$\frac{85}{2}$	$\frac{33}{2}$	$\frac{9}{2}$	$\frac{1}{2}$	0	0	-1	-3	$-\frac{9}{2}$	$-\frac{5}{2}$	$\frac{15}{2}$
$E_5^{(n)}(0)$	738	$\frac{1375}{4}$	137	$\frac{171}{4}$	$\frac{17}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$\frac{9}{4}$	13	$\frac{125}{4}$	$\frac{99}{2}$

4·962. $\dfrac{E_p^{(n)}(0)}{2^n}$

$n = 1(1)10$ $\quad$ $p = 1(1)10$ $\quad$ Called $\dfrac{1}{(2+\Delta)^n}0^p$ $\quad$ Jeffery 1862 (100)

Error: for $n = 4$, $p = 5$ the value should be 13/16.

4·97. $E_p^{(n)}(\frac{1}{2}n)$

The following short table may be used for identification. $E_{2q+1}^{(n)}(\frac{1}{2}n)$ is identically zero.

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$E_0^{(n)}(\frac{1}{2}n)$	1	1	1	1	1	1	1	1	1	1	1	1	1
$E_2^{(n)}(\frac{1}{2}n)$	$\frac{3}{2}$	$\frac{5}{4}$	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{3}{4}$	-1	$-\frac{5}{4}$	$-\frac{3}{2}$
$E_4^{(n)}(\frac{1}{2}n)$	6	$\frac{65}{16}$	$\frac{5}{2}$	$\frac{21}{16}$	$\frac{1}{2}$	$\frac{1}{16}$	0	$\frac{5}{16}$	1	$\frac{33}{16}$	$\frac{7}{2}$	$\frac{85}{16}$	$\frac{15}{2}$

4·99. Miscellaneous

Rothe 1800, which gives the Bernoulli numbers to B_{18}, obtains incidentally solutions to the difference equation

$$(m+1)p_m(n+1) + (m+2n-1)p_{m+1}(n) = (m+2n+1)p_{m+1}(n+1)$$

in which we have made some obvious slight changes of notation. The initial values may be taken as $p_m(1) = 1$ and $p_0(n) = 0$ for $n \neq 1$. Rothe tabulates, as fractions with factorized denominators, the values $p_m(n)$ for

$m = -9(1)-1$	$n = 1(1)6$	pp. 319–324
$m = 2$	$n = 1(1)14$	p. 313, Art. 4·622
$m = 3$	$n = 1(1)13$	p. 316

He also gives (p. 325) $[p_m(n)/m]_{m=0}$ for $n = 2(1)6$.

Note that

$$p_2(n) = \frac{1}{n}\sum_1^n \frac{1}{2n-1} = (-4)^{n-1}\frac{(n-1)!}{(2n-1)!}B_{n-1}^{(n+1)}(\tfrac{1}{2}) \qquad p_3(n) = \frac{3}{2n+1}\sum_1^n p_2(n)$$

SECTION 5

MATHEMATICAL CONSTANTS, π, e, M, γ, ETC.; MULTIPLES AND POWERS. ROOTS OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS. MISCELLANEOUS CONSTANTS. CONVERSION TABLES

This section gives references to many-figure values of basic constants. Some of these constants may also occur (as a rule to fewer figures) as entries in larger tables dealt with in other sections; such tables are not usually listed here, but a cross-reference is given. Section 4 should also be consulted; many constants connected with the Bernoullian functions are there considered. Also included in this section are references to miscellaneous constants, less commonly needed; these may have been evaluated only to a few decimals. In addition, the section deals with powers and multiples of constants. Conversion tables, being multiples of conversion factors, are also included, but our lists of these are extremely selective.

Numerical values of several of the constants are given (to at most 35 decimals); these seem likely to be more useful than lists of references to less accurate values. Some of them have been freshly derived (from others) and many of these are available to 40, 60 or more decimals; all that have been adopted without some numerical check have the source indicated. When a numerical value is given, the corresponding article contains only references to values more accurate than the one given, or to certain historically interesting papers.

The subdivision of the section is as follows:

"Multiples" as used in article headings does not necessarily mean integral, or even rational, multiples. In some cases, where the number of references is small, a constant and multiples of that constant may be included in a single article; the heading does not necessarily indicate this.

5·0†. Collections of Constants

Only a few collections are listed; these are noted because of the number of figures given, because of the choice of constants given, or because of the large number of constants given. Brackets indicate the presence of constants in the collection concerned to which no reference is made in the corresponding article.

† In Section 5, items added in proof are on pages 109–110.

Number of dec. or fig.	Reference	Articles
52–707 dec.	**Uhler 1921, 1937, 1938, 1940, 1942*b***	5·1, 5·2, 5·3, 5·5
50–140 dec.	Paucker 1841	5·1, 5·6
32–707 fig.	**Peters & Stein 1922**	5·1, 5·2, 5·3, 5·4, 5·5, 5·6, 5·9
105 dec.	**Bretschneider 1843**	5·2
12–40 dec.	Gudermann 1830*a*, *b*, *c*	(5·1), 5·2, 5·3
30 dec.	Templeton–Maynard 1858	5·1
30 dec. or fig.	Brandenburg 1931	5·1
	Pearson 1931	5·1, (5·2, 5·3)
23–30 dec.	Burgess 1898	5·1, 5·2, 5·3, 5·5, 5·8
10–30 dec.	Jones 1893	5·1, (5·2, 5·3, 5·4, 5·5)
	Glover 1930	(5·1, 5·2), 5·3, (5·4, 5·6, 5·8)
20+ dec. or fig.	Fisher & Yates 1938, 1943	(5·1, 5·2, 5·3), 5·4
	Jordan–Eggert 1939	(5·1, 5·2, 5·3), 5·5
	New York W.P.A. 1941*a*, *b*, *c*, *d*	(5·1, 5·2), 5·3, 5·4
15 dec.	**Barlow–Comrie 1930, 1941**	5·1, (5·2, 5·3, 5·6)
	Becker & Van Orstrand 1924	(5·1, 5·2), 5·3, 5·9
13 dec.	Lorán 1934	5·1, (5·2, 5·3, 5·5, 5·6)
10 dec.	Dale 1903	(5·1, 5·2, 5·3, 5·4)
	MT.C. 1931	5·1, (5·2, 5·3, 5·4, 5·5, 5·6)
	Peters 1938	(5·1, 5·2, 5·3, 5·4), 5·5, (5·6), 5·9
3–7 dec.	Adams & Hippisley 1922	5·7

Low & Low 1938 (140) gives 6-decimal values of the first 9 multiples of several constants involving π.

Terry 1938 gives a number of constants expressed in duodecimal notation. These are not listed separately below.

B.A. 1873 (81) gives information on some earlier collections.

For information on the accuracy of many-figure values, in a few cases, see the corresponding articles dealing with individual constants. Peters & Stein 1922 is reliable, but it should be noted that some many-figure constants have been reproduced without complete checking; for instance, Shanks's 707-decimal value of π is given, although only 64 decimals were recomputed. Errors have been found in Gudermann 1830*a*, *b*, *c*; his values should not be used without checking.

5·1. Constants involving π

5·111†. $\pi = 3{\cdot}14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ldots$

Histories of the numerical evaluation of π, formerly also known as Ludolph's number, are given by Glaisher (*Mess. Math.*, **2**, 119–128, 1872; **3**, 27–46, 1873), Muir (*Encycl. Brit.*, 11th ed., **6**, 384, 1910) and Ball–Coxeter 1939 (335–349). The following table is adapted from that given by Glaisher.

Decimals calculated	Decimals correct	Reference	
208	152	Rutherford 1841	
205	200	Dase 1844	
250	248	Clausen 1847	
440	440	Rutherford	Rutherford 1853
530	>500	Shanks	Rutherford 1853

Decimals calculated	Decimals correct	Reference
607	*	Shanks 1853
333	330	Richter 1853
261	261	Lehmann 1853 (159)
400	330	Richter 1854*a*
400	400	Richter 1854*b*
500	500	Richter 1855
707	*	Shanks 1873*a*
707	> 500	Shanks 1873*b*

The asterisk * indicates erroneous values; details of these errors are given in Shanks 1873*b*. A study of Shanks's work on $\log_e n$ and γ (see Arts. 5·3 and 5·4) makes it clear that it is unsafe to accept π as known to more than 500 decimals.

One modern calculation to 800 decimals has been projected and may be in progress. Uhler 1940 (208) has also verified Shanks's value to 333 decimals.

Further many-figure values are given as follows:

707 dec.	From Shanks 1873*b*	Peters & Stein 1922 (1)
208 dec.	From Rutherford 1841	Templeton–Maynard 1858 (169)
200 dec.	Original	Duarte 1920 (108)
40 dec.		Ramanujan 1911 (*Coll. Papers*, 11, 1927)
36 dec.		Brandenburg 1931 (336)

5·112†. Multiples of π

51 dec.	$\frac{1}{4}\pi$, $\frac{1}{8}\pi$		Paucker 1841 (10)
48 dec.	$\frac{1}{4}\pi$		Peters & Stein 1922 (xviii)
36 dec.	$\dfrac{\pi}{180n}$;	$n = 1, 60, 60^2$	Brandenburg 1931 (336)
35 dec.	$\dfrac{n\pi}{648\,000}$;	$n = 1(1)60(60)600(600)3600$	Herrmann 1848*b* (481)
32 dec.	$n\pi$;	$n = 1(1)9$	Peters & Stein 1922 (1)
30 dec.	$n\pi$;	$n = 1(1)100$	Kulik 1848 (419, 420)
30 dec.	$n\pi$; π/n;	$n = 2, 4, 36, \frac{2}{3}, \frac{4}{3}$ $n = 2, 3, 4, 6, 8, 12, 24, 180, 360, 10\,800, 648\,000$	Templeton–Maynard 1858 (170)
23 dec.	$\dfrac{\pi}{180n}$;	$n = 1, 60, 60^2$	Hayashi 1930*b* (42)
16 dec.	$\frac{1}{2}n\pi$;	$n = 1(1)9$	Hoüel 1901 (19)
15 dec.	$\dfrac{\pi}{180n}$;	$n = 1, 60, 60^2$	Zimmermann 1929 (203) Barlow–Comrie 1930 (207), 1941 (257)
13 dec.	$n\pi$; π/n;	$n = 2, 3, 4, 6, 12, 18, 24, 36, 108, 144, 648, \frac{2}{3}, \frac{4}{3}, \frac{3}{4}$ $n = 2, 3, 4, 6, 8, 9, 12, 18, 36, 108, 144, 216, 648$	Lorán 1934 (17)

31 dec.	$\pi/2\sqrt{2}$	Glaisher 1912c (97–99); Davis 1935 (285)
31 dec.	$\pi\sqrt{2}$, $\pi/\sqrt{2}$	Templeton–Maynard 1858 (170)
13 dec.	$\pi\sqrt{2}$, $\pi/\sqrt{2}$	Lorán 1934

See also Art. 4·651.

10 dec.	π/n; $n = 1(1)9$	Zimmermann 1929 (203)
6 dec.	π/n; $n = 1(1)500$	Peters 1937 (186)
6 dec.	$k\pi/n$; $n = 1(1)7, 16, 24, 32, 180$, $k = 1(1)9$	Low & Low 1938 (139)
6 dec.	$k\pi/n$; $n = 1(1)10$, $k = 1(1)10$	Potin 1925 (96)

See also Art. 5·91 and Sub-section 7·8.

5·121. $\pi^2 = 9{\cdot}86960\ 44010\ 89358\ 61883\ 44909\ 99876\ 15113 \ldots$

261+ dec.	Uhler 1938 (29); he announces agreement with
217 dec.	Serebrennikov 1906
30 dec.	Templeton–Maynard 1858 (171)

Templeton–Maynard also gives $6\pi^2$.

5·122. π^n, $n > 2$. $\pi^3 = 31{\cdot}00627\ 66802\ 99820\ 17547 \ldots$

38–41 fig.	$n = 1(1)16$	Hayashi 1930a (2, 4)
32 fig.	$n = 1(1)32$	Peters & Stein 1922 (2)
22–24 fig.	$n = 1(1)12$	Glaisher 1877b (140)

Glaisher also gives $(\pi^{12} - 1)/(\pi - 1)$ to 24 figures.

21–22 fig.	$n = 1(1)12$	Jones 1893 (38)
13 dec.	$n = 1, 2, 4$	Lorán 1934
5 dec.	$n = 1(1)16$	Hayashi 1930b (64)

See also Section 4, in particular Art. 4·022 and Sub-section 4·6.

5·123. $(k\pi)^n$, $k \neq 1$, $n > 2$

25–30 d. correct	$(\frac{1}{2}\pi)^n$	$n = 1(1)20$	Brandenburg 1931 (336)
20 dec.	$\left(\dfrac{\pi}{180k}\right)^n$	$\left\{\begin{array}{l} k = 1, 60, 60^2 \\ n = 1(1)11, 5 \text{ or } 3 \end{array}\right\}$	Jones 1893 (38)

5·124. $\dfrac{(k\pi)^n}{n!}$

40–14 dec.	$\pi^n/n!$	$n = 1(1)16$	Hayashi 1930a (3, 5)
35 dec.	$(\frac{1}{12}\pi)^n$	$n = 1(1)22$	Miller (MS.)
30 dec.	$(\frac{1}{2}\pi)^n/n!$	$n = 1(1)20$	Brandenburg 1931 (336)
28 dec.	$(\frac{1}{2}\pi)^n/n!$	$n = 1(1)30$	Euler 1748 (99), 1785 (273)
24 dec.	$(\frac{1}{2}\pi)^n/n!$	$n = 1(1)28$	Andoyer 1915 (2, 3)
24 dec.	$(\frac{1}{4}\pi)^n/n!$	$n = 1(1)22$	Peters & Stein 1922 (95)
22 dec.	$(\frac{1}{2}\pi)^n/n!$	$n = 1(1)26$	Hobson 1891 (141)
20 dec.	$\dfrac{1}{n!}\left(\dfrac{\pi}{180k}\right)^n$	$\left\{\begin{array}{l} k = 1, 60, 60^2 \\ n = 1(1)8, 5 \text{ or } 3 \end{array}\right\}$	Jones 1893 (38)

See also Callet 1795 (27).

5·13. $\frac{1}{\pi}$ and Multiples

5·131†. $\frac{1}{\pi} = 0{\cdot}31830\ 98861\ 83790\ 67153\ 77675\ 26745\ 02872\ldots$

253+ dec.	Uhler 1938 (29, J. W. Wrench)
137 dec.	Paucker 1841 (10)
Three further (incorrect) figures are also given.	
52 dec.	Peters & Stein 1922 (1)
36 dec.	Brandenburg 1931 (336)

5·132†. Multiples of $\frac{1}{\pi}$

134 dec.	$180/\pi$, $200/\pi$	Glaisher 1873*d* (310)
131 dec.	$648\ 000/\pi$	Glaisher 1873*d* (310)
Further digits given are incorrect.		
51 dec.	$1/4\pi$	Paucker 1841 (10)
36 dec.	$2/\pi$, $4/\pi$	Brandenburg 1931 (336)
34–31 d.	$180n/\pi$; $n = 1, 60, 60^2$	Brandenburg 1931 (336)
Further digits given are incorrect.		
35 dec.	$4/\pi$, $1/2\pi$	Egersdörfer & Egersdörfer 1936 (54)
32 dec.	n/π; $n = 1(1)9$	Peters & Stein 1922 (1)
30 dec.	n/π; $n = 2, 4, 6, 180, 360, 10\ 800, 648\ 000$	Templeton–Maynard 1858 (171)
30 dec.	$180/\pi$, $1/2\pi$	Pearson 1931 (262)
See also Arts. 5·92.		
30 dec.	$1/n\pi$; $n = 2, 4, 6$	Templeton–Maynard 1858 (173)
13 dec.	$1/n\pi$; $n = 2, 3, 4, 6, 18, 24, 36, 48, 72, 96, 144$	Lorán 1934
12 dec.	n/π; $n = 1(1)100$	Kulik 1848 (419)
10 dec.	$1/n\pi$; $n = 1(1)9$	Zimmermann 1929 (205)
7 dec.	$108/x\pi$; $x = {\cdot}15({\cdot}05)1({\cdot}1)2({\cdot}25)4({\cdot}5)10(1)70$	Ives 1931 (291)
6 dec.	$m/n\pi$; $m = 1(1)10$, $n = 1(1)10$	Potin 1925 (97)
30 dec.	$\sqrt{2}/\pi$, $1/\pi\sqrt{2}$	Templeton–Maynard 1858 (174)
13 dec.	$\sqrt{2}/\pi$, $1/\pi\sqrt{2}$	Lorán 1934
See also Sub-section 7·8.		

5·14†. Negative Integral Powers, π^{-n}. $1/\pi^2 = 0{\cdot}10132\ 11836\ 42337\ 77144\ldots$

51 dec.	$n = 2$	Paucker 1841 (10)
$1/6\pi^2$ is also given.		
32 dec.	$n = 1(1)32$	Peters & Stein 1922 (2)
30 dec.	$n = 2$	Templeton–Maynard 1858 (174)
$1/2\pi^2$ and $1/6\pi^2$ are also given.		
24–27 dec.	$n = 1(1)12$	Glaisher 1877*b* (141)
Glaisher also gives $(1 - \pi^{-12})/(1 - \pi^{-1})$ to 25 decimals.		
20 dec.	$n = 1(1)12$	Jones 1893 (38)

5·15. Fractional Powers of π, etc.

5·151. $\sqrt{\pi} = 1{\cdot}77245\ 38509\ 05516\ 02729\ 81674\ 83341\ 14518 \ldots$

140 dec.	$\sqrt{\pi}$	Paucker 1841 (10)

Not more than 136 decimals are correct. $2\sqrt{\pi}$ is also given, to 51 decimals.

60 dec.	$\sqrt{\pi}$	Peters & Stein 1922 (xxiv)
30 dec.	$n\sqrt{\pi}$; $n = 2, 1, \frac{1}{2}, \frac{1}{8}$	Templeton–Maynard 1858 (174)
30 dec.	$\frac{1}{2}\sqrt{\pi}$	Burgess 1898 (258)

5·152. $\sqrt{2\pi} = 2{\cdot}50662\ 82746\ 31000\ 50241\ 57652\ 84811\ 04525 \ldots$

$\sqrt{\tfrac{1}{2}\pi} = 1{\cdot}25331\ 41373\ 15500\ 25120\ 78826\ 42405\ 52262 \ldots$

60+ dec.	$\sqrt{2\pi}$	Miller (MS.)
30 dec.	$\sqrt{\frac{1}{2}\pi}$	Templeton–Maynard 1858 (174)

5·153. $\sqrt{n\pi}$

15 dec.	$n = 1, 2, 3, 5, 6$	New York W.P.A. 1940*d* (225)
6 fig.	$n = 2(2)80$	Wrinch & Wrinch 1923 (848) with 1924 (64)

5·154. $1/\sqrt{\pi} = 0{\cdot}56418\ 95835\ 47756\ 28694\ 80794\ 51560\ 77258 \ldots$

$2/\sqrt{\pi} = 1{\cdot}12837\ 91670\ 95512\ 57389\ 61589\ 03121\ 54517 \ldots$

66 dec.	$1/\sqrt{\pi}$, $2/\sqrt{\pi}$	Paucker 1841 (10, 11)

$1/6\sqrt{\pi}$ is also given.

60+ dec.	$1/\sqrt{\pi}$	Miller (MS.)
30 dec.	$n/\sqrt{\pi}$; $n = 2, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{8}$	Templeton–Maynard 1858 (175)

5·155. $1/\sqrt{2\pi} = 0{\cdot}39894\ 22804\ 01432\ 67793\ 99460\ 59934\ 38186 \ldots$

$\sqrt{2/\pi} = 0{\cdot}79788\ 45608\ 02865\ 35587\ 98921\ 19868\ 76373 \ldots$

60+ dec.	$1/\sqrt{2\pi}$	Miller (MS.)
35 dec.	$1/\sqrt{2\pi}$	Egersdörfer & Egersdörfer 1936 (54)
30 dec.	$\sqrt{2/\pi}$	Templeton–Maynard 1858 (175)
28 dec.	$2^{-n}/\sqrt{2\pi}$; $n = 2, 3, 6, 7, 9, 14, 15, 18, 22, 24, 25, 33, 34, 37$	Egersdörfer & Egersdörfer 1936 (54)

5·156. $\sqrt{2/\pi x}$

18 dec.	$x = 1(1)40$	Airey (MS.)
12 dec.	$x = 0(\cdot 01)2,\ 10(\cdot 1)20$	Hayashi 1930*a* (29, 48)
12 dec.	$x = 2(1)10$	Hayashi 1930*b* (174)

Lorán 1934 gives $\sqrt{10/\pi}$ and $6\sqrt{10/\pi}$ to 13 decimals.

5·157. Other Fractional Powers

Miller (MS.) has values to the accuracy indicated in each line.

51 dec.	$\sqrt[3]{\pi}$	= 1·46459 18875 61523 26302 01425 27263 79039 ...
42 dec.	$\sqrt[3]{\pi^2}$	= 2·14502 93971 11025 60007 74441 00941 23559 ...
48 dec.	$\sqrt[3]{1/\pi}$	= 0·68278 40632 55295 68146 70208 33158 16459 ...
48 dec.	$\sqrt[3]{1/\pi^2}$	= 0·46619 40770 35411 61438 19884 54914 46405 ...
34 dec.	$\sqrt[3]{\pi/6}$	= 0·80599 59770 08234 82035 ...
34 dec.	$\sqrt[3]{6/\pi}$	= 1·24070 09817 98800 03333 ...
34 dec.	$\sqrt[3]{\pi^2/36}$	= 0·64962 95149 53458 99316 ...
34 dec.	$\sqrt[3]{36/\pi^2}$	= 1·53933 89262 36506 33160 ...
44 dec.	$\sqrt[4]{\pi}$	= 1·33133 53638 00389 71279 75349 17950 28085 ...
44 dec.	$\sqrt[4]{1/\pi}$	= 0·75112 55444 64942 48285 87030 04776 22769 ...

Cube roots of the quantities indicated are given as follows:

51 dec.	π, 36π, π^2, $6\pi^2$	Paucker 1841 (11)
50 dec.	$6/\pi$	
13 dec.	36π, $\frac{4}{3}\pi$, $6\pi^2$, $3/4\pi$	Lorán 1934
11; 8 d.	36π; $6\pi^2$	Templeton–Maynard 1858 (175)
10 dec.	$\frac{4}{3}\pi$, $3/4\pi$	MT.C. 1931 (243)

New York W.P.A. announces the computation of a table giving π^x to 15 decimals for $x = 0(\cdot 001)1$.

5·16. $\text{Log}_{10}\,\pi$ and $n \log_{10} k\pi$

$$\text{Log}_{10}\,\pi = 0{\cdot}49714\ 98726\ 94133\ 85435\ 12682\ 88290\ 89887\ldots$$

Only references to values with more than 30 decimals are listed. See also Arts. 4·022 and 4·68.

214+ dec.	$\log_{10}\pi$, $\frac{1}{2}\log_{10}2\pi$	Uhler 1938 (27, 28)
61 dec.	$\log_{10}\pi$, $\frac{1}{2}\log_{10}\pi$	Peters & Stein 1922 (1, xxiii)
40 dec.	$\frac{1}{2}\log_{10}2\pi$	Duarte 1927 (xiii)
36 dec.	$\frac{1}{2}\log_{10}2\pi$	Duarte 1933 (xvi)
32 dec.	$n\log_{10}\pi$; $n = 1(1)9$	Peters & Stein 1922 (2)
32 dec.	$\frac{1}{2}\log_{10}k\pi$; $k = \frac{1}{2}$, 1, 2	Thoman 1867 (54)

Templeton–Maynard 1858 (170) gives 30-decimal values of about fifty such constants.

5·17. $\text{Log}_e\,\pi$ and $n \log_e k\pi$

$$\text{Log}_e\,\pi = 1{\cdot}14472\ 98858\ 49400\ 17414\ 34273\ 51353\ 05871\ldots$$

213+ dec.	$k = n = 1$	Uhler 1938 (27)
48 dec.	$k = n = 1$	Glaisher 1883*b* (137), 1891*b* (384)
48 dec.	$k = n = 1$	Peters & Stein 1922 (1)
35− dec.	$k = n = 1$	Euler 1785 (255); Glaisher 1891*b* (368)

32 dec.	$k = 1, n = 1(1)9$	Peters & Stein 1922 (2)
30 dec.	$k = 1, n = 20, 22$	Glaisher 1891*b* (365, 367)
25 dec.	$k = 2, n = \frac{1}{2}$	Euler 1770 (163)
20 dec.	$k = \frac{1}{2}, \frac{1}{8}, n = 1$	Callet 1795 (50, 52)

See also Arts. 4·022 and 4·68.

5·2. Constants involving *e*. Circular and Hyperbolic Functions

5·211. $e = 2{\cdot}71828\ 18284\ 59045\ 23536\ 02874\ 71352\ 66249\ldots$

Historical accounts of the calculation of *e* are given in D. H. Lehmer 1926, in Mitchell & Strain 1936 and in Uhler 1937.

707 dec. — Lehmer 1926 (142)
707 dec. — Uhler 1937 (432)

Uhler checks Lehmer's value to 477 decimals and reproduces the remainder.

225 dec. — Studnička 1892, 1897 (Tichánek)

Error: the 43rd decimal should read 0, not 6.

223 dec. — Boorman 1884*a*; Van Orstrand 1921 (79)

In all, 346 decimals are given, the last 123 being incorrect. The 32nd decimal should read 6, not 0. Mitchell & Strain 1936 (493) gives a facsimile reproduction of Boorman's value.

187 dec. — Shanks 1854 (397)

Further digits, numbering 18, are incorrect.

137 dec. — Shanks 1853
137 dec. — Glaisher 1871*c*, 1883*a* (252)
137 dec. — Boorman 1882*b*
113 dec. — Studnička 1894 (Tichánek, Minks)

Error: the 43rd decimal should read 0, not 6.

105 dec. — Bretschneider 1843 (28); Peters & Stein 1922 (12, v)

5·212. Multiples of *e*

32 dec.	ne; $n = 1(1)9$	Peters & Stein 1922 (12)
9 dec.	ne/k; $n = 1(1)10, k = 1(1)10$	Potin 1925 (444)

$$\frac{e}{2\sqrt{2}} = 0{\cdot}96105\ 77570\ 39779\ 20621\ 592$$

Miller (MS.) has 35 decimals. Burgess 1898 (279) gives 23 decimals, the 22nd being erroneous.

5·213. $\dfrac{1}{e-1} = \dfrac{1}{1+}\ \dfrac{2}{2+}\ \dfrac{3}{3+}\ldots = 0{\cdot}58197\ 67068\ 69$

This value is given in Glaisher 1875 (259).

5·216. $e^{e^p} = e_2(p)$, $e_3(p) = e^{e_2(p)}$, etc.

4 fig.	$e_n(p)$	$n = 2, 3, 4$	$p = -4(1)+5$	Miller 1940 (286)

Epstein 1939 gives the coefficients K_n in the expansion of $e_2(p)/e$ in powers of p to p^{20}. See also Art. 4·676, where some values are given.

5·22. Powers of e

Only a few many-figure values are listed here. See Section 10 for the general exponential function.

5·221. $\frac{1}{e} = 0{\cdot}36787\ 94411\ 71442\ 32159\ 55237\ 70161\ 46086\ldots$

477 dec.			Uhler 1937 (432)
105 dec.			Bretschneider 1843 (28); Van Orstrand 1921 (79); Peters & Stein 1922 (12, v)
32 dec.	n/e	$n = 1(1)9$	Peters & Stein 1922 (12)
9 dec.	n/ke	$n = 1(1)10,\ k = 1(1)10$	Potin 1925 (444)

5·222. Positive Integral Powers, e^n

290− fig.	$n = 10$	Uhler 1940 (212) with 1937
258 fig.	$n = 2(2)10$	Uhler 1937 (432, 433)
116 fig.	$n = 100$	Uhler 1937 (433)
42 fig.	$n = 1(1)100$	Van Orstrand 1921 (14)
32 fig.	$n = 1(1)32$	Peters & Stein 1922 (12)

5·2221. Du Bois-Reymond's Constants, C_m

$$C_m = \int_0^\infty \left| \frac{d}{dt}\left(\frac{\sin t}{t}\right)^m \right| dt - 1$$

Watson 1933 (143, 145) gives

$$\begin{aligned}
C_2 &= \tfrac{1}{2}(e^2 - 7) &&= 0{\cdot}19452\ 80495 \\
C_3 & &&= 0{\cdot}02825\ 17642 \\
C_4 &= \tfrac{1}{8}(e^4 - 4e^2 - 25) &&= 0{\cdot}00524\ 07047 \\
C_6 &= \tfrac{1}{32}(e^6 - 6e^4 + 3e^2 - 98) &&= 0{\cdot}00022\ 06747
\end{aligned}$$

The calculation of C_3 involves zeros of $\tan x = x$ (Art. 5·7111).

5·223. Negative Integral Powers, e^{-n}

294− dec.	$n = 10$	Uhler 1940 (212) with 1937
258 dec.	$n = 10$	Uhler 1937 (433)
83 dec.	$n = 100$	Uhler 1937 (433)
52 dec.	$n = 0(1)50$	Van Orstrand 1921 (27, 28)
62 dec.	$n = 50(1)100$	Van Orstrand 1921 (27, 28)
32 dec.	$n = 1(1)32$	Peters & Stein 1922 (12)

5·224. Fractional Powers of e, etc.

120 dec.	$e^{\pm 0{\cdot}1}$		Van Orstrand 1921 (79)
72 dec.	$e^{\pm 1/2}$		Peters & Stein 1922 (12)
55 dec.	$e^{\pm 1/\sqrt{2}}$	$e^{\pm 0{\cdot}1/\sqrt{2}}$	Miller (MS.)
23 dec.	$e^{\pm 1/\sqrt{2}}$	$e^{\pm 0{\cdot}1/\sqrt{2}}$	Airey (MS.)

5·23. Hyperbolic Functions

105 dec.	sinh 1	cosh 1	Bretschneider 1843 (28); Peters & Stein 1922 (60, vii)
24 dec.	sinh 1	cosh 1	Gudermann 1830*a* (6)

A further (incorrect) digit is also given.

27 dec.	sinh 0·01	cosh 0·01	Airey (MS.)

$$\tanh 1 = \frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{7+}\cdots = 0{\cdot}76159\ 41559\ 55764\ \ldots$$

An 11-decimal value is given in Glaisher 1875 (255).

55 dec.	$\sinh\frac{1}{\sqrt{2}}$, $\sinh\frac{0{\cdot}1}{\sqrt{2}}$	$\cosh\frac{1}{\sqrt{2}}$, $\cosh\frac{0{\cdot}1}{\sqrt{2}}$	Miller (MS.)
20 dec.	$\sinh\frac{1}{\sqrt{2}}$	$\cosh\frac{1}{\sqrt{2}}$	Airey 1935*d* (721)
18 dec.	$\sinh\frac{0{\cdot}1}{\sqrt{2}}$	$\cosh\frac{0{\cdot}1}{\sqrt{2}}$	Airey 1935*d* (721)
19 dec.	$\sinh\frac{10}{\sqrt{2}}$	$\cosh\frac{10}{\sqrt{2}}$	Airey (MS.)
18 dec.	$\sinh\frac{0{\cdot}01}{\sqrt{2}}$	$\cosh\frac{0{\cdot}01}{\sqrt{2}}$	Airey (MS.)

See also Art. 10·49.

5·24. Circular Functions

5·241. Sin x and cos x

477 dec.	sin 1	cos 1		Uhler 1937 (433, 434)
105 dec.	sin 1	cos 1		Bretschneider 1843 (28); Van Orstrand 1921 (79); Peters & Stein 1922 (60, vii)
284+ dec.	sin 10	cos 10	cos 20	Uhler 1940 (212) with 1937
212 dec.	sin 10	cos 10	cos 20	Uhler 1937 (434)
72 dec.	sin 50	cos 50		Miller (MS.)
72 dec.	sin 100, sin 200	cos 100, cos 200		Uhler 1937 (434)
120 dec.	sin 0·1	cos 0·1		Van Orstrand 1921 (79)

5·2421. Sin $\frac{x}{\sqrt{2}}$ and cos $\frac{x}{\sqrt{2}}$

55 dec.	$\sin\frac{1}{\sqrt{2}}$, $\sin\frac{0{\cdot}1}{\sqrt{2}}$	$\cos\frac{1}{\sqrt{2}}$, $\cos\frac{0{\cdot}1}{\sqrt{2}}$	Miller (MS.)

20 dec.	$\sin \frac{1}{\sqrt{2}}$	$\cos \frac{1}{\sqrt{2}}$	Airey 1935*d* (721)
18 dec.	$\sin \frac{0{\cdot}1}{\sqrt{2}}$	$\cos \frac{0{\cdot}1}{\sqrt{2}}$	
22 dec.	$\sin \frac{10}{\sqrt{2}}$	$\cos \frac{10}{\sqrt{2}}$	Airey (MS.)
18 dec.	$\sin \frac{0{\cdot}01}{\sqrt{2}}$	$\cos \frac{0{\cdot}01}{\sqrt{2}}$	

See also Art. 7·9.

5·2422. Sin $\frac{\sqrt{3}x}{2}$ and cos $\frac{\sqrt{3}x}{2}$

19 dec.	$\sin \frac{\sqrt{3}}{2}$	$\cos \frac{\sqrt{3}}{2}$	Airey (MS.)
17 dec.	$\sin \frac{\sqrt{3}}{20}$	$\cos \frac{\sqrt{3}}{20}$	

5·243. Sin $\log_e x$ and cos $\log_e x$

15 dec.	$x = 2$	Airey 1935*e* (733)
14 dec.	$x = 5, 10$	
13 dec.	$x = 100, 1000$	

For $x = 10$, the two values given by Airey should be interchanged. Airey (MS.) gives about 3 further decimals. See also Arts. 5·55 and 7·9.

5·25. $\text{Tan } 1 = \frac{1}{1-}\,\frac{1}{3-}\,\frac{1}{5-}\,\frac{1}{7-}\cdots = 1{\cdot}55740\ 77246\ 54902\ldots$

An 11-decimal value is given in Glaisher 1875 (255). Miller (MS.) has 25 decimals.

5·27. $\text{Log}_{10} \sin 1 = \bar{1}{\cdot}92503\ 91454\ 39504\ 49$

$$\text{Log}_{10} \cos 1 = \bar{1}{\cdot}73263\ 68209\ 97778\ 33$$

These values are given in Andoyer 1911 (xxi).

5·28. $\text{Tan}^{-1} \frac{1}{x}$

709 dec.	$x = 5^*, 239$	Shanks 1873*a**, 1873*b*

The asterisk indicates an erroneous value, corrected in the second reference.

609 dec.	$x = 5, 239$	Shanks 1853

Individual terms in the expansions of $\tan^{-1} 1/5$ and $\tan^{-1} 1/239$ are also given in full to 530 decimals.

530 dec.	$x = 5, 239$	Rutherford 1853 (Shanks and Rutherford)
335; 330 dec.	$x = 577$; 2449, 4999, 8749	Uhler 1940
261 dec.	$x = 2, 3, 5, 7$	Lehmann 1853 (158)

The numbers tabulated are multiples (by 4 or 8) of the constants concerned.

250 dec.	$x = 3, 5, 7, 239$	Clausen 1847
215; 216 dec.	$x = 451$; 10 081	Uhler 1940

5·29. Integrals

Bretschneider 1861 (131) gives 37-decimal values, and Bretschneider 1843 gives 35-decimal values, of $\text{Li}\,e = \text{Ei}\,1$, $\text{Li}\,(1/e) = \text{Ei}\,(-1)$, Chi 1, Shi 1, Ci 1 and Si 1. In both papers 20-decimal values of the same functions for arguments 1(1)10 are also given. Glaisher 1870 (371) gives 43-decimal values of $\text{Ei}(\pm 2)$, Si(2) and Ci(2).

For definitions see Section 13.

5·3. Logarithms, mainly of Integers, the Modulus *M*, etc.

5·31. $M = \log_{10} e = 0{\cdot}43429\ 44819\ 03251\ 82765\ 11289\ 18916\ 60508\ldots$

Decimals given	Decimals correct	Reference
330	330	Uhler 1940 (211)
325	325	Uhler 1942*b* (T. 4)
277	272	Adams 1887 (25); Glaisher 1911*a* (876)
282	263	Adams 1878*b* (93); Peters & Stein 1922 (7)
106	105	Parkhurst 1889 (82)
205	102	Shanks 1872*a* (28)
61	61	Stuckey 1914 (30)
205	59	Shanks 1854 (398)
137	59	Shanks 1853
42	42	Duarte 1933 (vii)
40	40	Thoman 1867 (54)

5·311. $\text{Log}_{10} \dfrac{e}{2\sqrt{2}} = M - \dfrac{3}{2}\log_{10} 2 = \bar{1}{\cdot}98274\ 94884\ 07280\ 03483\ 052$

This value is given in Burgess 1898 (279).

5·321. $\text{Log}_e\, n$. $\text{Log}_e\, 2 = 0{\cdot}69314\ 71805\ 59945\ 30941\ 72321\ 21458\ 17656\ldots$

$1/M = \log_e 10 = 2{\cdot}30258\ 50929\ 94045\ 68401\ 79914\ 54684\ 36420\ldots$

In many cases extra (incorrect) digits are also given.

330–331 d.	2, 3, 5, 7, 10, 17		Uhler 1940 (209)
325 dec.	10		Uhler 1942*b* (T. 4)
290–329 d.	11, 13, 17, 19, 23, 29, 31, 37, 101, 9901		Uhler 1943 (321)
273–274 d.	2, 3, 5, 7, 10		Adams 1887 (24) with 1878*b* (92)
274+ dec.	17		Uhler 1938 (28)
262+ dec.	2, 3, 5, 7, 10		Adams 1878*b* (92); Peters & Stein 1922 (7, 152)
262 dec.	1000, i.e. $3/M$		Adams 1877*b* (15), 1878*b* (94)
	500		Adams 1878*b* (93)
260± dec.	$1 \pm 1 . 10^{-p}$	$p = 1(1)29$	Uhler (MS.)
213+ dec.	71, 113		Uhler 1938 (29)
201+ dec.	2, 3, 5, 10		Shanks 1872*a* (28)

Errors occur in the 103rd and 104th places; see Adams 1878*b*.

163; 153 dec.	100!	Uhler 1943 (325); 1942*a* (61)
155+ dec.	41 (primes) 97	Uhler 1943 (323)
137–148 dec.	2 (1) 11 (primes) 31, 47, 53, 71, 97 (primes) 113	Uhler 1942*b* (T. 1, 4)

The main tables in Uhler 1942*b* are 137-decimal radix tables; see Art. 11·3.

140 ± dec.	250!, 400!, 625!	Uhler (MS.)
100 dec.	2, 3, 5, 10	Glaisher 1871*b* (521)

Shanks 1853, 1854 give values correct only to 59 decimals; see the note at the end of Art. 5·41. Peters & Stein 1919 (9) gives $1/M$ to 49 decimals. Duarte 1933 (vii) gives $1/M$ (also $\log_{10} 2$ and $\log_{10} 5$) to 42 decimals.

Stuckey 1914 (30) gives 60-decimal values of the logarithms and cologarithms of 2, 3, 5, 7, 10. For 2 and 5 all values are 6 units in error in the final decimal; in colog 10 the 28th decimal should be 3, not 5.

See also Arts. 11·11 and 11·3.

5·322. $\log_{10} n$

230 dec.	2, 3, 5, 7, 17	Uhler 1938 (28)
110 dec.	71, 113	Uhler 1938 (29)
100+ dec.	1 (1) 109	Parkhurst 1889

Given piecemeal on pages 1 (1) 4 (4) 16, 35, 52 (1) 62. Parkhurst also gives a radix table to the same accuracy. Errata; see Uhler, *M.T.A.C.*, 1, 121, 1943.

See also Section 6.

5·323. $\log_{10} f$

274 dec.	$f = \frac{10}{9}, \frac{25}{24}, \frac{81}{80}, \frac{50}{49}, \frac{126}{125}$	Adams 1878*b* (91–92) with 1887 (24)
263–270 d.		Adams 1878*b* (91–92)

A few further (incorrect) digits are also given.

5·331. Multiples nM and $\frac{n}{M}$

These are, or may be used as, conversion tables from natural to common logarithms, and vice versa. Only the more important tables of integral multiples are listed here; other multiples are given in Arts. 10·31 and 10·32.

Dec.	n	
137	20 (10) 90 and 110, n/M only	Uhler 1942*b* (T. 1)
100	2, 4, 8, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, nM only	Parkhurst 1889 (8, 12, etc.)
61	1 (1) 100	Peters & Stein 1922 (8–11)
40	1 (1) 100, nM only	Duarte 1927 (125)
40	1 (1) 10	Duarte 1933 (95)
34	10 (1) 20 (2) 52 (4) 96, nM only	Parkhurst 1889 (32)
30	$\frac{1}{2}n^2M$, n = 10 (1) 20 (2) 52 (4) 96	Parkhurst 1889 (32)
33	1 (1) 30, n/M only	Hoppe 1876 (10)
24	1 (1) 100	Peters & Stein 1919 (28)
21	1 (1) 100	Peters 1911*a* (52, 54)
20	1 (1) 20, n/M only	Potin 1925 (654)

The complementary form is also given.

Dec.	n	
20	$1(1)20$, n/M only	Leonelli 1803
20	$1(1)10$, n/M only	New York W.P.A. 1941*a*, *b*, *c*, *d*
20	$1(1)9$, n/M only	Steinhauser 1880
20	$1(1)9$, n/M	Hoüel 1901 (18)
18	$1(1)9$	Holtappel 1938 (132)
16	$1(1)9$, nM	Hoüel 1901 (19)
12	$11(1)110$	Thoman 1867 (46, 47)
11	$1(1)100$	Gudermann 1832*g* (375, 376)
10	$-500(1)+500$, nM only	Glaisher 1883*a* (254)
10	$1(1)100$	Bauschinger & Peters 1936 (365); Bruhns 1922 (186); Glover 1930 (678)
10	$1(1)100$, nM only	Becker & Van Orstrand 1924 (259)

5·332. Multiples $n \log_e 2$

12 dec.	$n = 1(1)20$	Gudermann 1830*a* (39)

5·341. Powers $M^{\pm n}$

32 dec.	M^n	$n = 1(1)32$	Peters & Stein 1922 (7)
32 fig.	M^{-n}	$n = 1(1)32$	Peters & Stein 1922 (7)

5·342. Powers $(\log_e 2)^n$

15 dec.	$n = 1(1)12$	Bessel 1812*a* (*Abh.*, **2**, 339, 1876)

5·351. $\mathrm{Log}_{10} M = \bar{1}\cdot63778\ 43113\ 00536\ 78912\ 29674\ 98645\ 10939\ldots$

This value is taken from Peters & Stein 1922.

100+ dec.	Parkhurst 1889 (82)
50 dec.	Peters & Stein 1919 (9), 1922 (7)

5·352. $\mathrm{Log}_{10} 2M = \bar{1}\cdot93881\ 43069\ 64517\ 98433\ 67063\ 93$

This value is given in Thoman 1867 (42).

5·36. $\mathrm{Log}_e M = \bar{1}\cdot16596\ 75547\ 52044\ 20019\ 67869\ 52142\ 46090\ldots$

54 dec.	Peters & Stein 1922 (7)

5·4. Euler's Constant γ, $e^{\pm\gamma}$, etc.

At one time also known as Mascheroni's constant.

5·41. $\gamma = 0\cdot57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243\ldots$

The value quoted is from Adams 1878*b*. Many of the following details have been extracted from a paper "On the History of Euler's Constant" by J. W. L.

Glaisher (*Mess. Math.*, **1**, 25–30, 1871); this should be consulted for details of references marked with an asterisk *, which are not listed in our Part II.

5+ dec.	*Euler 1734–35
15 dec.	Euler 1770 (154)
19 dec.	*Mascheroni 1790

Mascheroni gives 32 decimals, 13 incorrect; his value has been quoted in full by Lacroix, Bretschneider, Bessel and Grunert. Legendre 1816 (436) and 1826 (506) quote 19 decimals.

21 dec.	Soldner 1809
40 dec.	Gauss 1813 (*Werke*, **3**, 154, 1866, Nicolai)
34 dec.	*Lindman 1857
41 dec.	Oettinger 1862
59 dec.	Shanks 1867

This value contains an error in the 50th decimal. Later attempts by Shanks, prior to 1872*b*, give no improvement.

99+ dec.	Glaisher 1871*b* (522)
101 dec.	Shanks 1872*b* (31)

Further digits given are incorrect.

263+ dec.	Adams 1877*b* (15), 1878*b* (93, 94)

All the above are original publications. Others are

40 dec.	(Nicolai)	Bretschneider 1861 (131)
50 dec.	(Adams)	Nielsen 1906 (9)
263 dec.	(Adams)	Peters & Stein 1922 (2)

Glaisher 1871*b* contains a masterly account of detection of errors in Shanks's work on logarithms (Arts. 5·31 and 5·32) and on Euler's constant. Adams 1878*b* also has details of errors by Shanks.

5·44. Fractional Powers of γ

New York W.P.A. announces the computation of a table giving γ^x to 15 decimals for $x = 0(\cdot001)1$.

5·46. Constants involving γ

$-\log_e \gamma$	$= 0\cdot54953\ 93129\ 81644\ 82234$	New York W.P.A. 1941*a*, *b*, *c*, *d*
$\log_{10} \gamma$	$= \bar{1}\cdot76133\ 81087\ 83167\ 61054$	Thompson 1924 (Part 9, vii)
$\gamma + 2\log_e 2$	$= 1\cdot96351\ 00260\ 21423\ 47944\ 09(9)$	Gauss 1813 (*Werke*, **3**, 155, 1866)
$\log_e 2 - \gamma$	$= 0\cdot11593\ 15156\ 58412\ 44881\ 07200$	
	36 decimals are given in	Aldis 1899 (206)
$n(\log_e 2 - \gamma)$	$n = 0(\cdot01)1$ 22–24 dec.	Airey (MS.)
$\dfrac{\gamma + \log_e(4/\pi)}{4\pi}$	$= 0\cdot06515\ 64533\ 06926\ 22763$	Schott & Watson 1916 (332)
e^{γ}	$= 1\cdot78107\ 24179\ 90197\ 98522$	Fisher & Yates 1938 (88), 1943 (96)
$e^{-\gamma}$	$= 0\cdot56145\ 94835\ 66885\ 16983$	Fisher & Yates 1938 (88), 1943 (96)

5·5. Mixed Constants involving π, e and M

5·511. $M\pi$, $M\pi^2$, $1/M\pi$, etc.

23 d.	$M\pi = 1{\cdot}36437\ 63538\ 41841\ 34748\ 578$	Airey (MS.)
20 d.	$M\pi/200$	Jordan–Eggert 1939 (417)
10 d.	$M\pi$, $M\pi^2$ and common logarithms	Peters 1938 (512)
5 d.	$nM\pi$, $n = 0(\cdot 25)5$	Fowle 1933 (55)
21 d.	$1/M\pi = 0{\cdot}73293\ 55988\ 79427\ 74087\ 3$	Airey (MS.)
20 d.	$1/4M\pi$	Schott & Watson 1916 (332)
24 d.	$\log_{10} M\pi = 0{\cdot}13493\ 41839\ 94670\ 64347\ 4237$	Gray 1876 (31)
20 d.	$\log_{10}(M\pi/200)$	Jordan–Eggert 1939 (417)

5·512. $\dfrac{\text{Log}_e 2}{4\pi}$ and $\dfrac{\frac{1}{6}+\log_e 2}{4\pi}$

$$\left.\begin{aligned}(\log_e 2)/4\pi &= 0{\cdot}05515\ 89000\ 38162\ 89835\\ (\tfrac{1}{6}+\log_e 2)/4\pi &= 0{\cdot}06842\ 18119\ 62487\ 50966\end{aligned}\right\}$$ Schott & Watson 1916 (332)

5·52. $e^{k\pi}$, etc.

See Arts. 10·61 and 10·71 for general tables of $e^{k\pi}$. Of these we now mention only **Van Orstrand 1921** (40–46), which gives the following values:

23 dec.	$k = \pm n/360$	$n = 1(1)360$
25 fig.	$k = +n/12$ }	All integral n from 14 to 46, but excluding primes and 25
25 dec.	$k = -n/12$ }	
25 fig.	$k = n$ }	$n = 2(1)10$
25 dec.	$k = -n$ }	

These values are not again listed in the separate articles below.

5·5211. Integral k

$$e^{\pi} = 23{\cdot}14069\ 26327\ 79269\ 00572\ 90863\ldots$$

62 dec.	Uhler (MS., Wrench)
52 dec.	Uhler 1921 (115)
40 dec.	Peters & Stein 1922 (1)
28 dec.	Bastien 1920 (65); Archibald 1921 (121)

$$e^{-\pi} = 0{\cdot}04321\ 39182\ 63772\ 24977\ 44177\ldots$$

55 dec.		Uhler 1921 (115)
51 dec.		Gauss 1866*b* (428)
50 dec.		Archibald 1921 (119)
50 dec.		Peters & Stein 1922 (1)
35 dec.	$2e^{-4\pi}$	Gauss 1866*b* (419); Archibald 1921 (120)
27 dec.	$2e^{-9\pi}$, $2e^{-16\pi}$	Gauss 1866*b* (419); Archibald 1921 (120)
15 dec.	$e^{-4\pi}$, $e^{-9\pi}$	Glaisher 1878*c* (127)

See also Van Orstrand 1921 (Art. 5·52 above).

5·5212. Rational k

$$e^{\pi/2} = 4{\cdot}81047\ 73809\ 65351\ 65547\ 30356\ldots$$

54 dec.		Uhler 1921 (115)
28 dec.		Bastien 1920 (65) Archibald 1921 (121)

Gauss 1866*b* (431) gives a 34-decimal value, which is correct only to 22 decimals.

$$e^{-\pi/2} = 0{\cdot}20787\ 95763\ 50761\ 90854\ 69556\ldots$$

54 dec.		Uhler 1921 (115)

$e^{\pi/3} = 2{\cdot}84965\ 39082\ 26361\ 49747\ldots$ $\qquad e^{-\pi/3} = 0{\cdot}35091\ 98071\ 78410\ 96756\ldots$

$e^{\pi/6} = 1{\cdot}68809\ 17949\ 64468\ 60061\ldots$ $\qquad e^{-\pi/6} = 0{\cdot}59238\ 48471\ 88388\ 98366\ldots$

54 dec.		Uhler 1921 (115)
35 fig.	$e^{\pm\pi/12}$	Miller (MS.)

$$e^{\pi/4} = 2{\cdot}19328\ 00507\ 38015\ 45655\ 97696\ldots$$

46 dec.		Peters & Stein 1922 (1)

$$e^{-\pi/4} = 0{\cdot}45593\ 81277\ 65996\ 23676\ 59212\ldots$$

50 dec.		Gauss 1866*b* (430) Archibald 1921 (119)
48 dec.		Peters & Stein 1922 (1)
57 dec. 32 dec. 28 dec.	$e^{-9\pi/4}$ $2e^{-25\pi/4}$ $2e^{-49\pi/4}$	Gauss 1866*b* (418, 431) Archibald 1921 (119, 120)
17 dec. 15 dec.	$e^{-9\pi/4}$ $e^{-25\pi/4}$	Glaisher 1878*c* (127)
28 dec.	$e^{\pm\pi/20}$	Airey (MS.)
40 dec.	$e^{\pm\pi/100}$	Hayashi 1930*a* (125)
25 dec.	$e^{\pm\pi/100}$	Airey (MS.)
24 dec.	$e^{\pm\pi/200}$	Airey (MS.)

See also Van Orstrand 1921 (Art. 5·52 above).

5·522. Irrational k

$e^{\pi\sqrt{43}}$ =	884 736 743 · 999 777 423	Gray 1876 (30, 31)
$e^{\pi\sqrt{67}}$ =	147 197 952 743 · 999 998 662 468	
$e^{\pi\sqrt{22}}$ =	2 508 951 · 998 2	Ramanujan (*Q.J. Math.*, **45**, 350–372, 1914, also in *Coll. Pap.*, 23–29, 1927)
$e^{\pi\sqrt{37}}$ =	199 148 647 · 999 978	
$e^{\pi\sqrt{58}}$ =	24 591 257 751 · 999 999 82	
$e^{\pi\sqrt{163}}$ =	262 537 412 640 768 743 · 999 999 999 999 250 072 597	Lehmer, *M.T.A.C.*, **1**, 31, 1943

See also Arts. 21·7.

5·53. Sinh $k\pi$ and cosh $k\pi$

54 dec.	$k = \frac{1}{8}$	Called σ_2 and σ_1	Uhler 1921 (115)

See also Art. 10·63.

5·541. $\pi e = 8{\cdot}53973\ 42226\ 73567\ldots$ $\qquad \pi/e = 1{\cdot}15572\ 73497\ 90921\ldots$

Miller (MS.) has 25 figures.

5·542. $\dfrac{e}{\sqrt{2\pi}} = 1{\cdot}08443\ 75514\ 19227\ldots$ $\qquad \log_{10}\dfrac{e}{\sqrt{2\pi}} = 0{\cdot}03520\ 45477\ 24194\ldots$

Burgess 1898 (279) gives 23 decimals. Miller (MS.) has 35 decimals.

5·543. $\sqrt{\tfrac{1}{2}\pi e} = 2{\cdot}06636\ 56770\ 61246\ldots$ $\qquad \sqrt{\tfrac{1}{2}\pi/e} = 0{\cdot}76017\ 34505\ 33140\ldots$

Miller (MS.) has 25 decimals. A 9-decimal value of $\sqrt{\tfrac{1}{2}\pi e}$ is given in Glaisher 1875 (256).

5·55. $\text{Sin}\ \log_e \pi = 0{\cdot}41329\ 21161\ 016$ $\qquad \text{Cos}\ \log_e \pi = 0{\cdot}91059\ 84992\ 126$

Airey 1935*e* (733) gives these values.

5·56. $\pi^e = 22{\cdot}45915\ 77183\ 61045\ 47342\ 71522\ 045$

The value quoted is given in Bastien 1920, and reproduced in Archibald 1921 (121), with a correction to the 22nd decimal, where Bastien has "r" in place of "1".

5·6. Algebraic Constants

Only a few many-figure values, or values of unusual constants, and tables giving solutions of algebraic equations, are listed in this sub-section. Others are listed in Art. 2·611 and as fractional powers, or as zeros of Chebyshev, Hermite, Laguerre or Legendre polynomials. Yet others (see Art. 5·63) can be obtained from tables listed in Section 7.

Mehmke–d'Ocagne 1909 (320–325) gives an account of tables for the solution of algebraic equations; Mehmke 1902 (1004–1006) gives a shorter account. In particular we may mention Gundelfinger 1897 for the solution of the trinomial equation $ax^n + bx = c$, and Guldberg 1872–73 for the solution of equations of degree 2, 3 and 5. See also Heger 1900.

5·61. Square Roots

50+ dec.	$n = 1(1)100$ and some others	Miller (MS.)
160+ dec.	$\sqrt{10}$	Miller (MS.)
120+ dec.	$\sqrt{2}$	Miller (MS.)
110 dec.	$\sqrt{2}$	De Morgan 1872 (293, Colvill, Steel)
66 dec.	$\sqrt{3}$	Peters & Stein 1922 (1)
64 dec.	$\sqrt{2}$	Peters & Stein 1922 (1)

See also Arts. 2·61, 2·611.

5·62. Cube Roots

56 dec.	$\sqrt[3]{6}$ and $\sqrt[3]{36}$	Paucker 1841 (11)
56 dec.	$\sqrt[3]{2}$ and $\sqrt[3]{4}$	Gray 1877 (54)
50 dec.	$\sqrt[3]{2}$	Martin 1877 (51)

See also Arts. 2·611 and 2·71.

5·63. Higher Quadratic Surds

By these we mean constants depending only on the extraction (possibly repeated) of square roots. Since sin $3n°$, cos $3n°$, etc. are expressible in this way for integral n, values can be obtained from tables of circular functions with argument expressed in degrees. Similar considerations apply to certain other angles.

Tables, to a few decimals, for the solution of quadratic equations are not listed in this *Index*; see the general references mentioned in Art. 5·6 above.

See also Art. 5·65.

5·631. Sin $3n°$ as Surds

Explicit expressions for $n = 1(1)30$ are given in Vega 1797 (this appears in Vol. I, p. 409 of the 1812 edition, the only one examined), in Gregory's editions, 1830 to 1858 (xxxix), of Hutton 1811, and in Hobson 1891 (72). The first gives 12-decimal values of the necessary surds, namely $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{15}$, $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{3}{2}}$, $\sqrt{\frac{5}{2}}$, $\sqrt{\frac{15}{2}}$, $\sqrt{5 \pm \sqrt{5}}$, $\sqrt{2}.\sqrt{5 \pm \sqrt{5}}$, $\sqrt{3}.\sqrt{5 \pm \sqrt{5}}$, $\sqrt{6}.\sqrt{5 \pm \sqrt{5}}$.

Gray 1876*b* gives 24-decimal values of the same surds.

5·632. Lemniscate Functions

Gauss 1866*b* (421) gives 35-decimal values of $6\sqrt{5} - 13 = a$, $340 - 152\sqrt{5} = b^2$, b, $a \pm b$ and 25- to 26-decimal values of $\sqrt{a \pm b}$. See also Art. 5·82.

5·64. The Solution of Cubic Equations

By simple changes of the variable any cubic equation may be reduced to one of the standard forms $x^3 \pm x - c = 0$, $x^3 - c = 0$. For roots of the last, see tables of cube roots, listed in Art. 2·71. The solutions of $x^3 \pm x - c = 0$ may be found by inverse interpolation in a table of $x^3 \pm x$; such tables are listed in Art. 2·25. Tables of the Associated Legendre Function $P_{3,2}(x) = 15(x^3 - x)$ (see Art. 16·41), or of the Everett interpolation coefficient $E_1'' = \frac{1}{6}(n^3 - n)$ (see Art. 23·17), can also be used in the same way; likewise different changes of the variable might also be used to adapt the equation given for solution by means of any other cubic polynomial for which a table is available.

The solutions may also be tabulated directly and such tables are listed below.

Katz 1933 gives values of x, with differences, satisfying $x^3 \pm x = c$ for

$$c = 0(\cdot 02)\cdot 3(\cdot 01)\cdot 37(\text{var.})1100$$

Jahnke & Emde 1933 uses the standard form $y^3 + 2 = 3py$. For

$$3p = -9\cdot 9(\cdot 1) + 1(\cdot 05)2(\cdot 02)2\cdot 8(\cdot 01)3$$

when one root is real and we may write the roots as $-2y'$, $y' \pm iy'' = se^{\pm i\sigma}$, Jahnke & Emde gives y', y'' and s to 4 or 5 figures, and σ to 5 decimals of a right angle. For

$3p = 3(\cdot 01)3\cdot 2(\cdot 02)4(\cdot 05)5(\cdot 1)10(\cdot 5)15$, all roots are real, and are given to 4 or 5 figures. See also Emde 1940.

Mehmke–d'Ocagne 1909 (320) gives further references and descriptions, including Guldberg 1872–73 and Gundelfinger 1897.

5·645. The Equation $x^3 - 2x - 5 = 0$

This is a well-known test equation for numerical methods; the most extensive calculations of the real root are as follows:

152 dec.	De Morgan 1860 (966, J. P. Hicks), 1864 (337)
101 dec.	De Morgan 1850 (290, W. H. Johnston)

5·65. The Solution of Quartic Equations

The evaluation of the complex square roots $\sqrt{1+ik}$ and $\sqrt{k+i}$ is equivalent to the computation of the real roots of the quartic equations $4x^4 \pm 4x^2 = k^2$ and $4x^4 \pm 4kx^2 = 1$. These roots are $\sqrt{\frac{\sqrt{1+k^2}\pm 1}{2}}$ and $\sqrt{\frac{\sqrt{1+k^2}\pm k}{2}}$; see Art. 2·9.

See also Emde 1940 and Guldberg 1873 for the solutions of more general quartic equations.

5·66. Roots of Equations of Higher Degree

Lewis 1879 gives $x = (\sqrt{4n^2+1} - 2n)^{1/2n}$ to 6 decimals for $n = 1(1)4$; these are roots of $x^{4n} + 4nx^{2n} = 1$.

See Gundelfinger 1897 for solutions of $ax^n + bx = c$ and Guldberg 1872*b* for solutions of $x^5 + cx + c = 0$.

5·7. Roots of Transcendental Equations

It is, of course, practically impossible to give a complete list of published solutions of transcendental equations, but we give in the following articles references to solutions of a number of fairly simple equations.

Values given in Jahnke & Emde 1933, but not in Jahnke & Emde 1938, are possibly also given in Emde 1940, which we have not seen.

5·71. Equations of the Type $\tan x = f(x)$ and $\tanh x = f(x)$

In this article we take $f(x)$ to be the quotient of two polynomials in x. Nearly all of the equations considered are closely connected with Bessel functions of half odd integral order (see Sections 17 and 18); it seems desirable, however, to deal with these here, in conjunction with other equations involving circular functions. In each case we denote the roots by x_n, in ascending order of magnitude, x_1 being the smallest non-zero root.

The British Association calculations are still in progress, so that B.A. MS. items are not final.

5·7111. Tan $x = x$ or $J_{3/2}(x) = 0$. Maxima and Minima of $\frac{\sin x}{x}$

12–15 dec.	x_n	$n = 1(1)14$	B.A. (MS., Miller, etc.)
10 dec.	x_n	$n = 1, 2$	Watson 1933 (145)
6 dec.	x_n, $\frac{\sin x_n}{x_n}$	$n = 1(1)16$	Lommel 1886 (651)

Lommel gives also the squares of the extreme values.

5–7 dec.	x_n	$n = 1(1)3$	Cauchy 1882 (*Œuvres* (1) 1, 202)

4; 6 d.	x_n	$n = 1(1)16;\ 17(1)35$	Silberstein 1923 (97)
6 dec.	$x_n,\ \left(\dfrac{\sin x_n}{x_n}\right)^2$	$n = 1(1)8$	Lommel 1885 (322)

Reproduced in Gray & Mathews 1895 (192) and in G., M. & MacR. 1922 (203).

4 dec.	x_n	$n = 1(1)36$	Hayashi 1930*b* (52)
4 dec.	$x_n,\ \dfrac{\sin x_n}{x_n}$	$n = 1(1)16$	Adams & Hippisley 1922 (84) Jahnke–Emde 1909 (3), 1933 (30)
4 dec.	x_n	$n = 1(1)6$	Jahnke–Emde 1909 (163), 1933 (276), 1938 (206)
1″	$180^\circ\ x_n/\pi$	$n = 1(1)9$	Euler 1748 (2, 320)
4; 6 d.	$\dfrac{x_n}{\pi};\ \left(\dfrac{\sin x_n}{x_n}\right)^2$	$n = 1(1)7$	Pernter 1906 (452)
4 dec.	x_n/π	$n = 1(1)7$	Verdet 1869 (266)

Reproduced also in Preston, e.g., 1890 (210), 1895 (255), 1912 (272).

4 dec.	x_n/π	$n = 1(1)6$	Rayleigh 1872, 1888, 1894 Lamb 1882 (197)

Each table contains at least one error. We give values for $n = 1(1)7$:

1·4303, 2·4590, 3·4709, 4·4774, 5·4815, 6·4844, 7·4865

The following item is also in error for $n = 6$:

5 fig.	$2\pi/x_n$	$n = 1(1)6$	Rayleigh 1872 (102), 1896 (267)

5·7112. Cot $x = -x$ or $J_{-3/2}(x) = Y_{3/2}(x) = 0$

10–15 d.	x_n	$n = 1(1)15$	B.A. (MS., Miller, etc.)

5·7115. Tan $x = \lambda x$

4–6 d.	$\lambda = 4;\ 8/5$	$x_n,\ n = 1(1)4;\ 1(1)3$	Cauchy 1882 (204, 209)
4 dec.	$\lambda = \tan\theta$	$x_n,\ n = 1(1)4,\ \theta = -90^\circ(5^\circ)+90^\circ$	McKay 1932 (22)

For $0 < \theta < 45^\circ$ the root x_1 is not real and is not given.

5·7116. Cot $x = \lambda x$

Roots x_n have been given as follows:

	λ	n		
10 d.	1	1	$x_1 = 0{\cdot}86033\ 35890$	Miller (MS.)
0″·1	1	1	$x_1 = 49^\circ\ 17'\ 36''{\cdot}5$	Adams & Hippisley 1922 (86), from Schlömilch 1904 (355)
4 dec.	$\tan\theta$	1(1)4	$\theta = -90^\circ(5^\circ)+90^\circ$	McKay 1932 (22)
4 dec.	$\tan\theta$	1(1)4	$\theta = 0(5^\circ)90^\circ$	McLachlan 1939 (272, McKay)
4 dec.	$1/u$	1(1)7	$u = 0, \cdot 1, \cdot 5, 1(1)10, \infty$	Newman & Green 1935

5·7117. Coth $x = x$

Casale 1939 gives the root to 22 decimals and the corresponding maximum value of x sech x to 21 decimals. Jahnke & Emde 1933 (31) gives $x = 1{\cdot}199678$. Plummer 1918 (47) writes $\tan\alpha = x$ sech x and transforms the equation to $\cos\alpha \log_e \cot\frac{1}{2}\alpha = 1$, and finds $\alpha = 33^\circ\ 32'\ 3''{\cdot}0$, and $\tan\alpha = 0{\cdot}6627434$. This equation is connected with the determination of the radius of convergence of a series solution of Kepler's equation. We find $x \doteq 1{\cdot}19967\ 864$ and x sech $x = 0{\cdot}66274\ 342$.

$$5\cdot712.\quad \operatorname{Tan} x = \frac{\lambda x}{\lambda - x^2} \quad \text{or} \quad \frac{d}{dx}\{x^{\frac{3}{2}-\lambda} J_{3/2}(x)\} = 0$$

$$\operatorname{Cot} x = -\frac{\lambda x}{\lambda - x^2} \quad \text{or} \quad \frac{d}{dx}\{x^{\frac{3}{2}-\lambda} J_{-3/2}(x)\} = 0$$

$$5\cdot7121.\quad \lambda = 1.\quad \operatorname{Tan} x = \frac{x}{1 - x^2} \quad \text{or} \quad \frac{d}{dx}\{x^{1/2} J_{3/2}(x)\} = 0$$

4 fig.	x_n	$n = 1(1)6$		Collo 1921 (46) Adams & Hippisley 1922 (87)
4 dec.	x_n/π	$n = 1(1)6$		Lamb 1882 (202)
4 dec.	$x_n/n\pi$	$n = 1(1)4$	Errors!	Silberstein 1923 (97) Hayashi 1930*b* (52)

$$\lambda = 1.\quad \operatorname{Cot} x = -\frac{x}{1 - x^2} \quad \text{or} \quad \frac{d}{dx}\{x^{1/2} J_{-3/2}(x)\} = 0$$

4 fig.	x_n	$n = 1(1)4$	Collo 1921 (45) Adams & Hippisley 1922 (87)

$$5\cdot7122.\quad \lambda = 2.\quad \operatorname{Tan} x = \frac{2x}{2 - x^2} \quad \text{or} \quad \frac{d}{dx}\{x^{-1/2} J_{3/2}(x)\} = 0$$

4 dec.	x_n/π	$n = 1(1)6$	Lamb 1882 (202)
5 fig.	$2\pi/x_n$	$n = 1(1)6$	Rayleigh 1872 (102), 1896 (267)

$$5\cdot7123.\quad \lambda = 3.\quad \operatorname{Tan} x = \frac{3x}{3 - x^2} \quad \text{or} \quad J_{5/2}(x) = 0$$

8–15 dec.	x_n	$n = 1(1)6$	B.A. (MS., Miller, etc.)
4 dec.	x_n	$n = 1(1)6$	Jahnke–Emde 1909 (163), 1933 (276), 1938 (206)
4 dec.	x_n/π	$n = 1(1)6$	Lamb 1882 (197) Adams & Hippisley 1922 (85)

$$\lambda = 3.\quad \operatorname{Cot} x = -\frac{3x}{3 - x^2} \quad \text{or} \quad J_{-5/2}(x) = -Y_{5/2}(x) = 0$$

5–15 dec.	x_n	$n = 1(1)7$	B.A. (MS., Miller, etc.)

$$5\cdot7126.\quad \lambda = \frac{4}{3} \text{ or } \frac{8}{7}.\quad \operatorname{Tan} x = \frac{4x}{4 - 3x^2} \quad \text{or} \quad \frac{8x}{8 - 7x^2}$$

4 dec.	$\lambda = 4/3$	x_n/π	$n = 1(1)6$	Lamb 1882 (202) Adams & Hippisley 1922 (85, 86)
4 dec.	$\lambda = 8/7$	x_n/π	$n = 1(1)6$	Lamb 1882 (202)

$$5\cdot713.\quad \operatorname{Tan} x = \frac{3\lambda x - x^3}{3\lambda - (\lambda + 1)x^2} \quad \text{or} \quad \frac{d}{dx}\{x^{\frac{5}{2}-\lambda} J_{5/2}(x)\} = 0$$

$$\operatorname{Cot} x = -\frac{3\lambda x - x^3}{3\lambda - (\lambda + 1)x^2} \quad \text{or} \quad \frac{d}{dx}\{x^{\frac{5}{2}-\lambda} J_{-5/2}(x)\} = 0$$

5·7132. $\lambda = 2.$ $\operatorname{Tan} x = \dfrac{6x - x^3}{6 - 3x^2}$ or $\dfrac{d}{dx}\{x^{1/2}J_{5/2}(x)\} = 0$

4 dec.	x_n/π	$n = 1(1)6$	Lamb 1882 (205)

5·7133. $\lambda = 3.$ $\operatorname{Tan} x = \dfrac{9x - x^3}{9 - 4x^2}$ or $\dfrac{d}{dx}\{x^{-1/2}J_{5/2}(x)\} = 0$

4 dec.	$x_1 = 3{\cdot}3422$		Rayleigh 1872 (98), 1896 (266)
5 fig.	$2\pi/x_n$	$n = 1(1)4$	Rayleigh 1872 (102), 1896 (267)

5·7134. $\lambda = 4.$ $\operatorname{Tan} x = \dfrac{12x - x^3}{12 - 5x^2}$ or $\dfrac{d}{dx}\{x^{-3/2}J_{5/2}(x)\} = 0$

4 dec.	x_n/π	$n = 1(1)6$	Lamb 1882 (198)

5·7135. $\lambda = 5.$ $\operatorname{Tan} x = \dfrac{15x - x^3}{15 - 6x^2}$ or $J_{7/2}(x) = 0$

6–12 dec.	x_n	$n = 1(1)53$	B.A. (MS., Gwyther)

$\lambda = 5.$ $\operatorname{Cot} x = -\dfrac{15x - x^3}{15 - 6x^2}$ or $J_{-7/2}(x) = Y_{7/2}(x) = 0$

4–12 dec.	x_n	$n = 1(1)53$	B.A. (MS., Gwyther)

5·714. $\dfrac{d}{dx}\{x^{-1/2}J_\nu(x)\} = 0$ for $\nu = \frac{7}{2}(1)\frac{13}{2}$

Rayleigh 1872 (102), 1896 (267) gives $2\pi/x_n$ to 4 figures, as follows:

$\nu = \frac{7}{2}$	$\operatorname{Tan} x = \dfrac{60x - 7x^3}{60 - 27x^2 + x^4}$	$n = 1, 2, 3$
$\nu = \frac{9}{2}$	$\operatorname{Tan} x = \dfrac{525x - 65x^3 + x^5}{525 - 240x^2 + 11x^4}$	$n = 1, 2$
$\nu = \frac{11}{2}$	$\operatorname{Tan} x = \dfrac{5670x - 735x^3 + 16x^5}{5670 - 2625x^2 + 135x^4 - x^6}$	$n = 1$
$\nu = \frac{13}{2}$	$\operatorname{Tan} x = \dfrac{72765x - 9765x^3 + 252x^5 - x^7}{72765 - 34020x^2 + 1890x^4 - 22x^6}$	$n = 1$

5·715. $\left.\begin{array}{l} J_\nu(x) = 0 \text{ or } \tan x = f_\nu(x) \\ J_{-\nu}(x) = \pm Y_\nu(x) = 0 \text{ or } \cot x = -f_\nu(x) \end{array}\right\} \nu = \frac{9}{2}(1)\frac{21}{2}$

$$f_\nu(x) \text{ or } -\frac{1}{f_\nu(x)} = \frac{x^{\nu-\frac{1}{2}} - \dfrac{(4\nu^2-1)(4\nu^2-9)}{2!\,8^2}x^{\nu-\frac{5}{2}} + \dots}{\dfrac{4\nu^2-1}{1!\,8}x^{\nu-\frac{3}{2}} - \dfrac{(4\nu^2-1)(4\nu^2-9)(4\nu^2-25)}{3!\,8^3}x^{\nu-\frac{7}{2}} + \dots}$$

according as $\nu - \frac{1}{2}$ is odd or even.

3–12 dec.	x_n	$n = 1(1)53$	B.A. (MS., Gwyther)

See Arts. 5·7111, 5·7112 for $\nu = \frac{3}{2}$, 5·7123 for $\nu = \frac{5}{2}$ and 5·7135 for $\nu = \frac{7}{2}$.

5·719. $\tan x = \dfrac{480x - 92x^3 + 5x^5}{480 - 252x^2 + 25x^4 - x^6}$

4 dec.	x_n/π	$n = 1(1)6$	Lamb 1882 (206)

5·72. Equations of the Type $\cos x = f(x)$

5·721. $\cos x = x$

One root $x = 42^\circ\ 20'\ 47''\cdot 3$ (or 0·73908 5133 radians, Miller MS.) is given in Adams & Hippisley 1922 (87) from Schlömilch 1904 (353).

T. H. Miller 1891 gives the real root to 8 decimals and 6 complex roots (with positive real parts) to about 4 decimals. Silberstein 1935 gives the real root to 5 decimals and 3 complex roots (with negative real parts) to 5 figures. All the roots in the last two items are reproduced by Archibald, *Scripta Math.*, **4**, 100, 1936.

5·722. $\sec x = 1 + x^2$

6 dec.	x_n	$n = 1(1)6$	Adams & Hippisley 1922 (86)

From Schlömilch 1904 (354).

5·725. $\sin x = x$

Hillman & Salzer 1943 gives 6-decimal values of the first 10 non-zero roots of $\sin z = z$ having both real and imaginary parts positive.

5·73. $\cosh x \cos x = \pm 1$

5·731. $\cosh x \cos x = 1$

The roots fall into two sets, according as $\tan \frac{1}{2}x = \pm \tanh \frac{1}{2}x$.

7 dec.	x_n	$n = 1(1)5$	Rayleigh 1894 (278); Adams & Hippisley 1922 (86); Silberstein 1923 (98)
4 dec.	x_n	$n = 1(1)5$	Hayashi 1930*b* (52)
4 dec.	$\frac{1}{2}x_n$	$n = 1(1)4$	Jahnke–Emde 1933 (31)
4 dec.	x_n	$n = 1(1)3$	Jahnke–Emde 1909 (3)

Rayleigh also gives 6-figure values of $\beta_n = (-1)^n\{(n+\frac{1}{2})\pi - x_n\}$ from which 7- to 13-decimal values of x_n may be derived.

5·732. $\cosh x \cos x = -1$

The roots fall into two sets, according as $\tan \frac{1}{2}x = \pm \coth \frac{1}{2}x$.

8 dec.	x_n	$n = 1(1)7$	Fletcher (MS.)
6 dec.	x_n	$n = 1(1)6$	Rayleigh 1894 (278); Adams & Hippisley 1922 (86); Silberstein 1923 (98)

In x_2 the final digit should be 1, not 8.

4 dec.	x_n	$n = 1(1)6$	Hayashi 1930*b* (52)
4 dec.	$\frac{1}{2}x_n$	$n = 1(1)5$	Jahnke–Emde 1933 (31)
4 dec.	x_n	$n = 1(1)4$	Jahnke–Emde 1909 (3)

Rayleigh also gives, to 6 or more figures, values of $a_n = (-1)^n\{(n-\frac{1}{2})\pi - x_n\}$ from which 7- to 13-decimal values of x_n may be derived. Fletcher has 9-figure values of these differences and of a number of related quantities such as $\tan \frac{1}{2}a_n$, etc. The values given by Greenhill (*Mess. Math.*, **16**, 119, 1886) contain gross errors.

5·74. $xe^x = 2$ and $e^{3x/4} = 1 + x$

$xe^x = 2$ or $\log \frac{1}{2}x = -x$	$x = 0{\cdot}8526$
$e^{3x/4} = 1 + x$ or $\log(1+x) = \frac{3}{4}x$	$x = 0{\cdot}73360$

These values are given in Adams & Hippisley 1922 (87), from Schlömilch 1904 (353).

5·745. $10^n \log_{10} x = x$ or $10^x = x^{10^n}$

A note on this equation appears in *M.T.A.C.*, **1**, 202–203, 1944, where values to more than 50 figures, by D. M. Knappen, Sir C. V. Boys and J. W. Wrench, Jr. are given or discussed.

5·8. Miscellaneous Constants

See also Art. 4·79.

5·81. $\rho = 0{\cdot}47693\ 62762\ 04469\ 87338\ldots$ $\qquad \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^\rho e^{-x^2}dx = \frac{1}{2}$

$\mathrm{Log}_{10}\,\rho = \bar{1}{\cdot}67846\ 03565\ 21217\ 91322\ldots \qquad \rho\sqrt{2} = 0{\cdot}67448\ 97501\ 96081\ldots$

Burgess 1898 (278–279) gives ρ to 24 decimals (correct only to 20 decimals); 23-decimal values of the following constants and their common logarithms are also given on page 279:

$$\rho,\quad 1/\rho,\quad \rho^2,\quad \rho\sqrt{2},\quad 2\rho\sqrt{\pi},\quad 2\rho/\sqrt{\pi},\quad 2\rho^2/\sqrt{\pi},\quad 1/\rho\sqrt{\pi},\quad 1/2\rho\sqrt{\pi},\quad \rho\sqrt{\pi},$$

$$1/\rho\sqrt{2},\quad \rho\sqrt[6]{\pi},\quad \rho\sqrt[10]{\tfrac{1}{4}\pi},\quad \rho\sqrt{\pi-2},\quad \rho\sqrt{\tfrac{1}{30}(15\pi-8)},\quad \rho\sqrt{\tfrac{1}{1600}(945\pi-128)},$$

$$\rho\sqrt{\tfrac{4}{3}},\quad \rho\sqrt[4]{\tfrac{4}{3}},\quad \rho\sqrt{\tfrac{113}{45}},\quad \rho\sqrt[6]{\tfrac{8}{15}},\quad e^{\rho^2},\quad e^{\rho^2}\sqrt{\pi},\quad e^{-\rho^2},\quad 2e^{-\rho^2}/\sqrt{\pi}.$$

The value of $\rho\sqrt{2}$ is erroneous beyond the 13th decimal, although $\log_{10}\rho\sqrt{2}$ is in agreement with the basic value of ρ.

5·82. The Lemniscate Constant ϖ

$$\tfrac{1}{2}\varpi = A = \int_0^1 \frac{dx}{\sqrt{1-x^4}} = 1{\cdot}31102\ 87771\ 46059\ 90680\ 320(7)$$

$$\frac{1}{2}\frac{\pi}{\varpi} = B = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = 0{\cdot}59907\ 01173\ 67796\ 10372$$

Values quoted are from Gauss 1866*b* (413), 1813 (*Werke*, **3**, 150, 1866); Stirling in 1730 gave 17-decimal values. Gauss 1866*b* (413) also gives:

17+ dec.	$(\frac{1}{2}\varpi)^n$	$n = 2, 4, 5, 6, 8$	14+ dec.	$\log_{10}\frac{1}{2}\varpi$
12; 15; 11 d.	$(\frac{1}{2}\varpi)^n$	$n = 9;\ 12;\ 13$	24 dec.	$\log_e \frac{1}{2}\varpi$
26 dec.	$\sqrt{\varpi/\pi}$	(p. 418)	25 dec.	$\log_e \varpi$

21 dec.	ϖ/π	Markoff 1896 (194)

Art. 21·2 refers to $A = \frac{1}{2}\varpi$ and $B = \pi/2\varpi$, not to the A and B below.

Gauss 1866*b* (420) also gives 20-decimal values of

$$A,\quad 2e^{-\frac{1}{4}\pi}B,\quad 2e^{-2\pi}A,\quad 2e^{-9\pi/2}B,\quad 2e^{-8\pi}A,\quad 2e^{-25\pi/2}B$$

where
$$A = \frac{\sqrt{\pi/\varpi}}{2^{7/4}\cos\frac{1}{8}\pi} \qquad B = \frac{\sqrt{\pi/\varpi}}{2^{7/4}\sin\frac{1}{8}\pi}$$

5·831. The Constants $\dfrac{3^{-2/3}}{\Gamma(\frac{2}{3})}$ and $\dfrac{3^{-1/3}}{\Gamma(\frac{1}{3})}$

See Art. 14·33 for a full discussion of $\Gamma(\frac{1}{3})$ and $\Gamma(\frac{2}{3})$, etc.; we give here only the dependent constants

$$\frac{3^{-2/3}}{\Gamma(\frac{2}{3})} = 0{\cdot}35502\ 80538\ 87817 \qquad \frac{3^{-1/3}}{\Gamma(\frac{1}{3})} = 0{\cdot}25881\ 94037\ 92807$$

These constants are connected with the Airy Integral (Art. 20·2). Comrie (MS.) has these constants and their common logarithms to 20 decimals.

5·835. Glaisher's Constants A_r

These are defined by the relation (cf. Art. 14·65)

$$1^{1^r}\ 2^{2^r}\ 3^{3^r}\ \ldots\ x^{x^r} \sim A_r x^{f(x,\,r)} e^{F(x,\,r)}$$

and it may be shown that

$$\log_e A_{r+1} = (-1)^{r+1}\int_0^1 V_r(x)\log_e\Gamma(x)\,dx$$

where $V_r(x)$ is a Bernoulli polynomial (Art. 4·4), and $f(x, r) = (-1)^{r+1}V_{r+1}(-x)$ and $F(x, r)$ are rational polynomials in x, given by Glaisher 1896 (110) to $r = 7$.

$r = 0$ gives Stirling's series for $n!$ $\qquad A_0 = \sqrt{2\pi}$

$r = 1 \qquad A_1 = A = 1{\cdot}28242\ 71291\ 00623$

$\qquad\qquad \log_e A = 0{\cdot}24875\ 44770\ 33784$

Both Glaisher (*Mess. Math.*, **24**, 16, 1894) and Barnes (*Quart. Journ. Math.*, **31**, 291, 1900) have expressed the view that this may be regarded as a "known" or "definite" constant of analysis.

Jeffery 1864 (101) gives $\log_{10} A$ and $\log_{10} A_2$ to 10 decimals.

Glaisher 1877*e* (43) gives $\log_{10} A$ to 11 decimals and A to 9 decimals.

Glaisher 1896 (123) gives $\log_e A_r$, $r = 0(1)4$, to 9+ decimals. The sign of $\log_e A_4$ should be negative.

5·84. Lebesgue's Constants ρ_n

Watson 1930 (314) gives $\rho_{9/2}$ to 14 decimals and C_n to 15 decimals for $n = 0(1)7$, where

$$\rho_n = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{|\sin(2n+1)\theta|}{\sin\theta}\,d\theta \sim \frac{4}{\pi^2}\left\{\log_e(2n+1) + C_0 + \sum_1^\infty(-1)^{r-1}\frac{C_r}{(2n+1)^{2r}}\right\}$$

5·85. Lehmer's and Khintchine's Constants

Lehmer's constant is

$$\xi = \cot\{\cot^{-1} 0 - \cot^{-1} 1 + \cot^{-1} 3 - \cot^{-1} 13 + \cot^{-1} 183 - \cot^{-1} 33673 + \ldots \pm \cot^{-1} n_\nu \mp \ldots\}$$

where $n_{\nu+1} = n_\nu^2 + n_\nu + 1$. This is the most slowly convergent continued cotangent. Lehmer 1938 (334) gives 80 decimals, beginning with 0·59263 27182 01636 ...

Khintchine's constant is the limit as $n \to \infty$ of the geometric mean of the first n partial quotients of "almost all" continued fractions. Lehmer 1939 (151) gives 2·685550 as the value of this constant to 6 decimals.

5·9. Conversion Tables

These are tables of multiples of conversion factors, themselves constants that have, in many cases, been considered in preceding articles of this section. In this relatively unimportant sub-section we give only a few tables of outstanding use, chosen without extensive search. Very many books of tables, particularly those giving circular functions, contain a number of conversion tables. Time to arc tables (and vice versa) are very common, but are not considered here.

See also Arts. 7·811–7·824, and, for conversions between natural and common logarithms, needing multiples of M or $1/M$, see Arts. 5·331 and 10·31.

5·91. Arc to Radians

5·911. $\dfrac{n\pi}{180}$. Degrees to Radians

	n		
32 dec.	1(1)100		**Peters & Stein 1922** (3)
27 dec.	1(1)360		J. C. Schulze 1776
25 dec.	1(1)100		Hayashi 1926 (233)
15 dec.	1(1)90, 100(20)360		New York W.P.A. 1943*b* (404)
15 dec.	1(1)90		New York W.P.A. 1942*d* (169)
11 dec.	0(·0⁷1)·0⁵1(·0⁵1)·0³1(·0³1) ·01(·01)1(1)359		**Peters 1919*b*** (52)
11 dec.	1(1)100(10)270(30)360		Köhler 1847 (320); Ligowski 1890 (55); Becker & Van Orstrand 1924 (320)
10 dec.	1(1)180		Bruhns 1922 (608)
10 dec.	1(1)30(10)100		Potin 1925 (381)
4 dec.	$\log \dfrac{n\pi}{180}$	1(·01)9·99	Huntington 1910 (28)

5·912. $\dfrac{m\pi}{10\,800}$ and $\dfrac{n\pi}{648\,000}$. Minutes and Seconds of Arc to Radians

	m	n	
35 dec.	1(1)10(10)60	1(1)10(10)60	Herrmann 1848*b* (481)
32 dec.	1(1)100	1(1)100	**Peters & Stein 1922** (4, 5)
27 dec.	1(1)60	1(1)60	J. C. Schulze 1776
25 dec.	1(1)60	1(1)100	Hayashi 1926 (238, 241)

	m	n	
21 dec.	1(1)100		Peters 1911*a* (53)
15 dec.	1(1)60	1(1)60	New York W.P.A. 1943*b* (405)
13 dec.	1(1)600		Börgen 1908 (34)
11 dec.	1(1)60	1(1)60	Köhler 1847 (320) Ligowski 1890 Becker & Van Orstrand 1924 (320)
10 dec.	1(1)60	1(1)60	Bruhns 1922 (608)
10 dec.	1(1)30(10)100	1(1)30(10)100	Potin 1925 (381)
7 dec.	1(1)5400	1(1)60	Ives 1929 (91), 1931 (273) Pryde 1930 (251)

These tables probably have a common origin; the one in Ives 1931 is known to be accurate, having only two very close rounding-off errors.

6 dec.	1(1)5400	Potin 1925 (110)

5·913. $\dfrac{n\pi}{200}$. Grades to Radians

Some tables give multiples of $\frac{1}{2}\pi$; these are not specially indicated below.

	n	
32 dec.	1(1)100	**Peters & Stein 1922** (6)
17 dec.	1(1)100	**B.A. 1, 1931** (2) New York W.P.A. 1942*a* (182), 1943*b* (410)
17 dec.	1(1)50	New York W.P.A. 1940*d* (225)
15 dec.	1(1)100	Hayashi 1926 (265)
10 dec.	1(1)100	Peters 1938 (508)
10 dec.	1(1)50(5)100	Potin 1925 (383)
8 dec.	1(1)100	Hoüel 1901 (19) Potin 1925 (98)
5–6 dec.	0(·01)100	Potin 1925 (133)

6 decimals are given from 0 to 7 and from 93 to 100.

6 dec.	0(·0001)·01(·01)1(1)100	Wijdenes 1937 (30)

5·914. $2n\pi$. Circumferences (Gones) to Radians

6 dec.	1(1)100	Potin 1925 (204)

5·92. Radians to Arc

5·921. $\dfrac{180n}{\pi}$. Radians to Degrees

	n	
15 dec.	0(·0³1)·0²1(·0²1)·01(·01)·1(·1) 1(1)10(10)100	New York W.P.A. 1942*d* (168), 1943*b* (406)
9 dec.	0(·0⁹1)·0⁷1(·0⁷1)·0⁵1(·0⁵1) ·0³1(·0³1)·01(·01)1(1)6	**Peters 1919*b*** (49)
7 dec.	0(·0⁴1)·0³1(·0³1)·0²1(·0²1) ·01(·01)·1(·1)1	Chrétien 1932 (40)
6 dec.	0(1)100	Potin 1925 (192)

5·922. Radians to Degrees, Minutes and Seconds

The first column gives the number of decimals of a second of arc which are tabulated.

17	$0(\cdot0^51)\cdot0^31(\cdot0^31)\cdot01(\cdot01)1\cdot6$	Airey (MS.)
10	$m.10^{-p}$; $m = 1(1)9$, $p = 0(1)5$	Hayashi 1926 (204)
9	$m.10^{-p}$; $m = 1(1)9$, $p = 0(1)5$	Hayashi 1921 (177)
7	$m.10^{-p}$; $m = 1(1)9$, $p = 1(1)5$	Ligowski 1890 (55)
7	$0(\cdot1)10(1)100$	Becker & Van Orstrand 1924 (174, 224, 321)
7	$m.10^{-p}$; $m = 1(1)9$, $p = 1(1)5$	
2	$0(\cdot0001)\cdot1(\cdot001)1\cdot6$	
6	$0(\cdot0^31)\cdot0^21(\cdot0^21)\cdot01(\cdot01)\cdot1(\cdot1)$ $1(1)10(10)100$	New York W.P.A. 1942*d* (168), 1943*b* (406)
5	$0(\cdot00001)\cdot001$	Hayashi 1926 (4, 12)
2	$\cdot001(\cdot0^31)\cdot1(\cdot0^21)3(\cdot01)10(\cdot1)$ $20(1)50$	
2	$0(\cdot0^31)\cdot1(\cdot0^21)3(\cdot01)6\cdot3(\cdot1)10$	Hayashi 1921 (2)

5·9221. $\dfrac{648\,000\,n}{\pi}$. Radians to Seconds of Arc

7 dec.	$1(1)10$	Potin 1925 (221)

5·923. $\dfrac{200n}{\pi}$. Radians to Grades

	n	
7 dec.	$m.10^{-p}$; $m = 1(1)9$, $p = 1(1)5$	Chrétien 1932 (40)
6 dec.	$0(\cdot01)1(1)100$	Potin 1925 (99, 192)
6 dec.	$0(\cdot01)2(1)10$	Wijdenes 1937 (31)
6 dec.	$0(\cdot01)1$	Hoüel 1901 (19)

5·924. $\dfrac{n}{2\pi}$. Radians to Circumferences (Gones)

6 dec.	$1(1)100$	Potin 1925 (202)

Items added in Proof

5·0. Vogel 1943 gives a good collection of constants involving π; in detail:

5·111, 5·131.	70 dec.	π, $1/\pi$		p. 233
5·112, 5·132.	35 dec.	$n\pi$, $n\pi/180$, n/π, $180n/\pi$	$n = 1(1)9$	
		π/n, $1/n\pi$	$n = 1(1)19$	
5·112.	10 dec.	$k\pi/n$, $k = \frac{1}{2}$, 1, 2	$n = 1(1)200$	p. 216

Lowan & Laderman 1943 gives 10-decimal values of $\int_0^1 x^p \sin n\pi x\,dx$ and $\int_0^1 x^q \cos n\pi x\,dx$.

5·132.	$p = 0$	$2/n\pi$,	$n = 1(2)99$	p. 137
	$p = 1$	$1/n\pi$,	$n = 1(1)100$	p. 137
	$p = 2$	$1/n\pi$,	$n = 2(2)100$	p. 138
5·14.	$q = 1$	$2/n^2\pi^2$,	$n = 1(2)99$	p. 142
	$q = 2$	$2/n^2\pi^2$,	$n = 1(1)100$	p. 143
	$q = 3$	$3/n^2\pi^2$,	$n = 2(2)100$	p. 143

The other integrals involve sums of quantities of the type $\pm k(n\pi)^{-r}$ with $r < 11$. For example,

with $n = 1(2)99$, $p = 2$ gives $(n\pi)^{-1} - 4(n\pi)^{-3}$, $q = 3$ gives $-3(n\pi)^{-2} + 12(n\pi)^{-4}$;
with $n = 1(1)100$, $p = 3$ gives $\pm(n\pi)^{-1} \mp 6(n\pi)^{-3}$, $q = 4$ gives $\pm 4(n\pi)^{-2} \mp 24(n\pi)^{-4}$.

SECTION 6

COMMON LOGARITHMS, ANTILOGARITHMS, ADDITION AND SUBTRACTION LOGARITHMS

6·0. Introduction

So many tables of logarithms of numbers to base 10 have been published that a high degree of selection has been necessary in some articles, particularly those concerned with working tables giving 4 to 8 figures. Fortunately Henderson 1926 provides a very full description of all but the most recent tables to 6 or more figures, with very useful indexes of logarithmic and antilogarithmic tables. The existence of this work has greatly simplified the present authors' task of selection and description; it has also allowed us to omit a number of tables without great regret, since anyone requiring fuller information may be referred to Henderson. It does not, however, deal with addition and subtraction logarithms, and our articles on these have been put together in the same way as most of the rest of our *Index*.

Articles 6·1 to 6·3 deal with logarithms (also lologs and antilologs), 6·4 to 6·6 with antilogarithms, and 6·7 to 6·9 with addition and subtraction logarithms.

For proportional logarithms, etc., see Arts. 9·8 ff.

6·1. Fundamental Tables of Logarithms

In this article we consider tables giving logarithms of at least 9000 consecutive integers (i.e. the numbers 1000 to 10,000, or a more extensive range) to 10 or more decimals. The number of such tables is not large, and the following list contains most that have come to our knowledge. For many-figure radix and special methods, see Arts. 6·2 to 6·25. For many-figure logarithms of fewer than 9000 integers, see Art. 6·3.

28 dec.	1(1)10,038(composite numbers)20,000		Sang (MS. 1848–90)
21 dec.	1000(1)10,000; see also Art. 6·21		Steinhauser 1880
20 dec.	10,000(1)100,000	δ^{2n}	**Thompson 1924–**

A monumental work, of which Part 2, for the range 20,000(1)30,000, remains to be published.

19 dec.	1(1)10,000		Prony (MS. 1794)
15 dec.	100,000(1)370,000	Δ^2	Sang (MS. 1848–90)

Errors up to about 2 in 15th decimal.

14 dec.	1(1)20,000; 90,000(1)100,000	Δ	Briggs 1624

Last figure approximate. On errors in range 1(1)20,000, see Bauschinger & Peters 1936; in range 90,090(1)100,000, see Thompson 1924–, Part 4 (1928). In some copies the table extends to 101,000.

12 dec.	10,000(1)200,000		Prony (MS. 1794)

A 14-decimal table with errors up to about 30 in the last decimal.

12 dec.	20,000(1)200,000		Bauschinger & Peters (MS.)
12 dec.	1(1)120,000		J. Thomson (MS. 1873)
10 dec.	1(1)100,000	Δ	Vlacq 1628

On lists of errors see Henderson 1926 (56). Table first published in De Decker 1627.

10 dec.	1(1)101,000	Δ(10,000 on)	Vega 1794

List of errors in Henderson 1926 (94), from Peters 1922.

10 dec.	10,000(1)100,000	Δ	Duffield 1897

Many errors in common with Vega.

10 dec.	1(1)100,000	Δ(10,000 on)	**Peters 1922**
10 dec.	1000(1)10,000	Sp.	Numerov 1932*b*

6·15. Working Tables of Logarithms

Accuracy and convenient provision for interpolation are the chief requirements. Only the working range is usually given. For example, Pryde 1930 contains 4 pages giving logarithms of 1(1)1000; this is very convenient, but the logarithms of 1(1)1000 are essentially included among those of 10,000(1)100,000.

8 dec.	20,000(1)200,000	IΔ, PPd	**Bauschinger & Peters 1936 (1910)**
8 dec.	12,000(1)120,000	PPd	**Service Géog. 1891**

Only known error: log 28917 is 1 final unit too small.

8 dec.	10,000(1)125,000	PP(d)	Mendizábal Tamborrel 1891
8 dec.	10,000(1)100,000		Sperotti 1911
8 dec.	10,000(1)100,000	Log Δ	J. Newton 1658
7; 10 d.	20,000(1)200,000; 1(1)1000	fPPd; no Δ	**Sang 1915 (1871)**
7 dec.	10,000(1)100,000	IΔ, (f?)PPd	**Peters 1940*a***
7 dec.	10,000(1)100,000	fPPd	Bruhns 1922

Reprinted in J. W. Glover 1930.

7 dec.	10,000(1)100,009	fPPd	Vega–Bremiker, 1856
7; 8 d.	10,000(1)100,000; 100,000(1)108,000	fPPd	Schrön 1860

PP_{100} of 40(1)409, etc., in same volume.

7; 8 d.	10,000(1)100,000; 100,000(1)108,000	PP	Pryde 1930
6 dec.	10,000(1)100,009	fPPd	Jordan–Eggert 1939
6 dec.	10,000(1)100,009	fPPd	**Bremiker 1924 (1852)**
6 dec.	1000(1)10,000	IΔ, fPPd	Allen 1941, Ives 1934
6 dec.	1000(1)10,000	PP	G. W. Jones 1893
5 dec.	10,000(1)100,000	(No interpolation needed!)	**Scott 1870, 1912** F. W. Johnson 1930, 1933
5 dec.	4000(1)40,009	IΔ, PP	Chappell 1915
5 dec.	1000(1)14,500	fPPd	Ligowski 1892
5 dec.	1000(1)12,000	Δ, (PP)	Service Géog. 1914
5; 6 d.	1000(1)10,000; 10,000(1)11,009	fPPd	Wijdenes 1937 Newcomb 1882
5; 6 d.	1000(1)10,000; 10,000(1)10,999	fPPd	Silberstein 1923
5 dec.	1000(1)10,080	Δ, fPPd	Ist. Idrog. 1916
5 dec.	1000(1)10,009	fPPd	**Bremiker 1937 (1872)** **Becker 1928 (1882)** Beetle 1933 Hedrick 1920

5 dec.	1000(1)10,000	Δ, fPPd	Carmichael & Smith 1931
$4\frac{1}{2}$ dec.	1000(1)10,000	(No interpolation needed!)	McLaren 1915
$4\frac{1}{2}$ dec.	200(1)2009	Δ, fPPd	**MT.C. 1931**
$4\frac{1}{2}$ dec.	200(1)2009	fPPd	Desvallées 1920

4-decimal logarithms of 1000(1)10,000 are given in Hannyngton 1912, Schubert 1913 and F. W. Johnson 1930, 1933.

6·16. Cologarithms (Logarithms of Reciprocals)

Colog $x = -\log x = \log(1/x)$.

27 dec.	11(1)110		Thoman 1867
6 dec.	1000(1)10,000	PPMD	Caillet 1916
5 dec.	4000(1)40,000	IΔ, PP	Chappell 1915
5 dec.	1000(1)10,080	Δ, fPPd	Ist. Idrog. 1916
4 dec.	1000(1)10,000	(No interpolation needed!)	F. W. Johnson 1930, 1933
4 dec.	200(1)2000	PPMD	Plummer 1941

For smaller 4-decimal tables see Bottomley 1909 and Dougall 1931.

6·17. $n \log x$

5 dec.	$n = 1(1)10$	$x = 1(1)100$	No Δ	Houzeau 1875*c*
4 dec.	$n = 2, 3, \frac{1}{2}, \frac{1}{3}$	$x = 1(1)99$	No Δ	G. W. Jones 1893

6·18. Lologs (Logarithms of Logarithms)

These are intended for use in raising numbers to non-simple powers, by the formula

$$\log\log x^n = \log\log x + \log n$$

Chappell 1915 tabulates $\log(-\log x)$ to 5 decimals for $x = \cdot 001\,(\text{var.})\,1$, and $\log\log x$ to 5 decimals for $x = 1\,(\text{var.})\,1000$, in both cases with proportional parts over most of the range. See also *M.T.A.C.*, **1**, 131, 1943.

6·19. Antilologs

If $x = \text{lolog}\, y$, $y = \text{antilolog}\, x$. A 4-figure table with proportional parts is given in Chappell 1915.

6·2. Radix Methods involving Logarithms

The tables described in the next few articles contain many-figure logarithms (or cologarithms) of a limited number of integers and of $1 \pm m \, . \, 10^{-n}$, where m and n are integral. We shall refer to them as radix tables with one-, two-, three-, ... figure groups according as m takes the values 1(1)9, 1(1)99, 1(1)999, ... For each table we have stated the quantities of which the logarithms are tabulated; we have not distinguished between logarithms and cologarithms.

On the various methods of evaluating the logarithm of a number by means of the logarithms of factors of the above kind, see Henderson 1926, Glaisher 1915 and Ellis 1881. Until the advent of Thompson 1924–, which is a 20-decimal table arranged for interpolation by Everett's formula in the usual way, a radix or other special method afforded the only means of obtaining many-decimal logarithms, and the radix method is still convenient. References to special methods are given in Art. 6·25.

8

We consider here only radix tables involving logarithms; these can be employed for finding antilogarithms also. Radix tables involving antilogarithms, which are capable of giving logarithms also, are considered in Art. 6·5.

6·21. Radix Tables with Groups of Three or Four Figures

The only logarithmic radix employing exclusively 4-figure groups is Steinhauser 1880, which most unfortunately contains many errors. These have frequently been investigated; for references see Henderson 1926 and Comrie 1932*a*.

The standard 3-figure radix is Gray 1876, which however starts weakly with a single digit. If not more than 20 decimals are required, a step may be saved by starting with the logarithm of a 4-figure or 5-figure number from Thompson 1924–.

21 dec.	$1000(1)9999$; $1+m.10^{-n}$, $m=1(1)9999$, $n=7(4)15$		Steinhauser 1880
24 dec.	$1(1)9$; $1+m.10^{-n}$, $m=1(1)999$, $n=3(3)15$		**Gray 1876**
21 dec.	$1+m.10^{-n}$, $m=1(1)1000$, $n=7(3)13$		**Thompson 1924–, Part 8 (1927)**
15 dec.	$1(1)999$; $1+m.10^{-n}$, $m=1(1)999$, $n=5, 8$		Steinhauser 1865
14 dec.	$11(1)99$; $1+m.10^{-n}$ $\begin{cases} m=1(1)99, & n=3 \\ m=1(1)999, & n=6, 9 \end{cases}$		Eilmann 1804
13 dec.	$100(1)999$; $1+m.10^{-n}$, $m=1(1)1000$, $n=5$	Δ	Andoyer 1922*a*

A table for second-difference correction is also given.

12 dec.	$1(1)9$; $1+m.10^{-n}$, $m=1(1)999$, $n=3(3)9$		Gray 1865
11 dec.	$1(1)999$; $1+m.10^{-n}$, $m=1(1)9999$, $n=6$	fPP	Steinhauser 1857
10 dec.	$1(1)999$; $1+m.10^{-n}$, $m=1(1)10,999$, $n=6$	$\mathrm{fPP}_{100}\mathrm{d}$	Pineto 1871
10 dec.	$100(1)999$; $1+m.10^{-n}$, $m=1(1)999$, $n=5$	Δ, $1/\Delta$	Scott 1870, 1897

6·22. Radix Tables with Two-Figure Groups

16 dec.	$1(1)9$; $1\pm m.10^{-n}$, $m=1(1)99$, $n=2(2)10$	Shortrede 1849*a*

$1-m.10^{-2}$, $m=50(1)99$ omitted.

15 dec.	$10(1)99$; $1+m.10^{-n}$, $m=1(1)99$, $n=3(2)9$	Leonelli 1803; Hoüel 1901
12 dec.	$1(1)9$; $1+m.10^{-n}$, $m=1(1)99$, $n=2(2)12$	Gray 1848

See also Lorán 1934 (13 decimals, mainly 2-figure groups).

6·23. Single-Figure Radix Tables

These are rather numerous, and naturally will not be used when radix tables with 2 to 4 figures are available. Consequently we confine ourselves to six tables giving 27 or more decimals. See Henderson 1926 for smaller ones.

61 dec.	$1(1)100(\text{primes})1097$; $1\pm m.10^{-n}$, $m=1(1)9$, $n=3(1)6$	**Peters & Stein 1922**
52 dec.	$1(1)9$; $1\pm m.10^{-n}$, $m=1(1)9$, $n=1(1)26$	**Peters & Stein 1919**
37 dec.	$1\pm m.10^{-n}$, $m=1(1)9$, $n=2(1)19$	**Duarte 1933**
36 dec.	$1(1)1000$	**Duarte 1933**
33 dec.	$1(1)9$; $1+m.10^{-n}$, $m=1(1)9$, $n=1(1)17$	**Duarte 1927**
28 dec.	$1(1)9$; $1\pm m.10^{-n}$, $m=1(1)9$, $n=1(1)14$	**Peters & Stein 1922**
27 dec.	$1(1)110$; $1\pm m.10^{-n}$, $m=1(1)9$, $n=2(1)14$	**Thoman 1867**

6·25. Special Methods (Logarithmic)

For further details see Henderson 1926. For special methods of essentially antilogarithmic character, see Art. 6·55.

12 dec.	433,300(1)434,299; auxiliary tables	Namur & Mansion 1877
12 dec.	100(1)1000; auxiliary tables	Jäderin 1914
10 dec.	1000(1)10,000	Numerov 1932*b*

For interpolation the tables give $f = M/x$, $1/f$ and $\log f$, where M = modulus and x = 1000(1)10,000, together with tables of supplementary corrections.

9 dec.	1000(1)10,000; addition logarithms	Gundelfinger & Nell 1891
6 dec.	10(1)100; auxiliary tables	Numerov 1932*a*

This paper also contains specimens of 8-decimal and 10-decimal tables on the same plan (see Numerov 1932*b* above), with explanation in Russian. The 6-decimal table is reproduced in Komendantov 1932 (34–36).

6·3. Many-Decimal Logarithms of Low Integers and Primes

230 dec.	2, 3, 5, 7, 17	Uhler 1938
110 dec.	71, 113	Uhler 1938
102 dec.	1(1)109	Parkhurst 1889

On errors see Uhler, *M.T.A.C.*, **1**, 121–122, 1943.

84 dec.	Primes to 113 (Calculated by Grimpen)	**Peters & Stein 1922**
61 dec.	1(1)100(primes)1097	Sharp 1717

Reprinted in various collections (see Henderson 1926); most recently in Peters & Stein 1922.

61 dec.	113(primes)1129 (Mainly from Sharp via Callet)	Parkhurst 1889
50 dec.	1(1)222	Byrne 1849
36 dec.	1(1)1000(primes)10,007	Duarte 1933
36 dec.	1(1)1100	Hobert & Ideler 1799

Degen 1824 gives to 30 decimals the first 9 multiples of the logarithms of integers up to 29, except 8, 9, 16, 25, 27 (themselves powers) and 10, 20 (trivial).

6·4. Fundamental Tables of Antilogarithms

We include here tables with 4-figure or 5-figure arguments giving antilogarithms to 11 or more figures. For many-figure tables with fewer than 10,000 arguments, see Art. 6·6.

14 fig.	0(·00001)·0335(·0001)1		Deprez 1939 (1–100, 133–166)
13; 12 fig.	0(·0001)·6999; ·7000(·0001)1		Guillemin 1912
11 fig.	0(·0001)1		Börgen 1908
11 fig.	0(·00001)1	Δ	Dodson 1742

fPP for 8-figure work are given. On errors, see Henderson 1926 (167).

6·45. Working Tables of Antilogarithms

7 fig.	0(·00001)1	IΔ, (f?)PPd	**Peters 1940*a***
7 fig.	0(·00001)1	fPP_{100}	Filipowski 1851
7 fig.	0(·00001)1	fPPd	Shortrede 1844, 1849*a*

7 fig.	0(·0001)1	Sp.	Dietrichkeit 1903
7 fig.	0(·0001)1		Dodson 1747
6 fig.	0(·0001)1	fPP	Chappell 1915
5 fig.	0(·00001)1	(No interpolation needed!)	**Scott 1870, 1912** F. W. Johnson 1930, 1933
$4\frac{1}{2}$ fig.	0(·001)·9(·0001)1	fPPd	**MT.C. 1931**
$4\frac{1}{2}$ fig.	0(·001)·9(·0001)1	fPPd	Desvallées 1920

4-figure antilogarithms of 0(·0001)1 are given in Hannyngton 1912, Schubert 1913 and F. W. Johnson 1930, 1933.

6·5. Antilogarithmic Radix Tables

Since $10^{\cdot 3271\ldots} = 10^{\cdot 3}10^{\cdot 02}10^{\cdot 007}10^{\cdot 0001}\ldots$, the antilogarithm of ·3271 ... may be found by multiplying together the antilogarithms of ·3, ·02, ·007, ·0001, ..., and similarly for other numbers. Hence the usefulness of antilogarithmic radixes giving the antilogarithms of $m.10^{-n}$, where $m = 1(1)9$, $n =$ all positive integers up to a convenient limit. The extensive tables of Deprez 1939 use groups of 3 (and 4) figures.

Antilogarithmic radixes can be used to obtain logarithms. For logarithmic radixes, which can be used to obtain antilogarithms, see Arts. 6·2 to 6·23. In general, logarithmic radixes are more useful than antilogarithmic.

14 fig.	0(·0001)1; $m.10^{-n}$, $m = 1(1)999$, $n = 7(3)13$	Deprez 1939
16 fig.	$m.10^{-n}$, $m = 1(1)9$, $n = 1(1)17$	Gernerth 1866
11 fig.	$m.10^{-n}$, $m = 1(1)9$, $n = 1(1)8$	Hoüel 1901
10 fig.	$m.10^{-n}$, $m = 1(1)9$, $n = 1(1)8$	Long 1714

6·55. Special Methods (Antilogarithmic)

For further details, see Henderson 1926. For special methods of essentially logarithmic character, see Art. 6·25.

21 fig.	$m.10^{-n}$, $m = 1(1)100$, $n = 7, 10$ (Used with log. table)	**Thompson 1924–,** **Part 9 (1924)**
15 fig.	0(·001)1; addition logarithms	**Prytz 1886**
13 fig.	0(·00001)·00432 (Used with log. table)	Andoyer 1922*a*
12–13 f.	0(·0001)1; auxiliary table	**Guillemin 1912**
12 fig.	·637780(·000001)·638660	Namur & Mansion 1877
11 fig.	0(·0001)1; addition logarithms	Börgen 1908

6·6. Many-Figure Antilogarithms of fewer than 10,000 Arguments

30 fig.	0(·01)1		Thompson (MS.)
20 fig.	0(·00001)·00179	Δ^3	Callet 1783, 1795

Gardiner 1742, plus 40 more values.

20 fig.	0(·00001)·00139	Δ^3	Gardiner 1742

Reproduced in 1785 edition of Hutton, and extended to ·00149 in 1811 edition.

16 fig.	0(·01)1		**Andoyer 1922*b***
15 fig.	0(·001)1		Prytz 1886
15 fig.	0(·01)1		Dietrichkeit 1903 (54)
13 fig.	0(·00001)·00432	Δ	Andoyer 1922*a*

6·7. Addition and Subtraction Logarithms

Let a and b be two numbers whose logarithms are known, a being the greater, and let $D = \log a - \log b = \log (a/b)$. Then since

$$\log (a \pm b) = \log a + \log (1 \pm b/a)$$

$\log (a \pm b)$ may be found, without finding a and b, if we have tables of $A = \log (1 + 1/x)$ and $S = -\log (1 - 1/x)$ with argument $D = \log x > 0$, by the formulæ

$$\log (a+b) = \log a + A \qquad \text{and} \qquad \log (a-b) = \log a - S$$

The quantities $A = \log (1 + 10^{-D})$ and $S = -\log (1 - 10^{-D})$ are addition and subtraction logarithms respectively, of the type now usually preferred and associated with the name of Zech. They have the advantage that D, A and S are all positive. It is to be noticed that the functional relationship between S and D is reciprocal, since $10^{-D} + 10^{-S} = 1$, and that D equals S when each equals $\log 2 = \cdot 3010 \ldots$ It is now usual to tabulate $S(D)$ for $D > \log 2$, and to use the table with argument and function interchanged when $D < \log 2$. The letters D, A, S are those used in the fundamental tables of Andoyer, the excellent working tables of Cohn and Becker, and the critical tables of Berkeley.

The above Zech-type logarithms differ in both notation and substance from the original Gauss-type logarithms, which they have superseded. Gauss's notation is

$$A = \log x \qquad B = \log (1 + 1/x) \qquad C = \log (1 + x) \qquad C = A + B$$

and he tabulates $B = \log (1 + 10^{-A})$ and $C = \log (1 + 10^{A})$ with argument $A \geqslant 0$. If $A = \log (a/b)$, this gives

$$\log (a+b) = \log a + B = \log b + C$$

For subtraction, use may be made of the fact that B and C are connected by the same reciprocal relation as D and S above, namely $10^{-B} + 10^{-C} = 1$. Alternatively A may be used as a subtraction logarithm, but in any case either B or C must be used as argument. Thus Gauss provided a direct tabulation of two addition logarithms, which might be called the small and large addition logarithms (respectively less than and greater than log 2), and no *direct* tabulation of a subtraction logarithm.

Addition and subtraction logarithms have been arranged in many ways, often differing from the Zech type only in that at least one of the arguments or functions is allowed to be negative. The Zech type is now so well established that we have not thought it worth while to describe all variants in detail. Art. 6·8 deals with the Zech type, and Art. 6·85 with all other types. The modern notation is used throughout, the A, B, C of Gauss being transliterated as D, A, $A+D$.

6·8. Zech-Type Addition and Subtraction Logarithms

16 d. A, S $D = 0(\cdot 01)9$ **Andoyer 1922b**

Also $S + \log D$ for $D = 0(\cdot 01)\cdot 6$, as aid to interpolation.

15 d. A $D = 2\cdot 64(\cdot 01)7\cdot 63$ Prytz 1886

11 d. A $D = 3\cdot 638(\cdot 001)5\cdot 637$ Börgen 1908

The tables of Prytz and Börgen are given incidentally in connection with their methods of calculating logarithms and antilogarithms.

10 d. $\begin{cases} A & D = 0(\cdot 0001)6\cdot 94 \\ S & D = \cdot 3(\cdot 0001)6\cdot 94 \end{cases}$ Peters (MS.)

7 d. $\begin{cases} A & D = 0(\cdot 0001)2(\cdot 001)4(\cdot 01)6\cdot 94 & I\Delta, \text{(f?)PPd} \\ S & D = \cdot 3(\cdot 0001)2(\cdot 001)4(\cdot 01)6\cdot 94 & I\Delta, \text{(f?)PPd} \end{cases}$ **Peters 1940a**

7 d.	A	$D = 0(\cdot 0001)2(\cdot 001)4(\cdot 01)6(\cdot 1)6\cdot 9$	fPPd	Zech 1910 (1849)
	S	$D = 0(10^{-7})\cdot 0003(10^{-6})\cdot 05(10^{-5})\cdot 30299$	PPd	

S has only 4–6 decimals for *D* < ·01.

6 d.	A	$D = 0(\cdot 0001)1\cdot 501(\cdot 001)3\cdot 01(\cdot 01)5(\cdot 1)6$	1Δ, PPd	**Cohn 1909**
	S	$D = \cdot 3(\cdot 0001)1\cdot 501(\cdot 001)3\cdot 01(\cdot 01)5(\cdot 1)6$	1Δ, PPd	
5 d.	A	$D = 0(\cdot 001)2\cdot 01(\cdot 01)5$	1Δ, fPPd	**Becker 1928**
	S	$D = \cdot 3(\cdot 0001)\cdot 501(\cdot 001)2\cdot 01(\cdot 01)5$	1Δ, fPPd	
5 d.	A	$D = 0(\cdot 001)1\cdot 65(\cdot 01)3(\cdot 1)5$	Δ, PP	Hoüel 1921
	S	$D = \cdot 3(\cdot 0001)\cdot 48(\cdot 001)1\cdot 5(\cdot 01)3\cdot 1(\cdot 05)$ $3\cdot 2(\cdot 1)5$	Δ, PP	
5 d. critical		Range of *D* for every value of *A* or *S* from ·30103 by interval ·00001 down to ·00001		Berkeley 1930
4½ d.	A	$D = 0(\cdot 001)\cdot 9(\cdot 01)2(\cdot 1)3\cdot 9$	Δ, fPPd	**MT.C. 1931**
	S	$D = \cdot 2(\cdot 001)1(\cdot 01)3(\cdot 1)3\cdot 9$	Δ, (fPPd)	
4½ d.	A	$D = 0(\cdot 001)\cdot 91(\cdot 01)2(\cdot 1)3\cdot 9$	fPPd	Desvallées 1920
	S	$D = \cdot 2(\cdot 001)1\cdot 21(\cdot 01)3(\cdot 1)3\cdot 9$	fPPd	
4 d.	A, S	See B.A. 1873 (117) for details		J. H. T. Müller 1844

It appears from Becker 1928 (page vi) that Zech published 4-decimal tables in modern form ($D > \cdot 3$ for *S*; compare Zech 1910 (1849) above). Becker gives no precise reference; it is perhaps Zech 1864.

6·85. Other Types of Addition and Subtraction Logarithms

$$D = \log x \qquad A = \log(1 + 1/x) \qquad A + D = \log(1 + x) \qquad S = -\log(1 - 1/x)$$

When *D* is preceded by an asterisk, the argument actually given in the table is $-D$, either in bar notation or in 10-added form. For example, Wittstein tabulates the small addition logarithm as $\log(1 + x)$ for $\log x = 3(\cdot 1)4(\cdot 01)6(\cdot 001)8(\cdot 0001)10$.

14 d.	$A, A+D$	$D = 0(\cdot 00001)\cdot 00103$	No Δ	Leonelli 1803

Leonelli calculated a large MS. table to 14 decimals on the lines of this specimen.

9 d.	A	$^*D = 3(\cdot 0005)3\cdot 3(\cdot 001)5\cdot 2(\cdot 01)$ $7(\cdot 1)9$	Δ or $\Delta/5$	Gundelfinger & Nell 1891

Given incidentally in connection with a logarithmic method.

7 d.	A	$^*D = 0(\cdot 0001)2(\cdot 001)4(\cdot 01)6(\cdot 1)7$	fPPd	**Wittstein 1866**
	$A+D$	$D = 0(\cdot 0001)4(\cdot 01)6(\cdot 1)7$	PPd	

Photographically reprinted (without acknowledgement) in William W. Johnson 1943.

7 d.	$A, A+D$	$D = 0(\cdot 0001)2(\cdot 001)3(\cdot 01)4(\cdot 1)5(1)7$	PP	Matthiessen 1818
6 d.	A	$^*D = 0(\cdot 0001)1\cdot 3(\cdot 001)2\cdot 8(\cdot 01)5(\cdot 1)6$	fPPd	Bremiker 1924
	$A+D$	$D = 0(\cdot 0001)\cdot 2509$	fPPd	
	$-S$	$D = \cdot 4(\cdot 0001)1\cdot 301(\cdot 001)2\cdot 81(\cdot 01)$ $5\cdot 1(\cdot 1)6$	fPPd	
6 d.	A	$^*D = 0(\cdot 001)2\cdot 8(\cdot 01)5(\cdot 1)6$	PP	G. W. Jones 1893
	$A+D$	$D = 0(\cdot 0001)\cdot 1999$	fPP	
	$-S$	$D = \cdot 4(\cdot 0001)\cdot 5(\cdot 001)2\cdot 8(\cdot 01)5(\cdot 1)5\cdot 9$	PP	
6 d.	A	$D = 0(\cdot 001)2(\cdot 01)4(\cdot 1)6$	1Δ	Gundelfinger 1902
	$A+D$	$D = 0(\cdot 001)2$	1Δ	

6 d.	$A+D$	$D = 0(\cdot 0001)2$	fPP_{100}	Gray 1849,
	$-S$	$^*D = \cdot 1001(\cdot 0001)1(\cdot 001)3$	$\mathrm{I}\Delta$, fPP	1870
6 d.	A	$^*D = 0(\cdot 0001)1;\ 1(\cdot 001)3$	fPP_{100}; $\mathrm{I}\Delta$	Gray 1870
5 d.	A	$^*D = 0(\cdot 001)2(\cdot 01)4(\cdot 1)5$	fPPd	Bremiker 1937
	$A+D$	$D = 0(\cdot 001)\cdot 219$	fPPd	
	$-S$	$D = \cdot 42(\cdot 001)2\cdot 01(\cdot 01)3\cdot 1(\cdot 1)5$	fPPd	
5 d.	A	$^*D = 0(\cdot 001)2(\cdot 01)4(\cdot 1)5$	$\mathrm{I}\Delta$, fPPd	Müller-Rajna 1935
	S	$^*D = 0(\cdot 00001)\cdot 051(\cdot 0001)\cdot 4$	$(\mathrm{I}\Delta)$, PPd	
		$(\cdot 001)1\cdot 8(\cdot 01)4(\cdot 1)5$		
5 d.	$A, A+D$	$D = 0(\cdot 001)2(\cdot 01)3\cdot 4(\cdot 1)5$	Δ	Gauss 1812

This table by Gauss has been widely reproduced.

For further references see B.A. 1873, Mehmke 1902 (998) and Mehmke–d'Ocagne 1909 (309).

6·9. Weidenbach's Logarithm

A table of $W = \log \dfrac{10^D + 1}{10^D - 1} = \log \dfrac{1 + 10^{-D}}{1 - 10^{-D}}$ with argument D may be regarded as a table of $\log \dfrac{x+1}{x-1}$ with argument $\log x$ or as a table of $\log \dfrac{1+x}{1-x}$ with argument $-\log x$, and may be useful in connection with formulæ such as $\tan \dfrac{A-B}{2} = \dfrac{a-b}{a+b} \cot \dfrac{C}{2}$. Notice that the functional relationship between D and W is reciprocal, since $10^{D+W} = 10^D + 10^W + 1$. Argument and function are equal when each is $\log(1+\sqrt{2})$, or $\cdot 3827\ldots$ In the notation of Art. 6·7, $W = A+S$. The asterisks below have the same meaning as in Art. 6·85. Thus Hammer's table is arranged as a table of $\log z$ with argument $\log x$, where $z = \dfrac{1+x}{1-x}$ and $x = \dfrac{z-1}{z+1}$, so that his argument is always negative.

6 dec.	$^*D = \cdot 00991(\cdot 00001)\cdot 1(\cdot 0001)$	$\mathrm{I}\Delta$, PPd	**Hammer 1902**
	$1\cdot 5(\cdot 001)2\cdot 7(\cdot 01)4(\cdot 1)7$		
4–5 dec.	$^*D = 0(\cdot 00001)\cdot 0001(\cdot 000001)$		
	$\cdot 005(\cdot 00001)\cdot 01$		
5 dec.	$^*D = \cdot 00001(\cdot 00001)\cdot 0001(\cdot 0001)$	$\mathrm{I}\Delta$	**Hoüel 1901**
	$\cdot 001(\cdot 001)1(\cdot 01)3(\cdot 1)5$		

Only 3 decimals up to ·001 and then 4 decimals up to ·05.

5 dec.	$D = \cdot 382(\cdot 001)2\cdot 002,\ 2(\cdot 01)3\cdot 6(\cdot 1)5\cdot 5$	Δ	Weidenbach 1829

Other 5-decimal tables in Prasse–Mollweide–Jahn 1837 and Rex 1884*a*.

4½ dec.	$^*D = \cdot 001(\cdot 001)1(\cdot 01)2(\cdot 1)3$	No Δ	Foxell 1934

SECTION 7

NATURAL TRIGONOMETRICAL FUNCTIONS. MISCELLANEOUS FUNCTIONS CONNECTED WITH THE CIRCLE AND THE SPHERE

7·0. Introduction

This section deals chiefly with tables of the natural values of the trigonometrical functions. Details are also given in Arts. 7·8–7·87 of a few tables, more of the nature of mensuration tables, which have come to our notice in the course of preparing the main articles.

In the case of the six staple trigonometrical functions we have considered separately tables with arguments in (1) radians, (2) degrees and decimals, (3) grades (hundredths of a right angle) and decimals, (4) degrees, minutes and seconds, (5) time. The arguments (4) and (5) are inconvenient for general mathematical purposes, particularly when calculating machines are used, but they are required in astronomy and navigation.

The discovery of logarithms early in the seventeenth century greatly retarded the development of tables of natural trigonometrical functions, but the increasing use of calculating machines in the last few decades has provided a new stimulus. In most cases adequate tables are now available. Attention may, however, be called to the fact that when many-figure tables in degrees and decimals are needed, nothing at all comparable with the classical and inaccessible tables of Briggs & Gellibrand (1633) is now purchasable; while tables with more than 10 figures, argument in grades, and interval less than $0^{g}{\cdot}1$, are represented solely by two separate calculations, by Prony and by Sang, which have remained in manuscript.

When the argument is in radians, a separate tabulation of all six trigonometrical functions is useful. In other cases the tabulation is essentially of three functions, $\sin x$, $\tan x$ and $\sec x$, since $\cos x$, $\cot x$ and $\operatorname{cosec} x$ may be conveniently obtained from the same tables if the complementary argument ($90° - x$, or its equivalent in other units) is printed.

The best tables are arranged semi-quadrantally, i.e. with argument running from 0 to 45° (or its equivalent), the complementary argument being also printed. This arrangement is adopted so that the sine and cosine of an angle, which are frequently both needed, may be given on the same line. But we shall index a table of $\sin x$ and $\cos x$, where x goes from 0 to 45°, as a table of $\sin x$, where x goes from 0 to 90°; and similarly in the cases of the other functions.

Tables of sines normally give a fixed number of decimals, except that sines of small angles are sometimes given also to a fixed number of significant figures. Tables of $\tan x$ and $\sec x$ present difficulties in description. Some tables give a fixed number of decimals or a fixed number of figures throughout, but neither of these systems is always desirable. Many tables have neither a fixed number of decimals nor a fixed number of figures for values of x approaching 90°, while the first (and usually most important) decrease in the number of decimals in $\tan x$ occurs anywhere at or after 45°. There appears to be need of some norm, and we venture to make the following remarks.

With respect to $\tan x$, some tables appear to us to decrease the number of decimals at too early a stage. An angle is in principle always well determined by its tangent, and particularly so when the tangent is large. It therefore seems very undesirable that the "resolving power" of the table should be decreased by too early a dropping

of decimals. If the first change is postponed until $\sec^2 x = 10$, or $x = 71^\circ\cdot58$, so that the derivative of $\tan x$ has 10 times its initial ($x = 0$) value, and thereafter a fixed number of significant figures is given, the first difference will never be smaller than its initial value. This method of treatment need not usually cause second differences to become appreciable during the fixed-decimal stage, for in some typical cases the second differences (in units of the last place) at 72° are as follows:

Dec.	Interval	Δ^2
8	0°·001	2·0
7	10″	1·5
6	0°·01	2·0
5	1′	0·5
4	0°·1 = 6′	2·0

The intervals are not less than those normally adopted for the corresponding numbers of decimals in the class of table with which this *Index* is primarily concerned.

If an angle has to be determined from a decidedly large tangent, say about 10, the computer may often prefer to evaluate the reciprocal and use the early part of the table of tangents, but it seems undesirable that the table used should demand this procedure at an unnecessarily early stage.

If the determination of an angle by its tangent were the only use of a table of tangents, the number of figures given could fall greatly as the angle approaches 90°; the number of figures in the tangent and its reciprocal (as given in the earlier part of the table) could be kept comparable. But the tangents may be required for other purposes, and we have suggested a fixed number of figures in the later part of the table.

Rather similar remarks apply to secants, except that an angle is not in principle always well determined by its secant. It seems desirable that secants and cosines of small angles should be given to the same number of decimals (not figures). The critical point for the first decrease in the number of decimals occurs when $\sec x \tan x = 10$, or $x = 72^\circ\cdot04$. The second differences are then substantially as for $\tan x$.

We therefore suggest as a canonical form of n-figure table of natural trigonometrical functions:

(1) $\sin x$ to n decimals throughout;
(2) $\tan x$ to n decimals up to at least 72° or 80^g, then to n figures;
(3) $\sec x$ to n decimals (i.e. $n+1$ figures) up to at least 72° or 80^g, then to n figures.

In our *Index* we have given detailed information about tables of tangents and secants which are not to either a fixed number of decimals or a fixed number of figures, whenever the tables have been available for inspection. Thus we learn from Arts. 7·22 and 7·23 that the excellent Lohse 1935 gives both tangents and secants to 5 decimals up to 82° and then to strictly 4 decimals up to 90°, while Carey & Grace 1927 gives tangents to 4 decimals up to 84°·99 and then to strictly 5 figures up to 90°. This kind of information is, however, often not derivable from published accounts of tables; in dealing with tables which we have not seen, and about which our information is inadequate for a better description, we have used the notation "n:" to indicate that the table is supposed in a general way to be an "n-figure" table, so that up to at least 45° the tangents are almost certainly to n decimals and the secants are probably to n decimals (but may be to $n-1$ decimals). See, for instance, Enberg & Långström 1925 in Arts. 7·32 and 7·33.

In giving information about provision for interpolation, we have used the letters σ and τ to signify that tables of the auxiliary functions $x \operatorname{cosec} x$ and $x \cot x$ respectively are provided. Details of the tabulation of these are given in Art. 7·61.

Proportional parts in sexagesimal tables may be arranged in various ways, and we have not always thought it worth while to go into details, so that it is not to be assumed in these tables that PP without numerical suffix necessarily means proportional parts for tenths of the interval.

Most authors tabulate tangents as well as sines, but tables of secants are not always given, and we have sometimes indexed such a table when the same author's sines and tangents appeared not to require mention.

For tables of sin nx, see Harmonic Analysis (Section 24). Other tables with two variables, such as traverse tables and tables of quantities like sin x tan y, are indexed in Section 9.

The principal divisions of the present section are as follows:

- 7·0 Introduction
- 7·1 Tables with Argument in Radians
- 7·2 Tables with Argument in Degrees and Decimals
- 7·3 Tables with Centesimal Argument
- 7·4 Tables with Argument in Degrees, Minutes, etc.
- 7·5 Tables with Argument in Time
- 7·6 Auxiliary Functions (mainly for Small Angles), All Arguments
- 7·7 Versed Sines, Haversines, etc., All Arguments
- 7·8 Miscellaneous Tables connected with the Circle and the Sphere
- 7·9 Miscellaneous Expressions involving Circular Functions

7·1. Tables with Argument in Radians

These have been published almost entirely during the present century (but see Hutton in Art. 7·11). For fundamental quantities such as sin 1 and cos 1, see Section 5.

7·11. Sin x and cos x (Radians)

Differences are unnecessary in most cases. Each function provides the derivative of the other, and use may be made of this in interpolation.

25 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 4(1)10$	**Van Orstrand 1921**
23 dec.	$x = 0(\cdot 001)1\cdot 6(\cdot 1)10(1)100$	
20–25 d.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 6(1)10$	Hayashi 1926 (4–12)
20; 18; 15 d.	$x = 0(\cdot 00001)\cdot 001$; $3(\cdot 01)9\cdot 99$; $10(\cdot 1)20(1)100$	Hayashi 1926 (4–220)
20 dec.	$x = 1(1)10$	Bretschneider 1843
15 dec.	$x = 0(\cdot 1)50$	**B.A. 1, 1931**

Cos 26·1, for ·56756 read ·56755; sin 47·6, for ·468 read ·458 (Comrie).

15 dec.	$x = 1(1)100$	**B.A. 1923**
15 dec.	$x = 0(\cdot 1)10$; $10(\cdot 1)20(\cdot 5)50$; $20(\cdot 2)40$	B.A. 1916 (Doodson); 1924; 1928
15 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 1(1)5$	New York W.P.A. 1940*b*
13 dec.	$x = 0(\cdot 0001)\cdot 8$	Δ Airey (MS.)

See Comrie, *M.T.A.C.*, **1**, 59–60, 1943.

12 dec.	$x = 0(\cdot 00001)\cdot 00999$	New York W.P.A. 1940*b*
12 dec.	$x = 0(\cdot 001)2$	Sang (MS. 1848–90)
10; 12 d.	$x = \cdot 001(\cdot 0001)\cdot 0999$; $\cdot 1(\cdot 001)2\cdot 999$	Hayashi 1926
11 dec.	$x = 0(\cdot 001)1\cdot 6$	**B.A. 1916** (Airey); **B.A. 1, 1931**

10 dec.	$x = 0(\cdot 1)10(1)100$		**Becker & V. O. 1924**
10 dec.	$x = 0(\cdot 0001)\cdot 01;\ 10(1)40$		Hayashi 1930*b*
9 dec.	$x = 0(\cdot 0001)2;\ 0(\cdot 1)10$		**New York W.P.A. 1940*a***
8 dec.	$x = 0(\cdot 001)25(1)100$		**New York W.P.A. 1940*b***
7 dec.	$x = 0(\cdot 00001)\pi/2$		Hutton (MS. 1783)
6 dec.	$x = 0(\cdot 001)1\cdot 609$	IΔ, fPPd	Burrau 1907
6 dec.	$x = 0(\cdot 01)2$	Δ	Ligowski 1890
	Computed by H. Zimmermann.		
5 dec.	$x = 0(\cdot 0001)\cdot 1(\cdot 001)1\cdot 6$	v	**Becker & V. O. 1924**
5 dec.	$x = 0(\cdot 001)1\cdot 569$	PPMD	Meyer & Deckert 1924
5 dec.	$x = 0(\cdot 01)10$		Hayashi 1930*b*
5 dec.	$x = 0(\cdot 0001)\cdot 1(\cdot 001)3(\cdot 01)6\cdot 3(\cdot 1)10$		Hayashi 1921
5 dec.	$x = \cdot 5(\cdot 001)1\cdot 6$		Chrétien 1932
	Sin x only. See also $x - \sin x$, Art. 7·66.		
4½ dec.	$x = 0(\cdot 001)1\cdot 57;\ 0(\cdot 1)7\cdot 9$	Δ; no Δ	**MT.C. 1931**
4 dec.	$x = 0(\cdot 001)2$		Campbell 1929
4 dec.	$x = 0(\cdot 01)3\cdot 2$ (Not earlier editions)		Allen 1941
4 dec.	$x = 0(\cdot 01)1\cdot 6$		Carmichael & Smith 1931
4 dec.	$x = 0(\cdot 01)1\cdot 58$		Russell 1916
4 dec.	$x = 0(\cdot 01)1\cdot 57$		Hall 1905
4 dec.	$x = \cdot 10(\cdot 01)1\cdot 57$		Knott 1905

7·12. Tan x (Radians)

20 dec.	$0(\cdot 00001)\cdot 001$	No Δ	Hayashi 1926
10 dec.	$\cdot 001(\cdot 0001)\cdot 1(\cdot 001)3(\cdot 01)10(\cdot 1)20(1)50$		
5 dec.	$1\cdot 5680(\cdot 0001)1\cdot 5707$ (p. 88)		
10 dec.	$0(\cdot 1)10$	No Δ	**New York W.P.A. 1943*b***
8 fig.	$0(\cdot 0001)2$	(δ^2)	
Abt. 5 f.	$0(\cdot 001)1\cdot 569$	PPMD	Meyer & Deckert 1924
5 dec.	$0(\cdot 01)10$; also (p. 47) $1\cdot 56(\cdot 001)1\cdot 59$	No Δ	Hayashi 1930*b*
5 dec.	$0(\cdot 0001)\cdot 1(\cdot 001)3(\cdot 01)6\cdot 3(\cdot 1)10$	No Δ	Hayashi 1921
5 dec.	$0(\cdot 001)1$	No Δ	Chrétien 1932
Abt. 4½ f.	$0(\cdot 001)1\cdot 57$	Δ	**MT.C. 1931**
4 dec.	$0(\cdot 001)2$		Campbell 1929
4 dec.	$0(\cdot 01)3\cdot 2$ (Not earlier editions)	No Δ	Allen 1941
4 d.; 5 f.	$0(\cdot 01)1\cdot 47;\ 1\cdot 48(\cdot 01)1\cdot 57$	No Δ	Hall 1905
4; 3 dec.	$\cdot 10(\cdot 01)1\cdot 10;\ 1\cdot 10(\cdot 01)1\cdot 57$	No Δ	Knott 1905
Abt. 4 f.	$0(\cdot 01)1\cdot 6$	No Δ	Carmichael & Smith 1931

7·13. Cot x (Radians)

10 dec.	$0(\cdot 1)10$	No Δ	**New York W.P.A. 1943*b***
8 fig.	$0(\cdot 0001)2$	(δ^2)	
Abt. 4½ f.	$0(\cdot 0001)\cdot 04(\cdot 001)1\cdot 57$	Δ	**MT.C. 1931**
4 dec.	$0(\cdot 01)3\cdot 2$ (Not earlier editions)	No Δ	Allen 1941
3; 4 dec.	$0(\cdot 01)\cdot 78;\ \cdot 79(\cdot 01)1\cdot 6$	No Δ	Carmichael & Smith 1931

7·14. Sec x (Radians)

5½; 4½ f.	$0(\cdot 001)1\cdot 25;\ 1\cdot 25(\cdot 001)1\cdot 57$	Δ	**MT.C. 1931**
5 dec.	$1\cdot 56(\cdot 001)1\cdot 59$	No Δ	Hayashi 1930*b* (47)

7·15. Cosec x (Radians)

4½; 5½ f.	0(·0001)·04(·001)·3; ·3(·001)1·57	Δ	**MT.C. 1931**

7·2. Tables with Argument in Degrees and Decimals

Tables with this highly desirable argument are not very common. A new edition of Briggs & Gellibrand 1633 is badly needed. Peters 1918 is quadrantally arranged, and the final digit is not definitive. If Peters's 8-figure table is published, it will satisfy a real need. For a few errors in the useful Buckingham 1935, see Miller 1942 and Comrie, *M.T.A.C.*, **1**, 88–92, 1943. Adequate tables to 4, 5 and 6 figures are available.

The New York W.P.A. workers had under consideration in May 1942 a table of circular functions to 15 decimals at interval 0°·001. See also Neville and Vogel in Part II.

7·21. Sin x (Degrees and Decimals)

30 dec.	1°	No Δ	Herrmann 1848*b*
19 dec.	0°·625	Δ	**Briggs & Gellibrand 1633**
15 dec.	0°·01	Δ	

Error at 64°·49 (Peters 1918).

10 dec.	0°·01	Δ	Engel 1920

Second-difference corrections for every 0°·0001 are given on each page, using the mean second difference for the half-degree concerned.

9; 8 dec.	0(0°·01)0°·06; 0°·06(0°·01)0°·58	No Δ	Peters 1918
8 dec.	0°·01	Δ	Buckingham 1935
8 dec.	1°	Sp.	Herget 1933

A one-page table for finding sines to 8 decimals in machine computation.

8 dec.*	0°·001 (See Comrie 1929, p. 39)		Peters (MS.)
7 dec.	0°·001	fPPd	**Peters 1918**

Maximum error 0·65 final units.

7 dec.	0°·01	No Δ	Oughtred 1657
6 dec.	0°·01	Δ	**Peters 1937**
5 dec.	0°·01	Δ	**Lohse 1935**
5 dec.	0°·01	IΔ	Dwight 1941
5 dec.	0°·01	No Δ	De Lella 1934
4½ dec.	0°·1	Δ	**MT.C. 1931**
4 dec.	0°·1	PPMD	Carey & Grace 1927
4 dec.	0°·1	Δ, fPPd	Bremiker 1937, 1907

7·22. Tan x (Degrees and Decimals)

30 dec.	1°	No Δ	Herrmann 1848*b*
10 dec.	0°·01	Δ	**Briggs & Gellibrand 1633**

Errors at 6°·24 and 36°·00 (Peters 1918) and 19°·29 (Bauschinger & Peters 1936).

10 dec.	0(0°·01)45°	Δ	Engel 1920

Second-difference corrections for every 0°·0001 are given on each page, as for the sines.

9; 8 dec.	0(0°·01)0°·06; 0°·06(0°·01)0°·58	No Δ	Peters 1918
8 dec.; 8 fig.	0(0°·01)45°; 45°(0°·01)90° *	Δ	Buckingham 1935
"8 fig."	0°·001 (See Comrie 1929, p. 39)		Peters (MS.)
7 d.; 6 d.; 6–7 fig.	0(0°·001)45°; 45°(0°·001)84°; 84°(0°·001)90°	PPd (f to 70°)	**Peters 1918**

* Maximum error 0·85 final units. Sometimes only 4 or 5 figures after 89°·42.

7 dec.	0°·01	No Δ	Oughtred 1657
6 d.; 5 d.; 4–6 fig.	0(0°·01)65°; 65°(0°·01)82°; 82°(0°·01)90°	τ, Δ	**Peters 1937**

The discarding of decimals in large tangents is done in such a way that $\Delta < 1000$ up to 89°·76.

5 d.; 4 d.	0(0°·01)82°; 82°(0°·01)90°	τ, Δ to 87°	**Lohse 1935**
5 d.; 4 d.; 4–6 fig.	0(0°·01)45°; 45°(0°·01)83°; 83°(0°·01)90°	IΔ	Dwight 1941
5 d.; 5 f.	0(0°·01)45°; 45°(0°·01)90°	No Δ	De Lella 1934
4½ d.; 4½ f.	0(0°·1)70°; 70°(0°·1)86°(0°·01)90°	Δ	**MT.C. 1931**
4 d.; 5 f.	0(0°·1)80°(0°·01)84°·99; 85°(0°·01)90°	PPMD to 80°	Carey & Grace 1927
4 dec.	0°·1	Δ, PPd	Bremiker 1937, 1907

7·23. Sec *x* (Degrees and Decimals)

10 dec.	0°·01	Δ	**Briggs & Gellibrand 1633**
7 dec.	0°·01	No Δ	Oughtred 1657
6 d.; 5 d.; 4–6 fig.	0(0°·01)45°; 45°(0°·01)82°; 82°(0°·01)90°	σ, Δ	**Peters 1937**

$\Delta < 1000$ up to 89°·76; see Art. 7·22.

5 d.; 4 d.	0(0°·01)82°; 82°(0°·01)90°	σ, Δ to 87°	**Lohse 1935**
5 d.; 4 d.; 4–6 fig.	0(0°·01)65°; 65°(0°·01)83°; 83°(0°·01)90°	IΔ	Dwight 1941
4½ d.; 4½ f.	0(0°·1)70°; 70°(0°·1)86°(0°·01)90°	Δ	**MT.C. 1931**

7·3. Tables with Centesimal Argument

Tables with argument in grades and decimals, or in decimals of the quadrant ($1^q = 100^g$), are indexed here, and all are treated as though the argument were expressed in grades. It may be noted that $1^g = 54'$, $0^g{\cdot}1 = 324''$, $0^g{\cdot}01 = 32''{\cdot}4$, $0^g{\cdot}001 = 3''{\cdot}24$. The last is the smallest interval so far used with centesimal argument, and no smaller interval appears necessary. Grades have found little favour in this country, but their use is increasing elsewhere, and in some connections they have great advantages. Further information about tables in grades is given by Archibald, *M.T.A.C.*, **1**, 33–44, 1943. Peters has manuscript tables to 7, 8 and 10 decimals at intervals $0^g{\cdot}001$, $0^g{\cdot}001$ and $0^g{\cdot}01$ respectively.

7·31. Sin *x* (Grades)

33 dec.	$0^g{\cdot}05$	Δ^2	Sang (MS. 1848–90)
25 dec.	1^g	Δ^8	Prony (MS. 1794)
22 dec.	$0^g{\cdot}01$	Δ^5	Prony (MS. 1794)
20 dec.	1^g	v^8	**Andoyer 1915**

Reproduced in Roussilhe & Brandicourt 1933.

15 dec.	$0^g.01$	Δ^2	Sang (MS. 1848–90)
15 dec.	$0^g.1$	No Δ	Callet 1795
10 dec.	$0(0^g.1)100^g$; $0(0^g.001)0^g.1$	No Δ; PP	Steinbrenner 1914
10 dec.	$0^g.1$	No Δ	Hayashi 1926
10 dec.	1^g; also $0(0^g.01)0^g.5$	No Δ	Houël 1901
8 dec.	$0^g.01$	Δ	**Roussilhe & Brandicourt 1933**
7 dec.	$0(0^g.001)3^g(0^g.01)97^g(0^g.001)100^g$	Δ	Hobert & Ideler 1799

There is also a 10-decimal table for the first grade only.

6 dec.	$0^g.001$	fPPd	**Peters 1938**
6 dec.	$0(0^g.001)3^g(0^g.01)97^g(0^g.001)100^g$		Enberg & Långström 1925
6 dec.	$0^g.01$ (Sexages. argument also shown)	fPP	Clouth 1906
6; 5; 6 dec.	$0(0^g.01)7^g$; $7^g(0^g.01)93^g$; $93^g(0^g.01)100^g$	No Δ	Potin 1925
5 dec.	$0^g.01$	fPPd	Wijdenes 1937
5 dec.	$0^g.01$		Balzer & Dettwiler 1919
5 dec.	$0^g.01$	Δ, fPPd	Steinbrenner 1914
5 dec.	$0^g.1$	No Δ	Hayashi 1930*b*
4 dec.	$0^g.1$	Δ, fPPd	F. G. Gauss 1937
4 dec.	$0^g.1$	Δ	Houël 1901
4 dec.	$0^g.1$		Gravelius 1886

7·32. Tan *x* (Grades)

20 dec.	$0(1^g)100^g$	v^{12} up to 50^g	**Andoyer 1915**

Reproduced in Roussilhe & Brandicourt 1933.

10 dec.	1^g; also $0(0^g.01)0^g.5$	No Δ	Houël 1901
8 dec.	$0(0^g.01)50^g$	Δ	**Roussilhe & Brandicourt 1933**
7 dec.	$0(0^g.001)3^g(0^g.01)97^g(0^g.001)100^g$	Δ	Hobert & Ideler 1799

There is also a 10-decimal table for the first grade only.

6 d.; 6 f.	$0(0^g.001)74^g$; $74^g(0^g.001)100^g$	τ, fPPd	**Peters 1938**
6:	$0(0^g.001)3^g(0^g.01)97^g(0^g.001)100^g$		Enberg & Långström 1925
6; 5; 4 d.	$0(0^g.01)7^g$; $7^g(0^g.01)99^g.93$; $99^g.94(0^g.01)100^g$	No Δ	Potin 1925 (278)
5 dec.	$0^g.01$	PPd	Wijdenes 1937
5 d.; 5 f.; 2 d.	$0(0^g.01)80^g$; $80^g(0^g.01)97^g$; $97^g(0^g.01)100^g$		Balzer & Dettwiler 1919
5 d.; 5–6 f.	$0(0^g.01)85^g$; $85^g(0^g.01)97^g$(var.)100^g	Δ, PPd	Steinbrenner 1914
4 dec.	$0^g.1$	Δ, PPd	F. G. Gauss 1937
4 d.; 4–5 f.	$0(0^g.1)80^g$; $80^g(0^g.1)100^g$	Δ	Houël 1901
4:	$0^g.1$		Gravelius 1886

7·33. Sec *x* (Grades)

20 dec.	$0(1^g)100^g$	v^{12} up to 50^g	**Andoyer 1915**

Reproduced in Roussilhe & Brandicourt 1933.

6:	$0(0^g.001)3^g(0^g.01)97^g(0^g.001)100^g$		Enberg & Långström 1925
4–6 dec.	$0(0^g.01)100^g$	No Δ	Potin 1925 (278, 496)
4 d.; 4–5 f.	$0(0^g.1)80^g$; $80^g(0^g.1)100^g$	Δ	Houël 1901

7·4. Tables with Argument in Degrees, Minutes, etc.

The classical work of Rheticus and Pitiscus with this traditional argument has been superseded by the fundamental tables of Andoyer. Peters 1939, Brandenburg 1931 (not the first edition, which according to Comrie 1932*a* contains many errors) and Peters 1929 (or Brandenburg 1932) are the standard tables to 8, 7 and 6 figures respectively.

7·41. Sin x (Degrees, Minutes, etc.)

33 dec.	sin and cos of $0(1'')100''(100'')1000''(1000'')45°$		**Van Orstrand & Shoultes 1922**
30 dec.	$1°$	No Δ	Herrmann 1848*b*
22 dec.	sin and cos of $10''(20'')34'\ 50''$	Δ^4 or Δ^5	Pitiscus 1613
21 dec.	$10'$	No Δ	**Peters 1911*a***
21 dec.	sin and cos of $0(1'')10'$	Δ^3	
17 dec.	$9'$	$v^6(10'')$, $0(18')30°$, $60°(18')90°$	**Andoyer 1915**
15 dec.	$10''$	Δ	
15 dec.	$10''$; $0(1'')1°$, $89°(1'')90°$	Δ^3; Δ^2	Pitiscus 1613 (Rheticus)
15 dec.	$15'$	No Δ	Legendre 1816, 1826 (T. 3)
10 dec.	$10''$	Δ	Rheticus 1596
10 fig.	$0(1'')12'$	Δ	Vega 1794

8 dec.	$1''$	$I\Delta$	**Peters 1939**
8 dec.	$1''$	No Δ	U.S. C. & G.S. 231, 1942
8 dec.	$1''$	PPdMD	Gifford 1914
8 dec.	$1'$	$\Delta/60$	Hudson & Mills 1941

On errors see *M.T.A.C.*, **1**, 86–87, 1943.

7 dec.	$0(1'')45°$; $45°(1'')90°$	No Δ	Hof 1943*a*, *b*
7 dec.	$10''$	Δ, fPPd	**Brandenburg 1931**
7 fig.	$0(10'')1°$	Δ, PPd	
7 dec.	$10''$	Δ, fPPd	Ives 1931
7 dec.	$10''$	No Δ	Benson 1927
7 dec.	$10''$	Δ	Jordan 1897
7 dec.	$10''$	$(I\Delta)$, PPd	Neill 1927
7 dec.	$10''$	PP	Jurisch 1884
7 dec.	$1'$	Δ	Pryde 1930

6 dec.	$10''$	Δ, fPPd	**Peters 1929**
6 dec.	$10''$	Δ, fPPd	**Brandenburg 1932**
6 fig.	$0(10'')1°$	Δ, fPPd	

Other 6-decimal $10''$ tables are Junge 1864 and Kerigan 1821 (see B.A. 1873).

6 dec.	$1'$	$I\Delta$ for $100''$	Norie 1894, 1904
6 dec.	$1'$	No Δ	Allen 1941

5 dec.	$10''$	Maximum $\Delta = 5$	**N.A.O. 1941**
5 dec.	$1'$	Δ	Gonggrijp 1916
5 fig.	$0(1')6°$	No Δ	
5 dec.	$1'$	Δ, fPPd	Carmichael & Smith 1931; F. G. Gauss 1938

5 dec.	1′	Δ, fPP$_{12}$d	Wolfe & Phelps 1942
5 dec.	1′	Δ/60	Chappell 1915
5 dec.	1′	fPPd	Ist. Idrog. 1916
5 dec.	1′	No Δ	Beetle 1933, Ives 1934 Hedrick 1920
$4\frac{1}{2}$ dec.	1′	Maximum Δ = 3	**MT.C. 1931**

7·42. Tan *x* (Degrees, Minutes, etc.)

30 dec.	1°	No Δ	Herrmann 1848*b*
17 dec.	9′	v^7(5″), 0(9′)30°	Andoyer 1915
15 dec.	10″	Δ	**Andoyer 1916**
10 dec.	10″	Δ	Rheticus 1596
11 f.; 10 d.	0(10′)84° 10′; 84° 20′(10′)90°	No Δ	Börgen 1908
8 d.; 8 f.; 5–7 fig.	0(1″)71° 30′; 71° 30′(1″)88° 12′; 88° 12′(1″)90°	τ, IΔ	**Peters 1939**
8; 7 dec.	0(1″)89° 25′ 37″; 89° 25′ 38″(1″)90°	PPMD	Gifford 1927
8 dec.	0(1′)45°	Δ/60	Hudson & Mills 1941
7 dec.	0(1″)45°	No Δ	Hof 1943*c*
7; 6–5 d.	45°(1″)84° 18′; 84° 18′(1″)90°	No Δ	Hof 1944
7 dec.	0(10″)84°; 84°(1″)90°	Δ, PPd; no Δ	**Brandenburg 1931**
7 fig.	0(10″)1°	Δ, PPd	**Brandenburg 1931**
7 d.; 8 f.	0(10″)84° 17′ 20″; 84° 17′ 30″(10″)90°	Δ, PPd	Ives 1931
7 d.; 8 f.	0(10″)84° 17′ 20″; 84° 17′ 30″(10″)90°	No Δ	Benson 1927
7 d.; 8 f.	0(1′)84° 17′; 84° 18′(1′)90°	No Δ	Pryde 1930, Ives 1929
6 d.; 6 f.	0(10″)60°; 60°(10″)90°	τ, Δ, PPd	**Peters 1929**
4–6 fig.	88° 40′(1″)90°	τ, Δ, PPd	**Peters 1929**
6 d.; 6–7 f.	0(10″)84° 17′ 20″; 84° 17′ 30″(10″)90°	Δ, PPd	**Brandenburg 1932**
6 fig.	0(10″)1°	Δ, fPPd	**Brandenburg 1932**
7 fig.	87°(1″)90°	No Δ	**Brandenburg 1932**
6 dec.	1′	No Δ	Allen 1941
5 d.; 5 f.	0(10″)63°; 63°(10″)82° 30′(1″)90°	No Δ	**N.A.O. 1941**
5 d.; 3–6 f.	0(1′)71°; 71°(1′)79°(10″)89° 55′(1″)90°	Δ, fPPd	F. G. Gauss 1938
5 dec.	1′	Δ	Gonggrijp 1916
5 fig.	0(1′)6°	No Δ	Gonggrijp 1916
5 d.; 6 f.	0(1′)85°; 85°(1′)90°	Δ/60	Chappell 1915
5 d.; 6 f.	0(1′)84° 17′; 84° 18′(1′)90°	PPd	Ist. Idrog. 1916
5 d.; 6 f.	0(1′)84° 17′; 84° 18′(1′)90°	No Δ	Ives 1934
5 d.; 5 f.	0(1′)45°; 45°(1′)90°	Δ, PPd	Carmichael & Smith 1931
5 d.; 5 f.	0(1′)45°; 45°(1′)90°	Δ, fPP$_{12}$d	Wolfe & Phelps 1942
5 d.; 5 f.	0(1′)45°; 45°(1′)90°	No Δ	Beetle 1933 Hedrick 1920
$4\frac{1}{2}$ d.; 4–5 f.	0(1′)80°; 80°(1′)90°	Δ	**MT.C. 1931**

7·43. Sec x (Degrees, Minutes, etc.)

17 dec.	9″	v^7(10″), 0(18′)30°	Andoyer 1915
15 dec.	10″	Δ	**Andoyer 1918**
10 dec.	10″	Δ	Rheticus 1596
8 d.; 10 f.	0(1″)89° 25′ 37″; 89° 25′ 38″(1″)90°	PPdMD	Gifford 1929
8 dec.	0(1′)45°	Δ/60	Hudson & Mills 1941
8 fig.	1′	No Δ	Pryde 1930, Ives 1929
6 d.; 6 f.	0(10″)60°; 60°(10″)90°	σ, Δ, PPd	**Peters 1929**
4–6 fig.	88° 40′(1″)90°	σ, Δ, PPd	**Peters 1929**
6 dec.	1′	No Δ	Allen 1941
5 dec.	1′	No Δ	C. F. W. Peters 1871
5 d.; 6 f.	0(1′)85°; 85°(1′)90°	Δ/60	Chappell 1915
5 d.; 6 f.	0(1′)84° 15′; 84° 16′(1′)90°	PPd	Ist. Idrog. 1916
5 d.; 6 f.	0(1′)84° 15′; 84° 16′(1′)90°	No Δ	Norie 1938 (369)
5 d.; 5–6 f.	0(1′)80°; 80°(1′)90°	PP	Inman 1906 (466), 1940 (474)
5 d.; 5 f.	0(3′)84° 15′; 84° 18′(3′)90°	PPMD	Dale 1903
4½ d.; 4–5 fig.	0(1′)80°; 80°(1′)90°	Δ	**MT.C. 1931**
5 fig.	1′	Δ, fPP$_{12}$d	Wolfe & Phelps 1942

7·5. Tables with Argument in Time

This argument is required in astronomy and navigation. $24^h = 360°$, $1^h = 15°$, $1^m = 15'$ and $1^s = 15''$. The tables of both N.A.O. and Comrie are semi-quadrantally arranged.

7 dec.	sin x	$0(1^s)6^h$	IΔ	**N.A.O. 1939*a*** (Comrie)
7 d.; 5–7 f.	tan x	$0(1^s)4^h$; $4^h(1^s)6^h$	τ, IΔ	**N.A.O. 1939*a*** (Comrie)
7 dec.	cos x	$0(0^s{\cdot}1)1^h$	fPPd	Schorr 1935*b*
7 dec.	tan x	$0(0^s{\cdot}1)1^h$	fPPd	Schorr 1935*a*

These two Hamburg tables were computed by W. Dieckvoss and H. Kox.

4½ dec.	sin x	$0(10^s)6^h$	Δ	**Comrie 1931**
4½ d.; 4–5 f.	tan x	$0(10^s)5^h$; $5^h(10^s)5^h\,40^m(1^s)6^h$	τ, Δ	**Comrie 1931**
4½ d.; 4–5 f.	sec x	$0(10^s)5^h$; $5^h(10^s)5^h\,40^m(1^s)6^h$	σ, Δ	**Comrie 1931**

7·6. Auxiliary Functions (mainly for Small Angles), All Arguments

7·61. Functions σ and τ

These are the standard auxiliary functions for dealing with cosecants and cotangents of small angles. They are defined by $\sigma = x \operatorname{cosec} x^{\mathrm{u}}$ and $\tau = x \cot x^{\mathrm{u}}$, where u is a unit of angular measure.

Degrees and Decimals: x cosec x° and x cot x°

6 fig. critical	To $1^\circ{\cdot}011$ and $1^\circ{\cdot}001$ respectively	Peters 1937 (182)
6 fig.	$0(0^\circ{\cdot}1)3^\circ$	Lohse 1935 (15)

Grades: x cosec x^g and x cot x^g

6 fig.	$0(0^g{\cdot}01)2^g$, x cot x^g only	Peters 1938

Both functions and their reciprocals are tabulated in Enberg & Långström 1925.

Seconds of Arc: x cosec x'' and x cot x''

8 fig.	$0(10'')2^\circ$, x cot x'' only	Peters 1939
6 fig.	$0(10'')1^\circ\,20'$	Peters 1929
6 fig. critical	To $1^\circ\,20'\,10''$ and $1^\circ\,20'\,5''$ respectively	Peters 1929 (22)

Seconds of Time: x cosec x^s and x cot x^s

8 fig.	$0(1^s)1800^s$ [$= 7^\circ\,30'$], x cot x^s only	IΔ	N.A.O. 1939*a*
5 fig. critical	To $1055^s{\cdot}5$ [$= 4^\circ\,24'$] and $1064^s{\cdot}6$ [$= 4^\circ\,26'$] respectively		Comrie 1931 (32)

7·62. Cosec $x - \dfrac{1}{x}$ and $\dfrac{1}{x} - \cot x$

These are alternative auxiliary functions for dealing with cosecants and cotangents of small angles.

Grades: $h = \operatorname{cosec} x^g - \dfrac{200}{\pi x}$ and $g = \dfrac{200}{\pi x} - \cot x^g$

20 dec.	$0(1^g)50^g$	v^9	Andoyer 1915

Reproduced in Roussilhe & Brandicourt 1933.

Seconds of Arc: $H = \operatorname{cosec} x'' - \dfrac{648{,}000}{\pi x}$ and $G = \dfrac{648{,}000}{\pi x} - \cot x''$

17 dec.	$0(9')15^\circ$	$v^8(10'')$, $0(18')15^\circ$, for H only	Andoyer 1915
15 dec.	$0(10'')15^\circ$	Δ	Andoyer 1918

7·63. $\dfrac{\sin x}{x}$, also $\dfrac{\cos x}{x}$ and $\dfrac{\sin^2 x}{x^2}$

In these functions x is in radians, however it may be expressed as an argument. The following tables give $\dfrac{\sin x}{x}$, and also other functions as mentioned. See also Arts. 5·71 ff.

8 dec.	$0(\cdot01)10(\cdot1)20(1)100$	No Δ	Hayashi 1930*a*

Also cos x/x to 8 decimals with the same arguments.

7 dec.	$0(2^g)2000^g$	Δ	Jørgensen 1916
7 dec.	$0(\cdot1)3\cdot1$	D^5	B.A. **2**, 1932 (1)
6 dec.	$0(1)50$	No Δ	Lommel 1886 (651)

Also $(\sin x/x)^2$ to 6 decimals with the same arguments.

4 dec.	0(·01)20(·02)40(·05)100	No Δ	Sherman 1933
4–5 fig.	0(·01)3·14	No Δ	Jahnke–Emde 1933 (32)
4 dec.	sin x/x, 0(15°)540°	No Δ	Schuster 1909 (105)
4; 5 d.	$(\sin x/x)^2$, 0(15°)180°; 180°(15°)540°		
4 dec.	0(1^g)200^g(2^g)2000^g(10^g)8000^g	No Δ	Stumpff 1939 (122)

Stumpff also gives smaller tables of x cosec x and x^2 cosec^2 x.

7·64. $\dfrac{\tan x}{x}$

4–5 fig.	0(·01)3·14	No Δ	Jahnke–Emde 1933 (32)

7·66. $x - \sin x$

This function is useful in dealing with sines of small angles, as in optics and astronomy.

11 dec.	0(·00001)·001	Δ	B.A. 1916 (Airey)
6 dec. critical	To x = ·181842 [= 10° 25′ approx.]		Chrétien 1932
5 dec. critical	To x = ·52661 [= 30° 10′ approx.]		

In the following tables $x - \sin x$ is expressed in sexagesimal measure; e.g. 30° − sin 30° equals 30° minus half a radian, or approximately 1° 21′ 7″·60.

0″·0001; 0″·01	0(1′)3°; 3°(1′)40°	Δ	Bauschinger 1901 (142)
0″·001; 0″·01	0(10″)2° 46′ 40″; 2° 46′ 50″(10″)30°	No Δ	Steinheil & Voit 1891 (271)

Sin x itself is given in sexagesimal measure to 0″·1 for x = 0(1′)90° in Schulze 1776 (3, 172), and centesimally in Sang (MS. 1848–90), see Art. 7·9.

See also Art. 7·84 and Legendre's $\int_0^\theta \sin^2\theta\, d\theta$ in Art. 14·98.

7·67. Tan $x - x$

This is sometimes called inv x, or involute function of x, and is used in connection with gearing.

13–16 dec.	0(1″)10′		Börgen 1908
12; 10 dec.	0(0°·01)0°·50; 0°·50(0°·01)1°	Δ	Buckingham 1935
8; 7 dec.	1°(0°·01)37°; 37°(0°·01)60°		

At 9°·15, for 6160 read 7160 (see Miller 1942).

6 d.; 5 d.; 4–6 fig.	0(0°·01)37°; 37°(0°·01)67°·5; 67°·5(0°·01)90°	Δ	Peters 1937

Peters also gives tan $x - x$ in degrees and decimals, as follows:

4 d.; 3–1 d.	0(0°·01)67°·5; 67°·5(0°·01)90°	Δ	Peters 1937

Merritt 1942 (136) gives tan $x - x$ to 6 decimals for sec x = 1(·001)2.

7·68. Berry Functions

Pritchard 1938 gives tables of the three functions

$$\frac{6(2x \operatorname{cosec} 2x - 1)}{(2x)^2} \qquad \frac{3(1 - 2x \cot 2x)}{(2x)^2} \qquad \frac{3(\tan x - x)}{x^3}$$

to 4 decimals for x = 0(1°)70°(0°·1)88° and to 3 decimals for x = 92°(1°)135°. See also Art. 10·48.

7·7. Versed Sines, Haversines, etc., All Arguments

7·71. Vers $x = 1 - \cos x$

Vers x stands for the versed sine of x. In navigation the haversine (see Art. 7·72) is now more used.

x in Radians

11 dec.	$x = 0(\cdot 00001)\cdot 001$	Δ	B.A. 1916 (Airey)

x in Degrees, Minutes and Seconds

7 dec.	$x = 0(10'')125°\ 59'\ 50''$	PPMD	Farley 1856
7 dec.	$x = 0(1')180°$	$PP_{60}MD$	Inman 1891 (T. 36)
7 dec.	$x = 0(1')90°$	Δ	Pryde 1930
6 dec.	$x = 0(10'')180°;\ 0(10'')90°$	PPMD	Mackay 1810; 1801
6 dec.	$x = 0(1')180°$	$PP_{60}MD$	Norie 1894, 1904 (T. 36)
6 dec.	$x = 0(1')90°$ (Column "Sum or Diff.")	No Δ	Aquino 1924 (10*)
5; 4 dec.	$x = 0(1')12°;\ 12°(1')90°$	No Δ	Hall 1920

x in Time

7 dec.	$x = 0(1^s)36^m$	No Δ	Inman 1891 (T. 36)

7·715. $2 \sin^2 \frac{1}{2}x \operatorname{cosec} 1''$

The above is the usual (slightly inaccurate) description of $1 - \cos x$ expressed in seconds of arc. The quantity occurs in some surveying and nautical tables in connection with the "reduction to the meridian" of circummeridian observations. For logarithmic values see Art. 8·715.

$0''\cdot 01$; $0''\cdot 1$	$x = 0(1^s)25^m;\ 25^m(1^s)30^m$	No Δ	Ambronn & Domke 1909 (81)
$0''\cdot 01$; $0''\cdot 1$	$x = 0(1^s)15^m;\ 15^m(1^s)42^m$	No Δ	C. F. W. Peters 1871
$0''\cdot 01$; $0''\cdot 1$	$x = 0(1^s)16^m\ 59^s;\ 17^m(1^s)29^m\ 59^s$	No Δ	Chauvenet 1863
$0''\cdot 01$	$x = 0(1^s)16^m\ 59^s$ From Chauvenet	No Δ	Nassau 1932
$0''\cdot 1$	$x = 0(2^s)40^m$	Δ	Wirtz 1918 (100)
$1''$	$x = 0(1^s)36^m$	No Δ	Inman 1891 (T. 20*a*)

7·716. $2 \sin^4 \frac{1}{2}x \operatorname{cosec} 1''$

For logarithmic values see Art. 8·716.

$0''\cdot 001$	$x = 0(10^s)30^m$	Δ	Albrecht 1908 (231); Ambronn & Domke 1909 (93)
$0''\cdot 01$	$x = 0(10^s)42^m$	No Δ	C. F. W. Peters 1871
$0''\cdot 01$	$x = 0(1^m)12^m(10^s)30^m$	No Δ	Chauvenet 1863
$0''\cdot 01$	$x = 0(1^m)5^m(10^s)34^m(20^s)36^m$	No Δ	Inman 1891 (T. 20*a*)
$0''\cdot 01$	$x = 0(20^s)40^m$	Δ	Wirtz 1918 (100)

7·717. $\sin^2 x \operatorname{cosec} 1''$, $\cos^2 x \operatorname{cosec} 1''$, $\sin x \cos x \operatorname{cosec} 1''$

$1''$	$x = 0(1')45°$	fPPd	Brandenburg 1932*a*

7·718. $\text{Tan}^2 \frac{1}{2}x$ cosec 1″

0″·01	$x = 0(1^s)30^m$	Δ	König 1933

7·72. Hav $x = \frac{1}{2}(1 - \cos x) = \sin^2 \frac{1}{2}x$

Hav x stands for half versed sine of x. The function is used mainly for nautical purposes, e.g. in solving spherical triangles by the formula

$$\text{hav } a = \text{hav } (b - c) + \sin b \sin c \text{ hav } A$$

7 dec.	$x = 0(10'')180°$	fPP_{100}	Hannyngton 1876

Time argument also given, to 2 decimals of 1^s (interval approximately 0^s·67).

7 dec.	$x = 0(10'')120°$	fPP	Andrew 1805
5 dec.	$x = 0(15'')120°(30'')135°(1')180°$; time argument also	No Δ	Norie 1938
5 dec.	$x = 0(30'')15°(15'')120°(30'')135°(1')180°$; time argument also	No Δ	Davis 1905, Inman 1906, 1940, Bowditch 1925
5 dec.	$x = 0(30'')120°(1')180°$; time argument also	No Δ	Burton 1941
4 dec.	$x = 0(6')180°$	PP_6MD	Hall 1905

7·8. Miscellaneous Tables connected with the Circle and the Sphere (Areas, Volumes, Chords, Segments, Regular Polygons, etc.)

Tables of this kind for engineering purposes must be very numerous, and we have not specially sought them out, but we give information about some which have come to our knowledge. (Trautwine 1937 was seen too late for insertion.) Tables which use eighths, twelfths, etc. instead of decimals have usually been ignored.

7·811–7·824. Quantities associated with the Whole Circle or Sphere

7·811. πx = Circumference of Circle of Diameter x

For many-figure multiples of π, see Art. 5·112. Todd 1853 gives πx to 5 decimals for $x = 0(\cdot 1)100$. Templeton–Maynard 1858 gives πx to 4 decimals for $x = 1(\cdot 1)100$. Tables to about 5 or 6 figures are very common. A 6-figure table for $x = 1(\cdot 1)100 \cdot 9$ is given in Low & Low 1938. Values of πx for $x = 1(1)1000$, to 2 decimals, are given in Clark 1891, Clark 1919, Kempe 1927 and Allen 1941. See also Oberg & Jones 1936 (55) and Templeton–Hutton 1900 (90).

Only values for $x = 100(1)1000$ are really needed, and this range has been primarily considered in describing the tables. Thus Kempe 1927 gives 3 decimals for $x = 4(1)31$, but the values are tenths of the 2-decimal values for $x = 40(10)310$, and are redundant. A similar remark applies elsewhere in Arts. 7·811–7·824.

Hütte 1920 and 1931 give πx to 5 figures for $x = 1(1)1100$ and $x = 1(1)1500$ respectively.

7·812. $2\pi x$ = Circumference of Circle of Radius x

6 dec.	$x = 1(1)100$	Potin 1925 (204)
2 dec.	$x = 1(1)100$	F. G. Gauss 1870, Schorr 1916

7·813. $\frac{1}{2}\pi x$ = Semi-Circumference of Circle of Diameter x

6-figure values are given for $x = 0(\cdot 1)100$ in H. Zimmermann 1929. See also tables for conversion from grades to radians (Art. 5·913).

7·814. πx^2 = Area of Circle of Radius x = Surface of Sphere of Diameter x

4 dec.	$x = 0(\cdot 1)100(1)600$	No Δ	Todd 1853
5; 6 fig.	$x = 0(\text{var.})10(\cdot 25)29(\cdot 5)101(1)178;$ $179(1)200$	No Δ	Oberg & Jones 1936 (82)
4–5 fig.	$x = 1(1)100$	No Δ	F. G. Gauss 1870 Schorr 1916

7·815. $4\pi x^2$ = Surface of Sphere of Radius x

Nearest integer	$x = 1(1)100$	No Δ	F. G. Gauss 1870 Schorr 1916

7·816. $\frac{1}{4}\pi x^2$ = Area of Circle of Diameter x

6 dec.	$x = 0(\cdot 1)100$	No Δ	Todd 1853
2 dec.	$x = 1(1)1000$	No Δ	Allen 1941
2 dec.	$x = 1(1)1000$	No Δ	Clark 1919
2 dec.	$x = 1(1)1000$	No Δ	Clark 1891
4 dec.	$x = 1(\cdot 1)100$	No Δ	Templeton–Maynard 1858
7 f, or 4 d.	$x = 0(\cdot 1)99\cdot 9$	No Δ	Templeton–Hutton 1900
6; 7 fig.	$x = 1(1)1128;\ 1129(1)1500$	No Δ	Hütte 1931
6 fig.	$x = 1(1)1100$	No Δ	Hütte 1920
6 fig.	$x = 0(\cdot 1)100$	No Δ	H. Zimmermann 1929
6 fig.	$x = 1(\cdot 1)100\cdot 9$	No Δ	Low & Low 1938
5–6 fig.	$x = 1(1)1000$	No Δ	Kempe 1927
5 fig.	$x = 0(\cdot 1)100\cdot 9$	No Δ	Castle 1910

See also Oberg & Jones 1936 (55).

7·817. $\frac{4}{3}\pi x^3$ = Volume of Sphere of Radius x

Nearest integer	$x = 1(1)100$	No Δ	F. G. Gauss 1870 Schorr 1916

7·818. $\frac{1}{6}\pi x^3$ = Volume of Sphere of Diameter x

4 dec.	$x = 0(\cdot 1)100(1)600$	No Δ	Todd 1853
7 fig.	$x = 1(1)200$	No Δ	Hütte 1920 (35), 1931 (42)
5 fig. 6; 7 fig.	$x = 0(\text{var.})10(\cdot 25)29(\cdot 5)57\cdot 5$ $x = 58(\cdot 5)101(1)124;\ 125(1)200$	No Δ	Oberg & Jones 1936 (82)

7·821. x/π = Diameter of Circle of Circumference x

For many-figure multiples of $1/\pi$, see Art. 5·132.

4; 3 dec.	$x = 1(1)100;\ 101(1)250$		Oberg & Jones 1936 (71)

7·822. $x/2\pi$ = Radius of Circle of Circumference x

6 dec.	$x = 1(1)100$	Potin 1925 (202)
3–4 fig.	$x = 1(1)100$	F. G. Gauss 1870, Schorr 1916

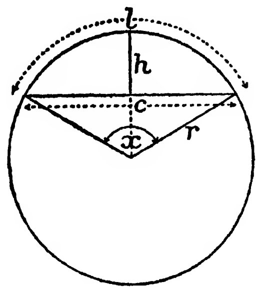
7·823. $x^2/4\pi$ = Area of Circle of Circumference x

2 dec.	$x = 1(1)100$	No Δ	F. G. Gauss 1870 Schorr 1916

7·824. $\frac{1}{2}x\sqrt{\pi}$ = Side of Square equal in Area to Circle of Diameter x

4; 3 dec.	$x = 0(\frac{1}{16})6(\frac{1}{8})11\frac{1}{4};\ 11\frac{3}{8}(\frac{1}{8})50(\frac{1}{4})100(\frac{1}{2})120$	Clark 1891
2 dec.	$x = 1(\frac{1}{2})60\frac{1}{2}$	Kempe 1927

7·83. Arcs, Chords, etc. of Circles



Articles 7·831 to 7·836 refer to functions of the angle x subtended at the centre of a circle of radius r by an arc of length l or a chord of length c. h stands for the height of the segment whose chord is c.

Other arguments occur in Arts. 7·837 and 7·838.

Common logarithms of l/r, c/r, h/r, l/c, h/c, h/l and hr/l^2 for $x = 0(1')180°$ are given to 5 decimals with PPd in Smoley 1937 (14).

7·831. l/r as a Function of x

See tables of the radian measure of x (Art. 5·91).

7·832. $c/r = 2 \sin \frac{1}{2}x$

Tables of chords are fairly numerous.

7 dec.	$x = 0(1')90°$	No Δ	Pryde 1930, Ives 1929
6 dec.	$x = 0(1')180°$		Gronwaldt 1850
5 dec.	$x = 0(0°{\cdot}1)90°$	No Δ	Castle 1910
4 dec.	$x = 0(10')180°$	fPPd	F. G. Gauss 1870 Schorr 1916
4 dec.	$x = 0(20')180°$	No Δ	Knott 1905
4 dec.	$x = 0(0^g{\cdot}1)200^g$	fPPd	F. G. Gauss 1937

7·833. $h/r = 1 - \cos \frac{1}{2}x$

4 dec.	$x = 0(10')180°$	fPPd	F. G. Gauss 1870 Schorr 1916
4 dec.	$x = 0(0^g{\cdot}1)200^g$	fPPd	F. G. Gauss 1937

7·834. $r/c = \frac{1}{2} \operatorname{cosec} \frac{1}{2}x$

7 dec.	$x = 0(1°)180°$ (7-decimal logarithms also given)	No Δ	Kempe 1927
6 fig.	$x = 0(1')10°(2')20°(10')30°(20')38°(30')50°$	No Δ	Ives 1931

Ives tabulates various other functions connected with road and railway curves.

7·835. $l/c = \frac{1}{2}x \operatorname{cosec} \frac{1}{2}x$

7 dec.	$x = 10°(1')30°(10')180°$		No Δ	Smoley 1937
7 dec.	$x = 0(1°)180°$	(7-decimal logarithms also given)	No Δ	Kempe 1927

7·836. $c/h = 2 \cot \frac{1}{2}x$

7 dec.	$x = 0(1°)180°$	(7-decimal logarithms also given)	No Δ	Kempe 1927

7·837. $2h/c$ as a Function of $c/2r$

Card 1943 tabulates F to 3 decimals without differences for $r/R = 0(\cdot001)\cdot8$. In our notation his F is $2h/c$ and his r/R is $c/2r$. The table is primarily intended for use in connection with spherical segments in optical work.

7·838. l/c as a Function of h/c

5 dec.	$h/c = 0(\cdot001)\cdot500$	No Δ	Clark 1919
5 dec.	$h/c = \cdot100(\cdot001)\cdot500$	No Δ	Clark 1891

7·84. Areas of Segments of a Circle

Sharp 1717 gives to 17 decimals with differences the ratio of the area of the segment to that of the whole circle, the argument being the ratio of the breadth (the quantity denoted by h in Art. 7·83) to the diameter, and taking the values $0(\cdot0001)\cdot5000$.

Pryde 1930 and Low & Low 1938 give to 6 decimals without differences the areas of segments of a circle of unit diameter with breadths $0(\cdot001)\cdot500$.

Clark 1891 gives the same table to 5 decimals. Clark 1919 also gives it to 5 decimals, except that the first 9 values are to 6 decimals.

Kempe 1927 gives the same function to 4 figures (not more than 6 decimals) for breadths $0(\cdot001)1$. The same work contains a table of $\frac{\text{segment area}}{(\text{chord})^2}$, given naturally to 7 figures and logarithmically to 7 decimals, for $0(1°)180°$ of angle subtended by the chord at the centre.

Wijdenes 1937 gives to 4 decimals, with Δ and fPPd, the ratio of the area of a segment of chord k and breadth p to the area pk of the symmetrical circumscribing rectangle, for $p/k = 0(\cdot01)\cdot50$.

Sang (MS. 1848–90) has an unpublished table of the areas of segments in a circle of unit radius, "measured in degrees of the surface of the circle, for each of the 40,000 minutes of the entire circumference". This is Sang's own description (Sang 1884*a*, p. 541). He refers to centesimal degrees and minutes. See also Knott 1915 (237) and Horsburgh 1914 (46). The quantity tabulated is $x - \frac{200}{\pi} \sin x^g$, to 8 decimals with second differences.

Schorr 1916 and Hütte 1920 give $\frac{1}{2}(x - \sin x)$ to 5 decimals without differences for $x = 0(1°)180°$. Oberg & Jones 1936 gives $\frac{1}{2}(x - \sin x)$ to 5 decimals for $x = 0(1°)90°$ and to 4 decimals for $x = 91°(1°)180°$. Smoley 1937 (220) gives to 5 decimals, with PPd, common logarithms of the same function for $x = 0(1')180°$.

See also Art. 7·66 and Legendre in Art. 14·98, as well as Röther & Lüdemann 1907 in Part II.

7·85. Regular Polygons

7·851. Angles π/n and $2\pi/n$

π/n to 6 dec. of radian; $180°/n$ to 4 dec. of $1°$	$n = 1(1)500$	**Peters 1937**
$2\pi/n$ to 4 dec. of radian	$n = 1(1)100$	MT.C. 1931 (36)
$360°/n$ to 3; 4 dec. of $1°$	$n = 1(1)90;\ 91(1)180$	Stumpff 1939 (24)
$360°/n$ to $1''$	$n = 3(1)60$	Smoley 1937 (247)

Vogel 1943 (212–217) gives $90°/n$, $180°/n$ and $360°/n$, where $n = 1(1)200$, all to 4 decimals of $1''$, 7 decimals of $1°$ and 10 decimals of a radian.

7·852. Sin $180°/n$ and cos $180°/n$

Note that cos $180°/n$ is the apothem (perpendicular from centre to side) of a regular n-gon when the circumscribing circle has unit radius, and sin $180°/n$ is the side when that circle has unit diameter.

6 dec.	$n = 1(1)500$	Peters 1937

Oberg & Jones 1936 gives on pp. 78–79 the values of $D \sin 180°/n$ to 3 decimals for $n = 3(1)26(2)32$ and $D = 0(\frac{1}{16})1(1)14$. See also Templeton–Maynard 1858 in Art. 7·853, as well as Smoley 1937 (247) and our Art. 24·21.

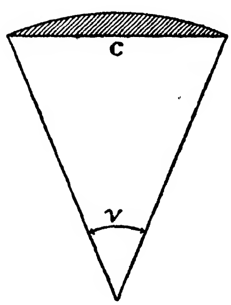
7·853. Apothems and Areas (Side = 1)

These quantities are respectively $\frac{1}{2}\cot\dfrac{\pi}{n}$ and $\frac{1}{4}n\cot\dfrac{\pi}{n}$.

7 dec.	$n = 3(1)12$	Pryde 1930 (438)

See Oberg & Jones 1936 (80) for these and various other quantities when $n = 3(1)10,\ 12(4)24,\ 32,\ 48,\ 64$. Templeton–Maynard 1858 (48) gives to 7 decimals for $n = 3(1)50$ the quantities circumradius/apothem, side/circumradius, circumradius/side and area/square of side. See also Hütte 1920 (131) and 1931 (196).

7·86. Surface of Zone of Sphere



c = diameter of bounding circle, v = central angle.

Kempe 1927 tabulates the quantity

$$\frac{\text{surface of zone}}{c^2} = \frac{\pi}{2(1+\cos\frac{1}{2}v)}$$

naturally to 6 figures and logarithmically to 7 decimals for $v = 0(1°)180°$, without differences.

7·87. Volume of Segment of Sphere

Kempe 1927 tabulates the quantity

$$\frac{\text{volume of segment}}{c^3} = \tfrac{1}{4}\pi(\operatorname{cosec}\tfrac{1}{2}v - \cot\tfrac{1}{2}v)^2(2\operatorname{cosec}\tfrac{1}{2}v + \cot\tfrac{1}{2}v)$$

naturally to 6 figures and logarithmically to 7 decimals, for $v = 0(1°)180°$, without differences. The notation is as in Art. 7·86.

7·9. Miscellaneous Expressions involving Circular Functions

$\sin \frac{x}{\sqrt{2}}$, $\cos \frac{x}{\sqrt{2}}$	12 dec.	$x = 0(\cdot 1)20$	No Δ	Airey 1935*d*

Some basic values to 18 and 20 decimals. See also Arts. 5·23 and 10·49.

$\sin(\log_e x)$, $\cos(\log_e x)$	10 dec.	$x = \cdot 1(\cdot 1)20$, also $\cdot 98(\cdot 001)1\cdot 02$	No Δ	Airey 1935*e*

Some basic values to 13–15 decimals.

$x \tan x$	4–5 fig.	$x = 0(\cdot 01)3\cdot 14$	No Δ	Jahnke–Emde 1933 (32)
$x \tan \frac{1}{2}\pi x$, $x \cot \frac{1}{2}\pi x$	4–5 fig.	$x = 0(\cdot 1)9$	No Δ	Jahnke–Emde 1933 (38)
$\pi \cot \pi x$	15 dec.	$x = 0(\cdot 01)\cdot 50$	No Δ	Davis 1933 (288)
$\frac{2}{\pi} \sin x$	10 dec.	$x = 0(0^g\cdot 01)100^g$		Sang (MS. 1848–90)
$\frac{1}{2}(\theta + \cos\theta)$, expressed in seconds		$\theta = -90^\circ(1')90^\circ$	Δ^2	Aldis 1902

About 10 figures (4 decimals of 1″ for $\pm\theta = 0(1')84^\circ\ 22'$, 5 decimals for $\pm\theta = 84^\circ\ 23'(1')89^\circ\ 24'$, then 6 decimals).

$\cos x - \sin x$	5 dec.	$x \doteq 0(1')45^\circ$	PP_6MD	Goussinsky 1943
$\cos^n x$, $\sin x \cos^{n-1} x$	4 dec.	$n = 2, 3, 4, \frac{1}{2}, \frac{3}{2}$, $x = 0(30')90^\circ$	No Δ	Potin 1925 (408)
$\sin^2 x$, $\tan^2 x$	5 d. or f.	$x = 0(1')90^\circ$		Pasquich 1817

Also see Section 21 for tables of $\sin^2 x$ and $\sin^{-1}\sqrt{x}$ ($k = \sin\theta$, tabulations of k^2 against θ and θ against k^2). For tables of $\sin^2 \frac{1}{2}x$, see Art. 7·72.

SECTION 8

LOGARITHMS OF TRIGONOMETRICAL FUNCTIONS

8·0 Introduction

This section deals with the common logarithms of trigonometrical functions.

When the argument x is in radians, separate tables of $\log \sin x$, $\log \cos x$ and $\log \tan x$ are desirable. Since $\log \operatorname{cosec} x$, $\log \sec x$ and $\log \cot x$ differ from the former only in sign, tables of them may be dispensed with.

When the argument is in other customary units (degrees, grades, hours, etc.) a table of $\log \sin x$ serves as a table of $\log \cos x$, particularly if the complementary argument is printed; thus $\log \sin x$ and $\log \tan x$ are the staple functions. We shall also list a few tables of $\log \sec x$.

The best arrangement is the semi-quadrantal one, giving on each line say

$$x, \quad \log \sin x, \quad \log \tan x, \quad \log \cot x, \quad \log \cos x, \quad \text{complement of } x$$

where x goes from 0 to 45° (or its equivalent in other units). But we shall always describe such a table as giving $\log \sin x$ and $\log \tan x$, where x goes from 0 to 90° (or its equivalent).

Besides giving the usual indications concerning differences and proportional parts, we have indicated whether the auxiliary functions S and T for small angles (see Art. 8·6) are provided. Details of range and interval for these should be sought in Art. 8·6. Since $\log \sin x$, $\log \tan x$ and $\log \cot x$ all have infinities at $x = 0$, special provision, either by auxiliary functions or by narrower intervals or by both, is usually made for small angles, and the indications Δ, PP do not imply that differences or proportional parts are necessarily given for small angles. It has not been found desirable to indicate where differences or proportional parts commence, or proportional parts become complete, but the indication fPP has been given when the special provision has seemed adequate and the proportional parts are complete in the remainder of the table. As in Section 7, PP without numerical suffix does not necessarily imply proportional parts for tenths of the interval when the argument is sexagesimal.

Fundamental tables with any kind of argument are few, and all known to us have been mentioned. But working tables with 4 to 7 decimals and sexagesimal argument are very numerous, and our lists of such tables are highly selective. The sexagesimal argument (degrees, minutes, etc.) is used in astronomical and nautical calculations, but for general mathematical purposes a decimal subdivision of either the degree or the grade (hundredth of a right angle) is greatly to be preferred; in some connections it is almost essential.

The main subdivisions of the present section are:

8·1. Tables with Argument in Radians

These are far from numerous. The reader may be reminded that very extensive tables of *natural* sines and cosines with radian argument exist—see Art. 7·1.

8·11. Log sin x and log cos x (Radians)

5 d.	Both fns.	0(·0001)·1(·001)1·6(·1) 10(1)100	v to 1·6	**Becker & Van Orstrand 1924**
5 d.	log sin x	0(·001)1·569	PPMD	Meyer & Deckert 1924
5 d.	log cos x	0(·001)1·539	PPMD	Meyer & Deckert 1924
4½ d.	Both fns.	0(·1)7·9	No Δ	**MT.C. 1931**
4½ d.	log sin x	0(·0001)·04(·001)1·57	S, Δ	**MT.C. 1931**
4½ d.	log cos x	0(·001)1·57	Δ	**MT.C. 1931**
4 d.	Both fns.	0(·01)3·2 (Not in earlier editions)	No Δ	Allen 1941
4 d.	Both fns.	0(·01)1·6	No Δ	Carmichael & Smith 1931
4 d.	Both fns.	·10(·01)1·57	No Δ	Knott 1905

8·12. Log tan x and log cot x (Radians)

Since log cot x = −log tan x, the two functions differ trivially.

5 dec.	log tan x	0(·001)1·569	PPMD	Meyer & Deckert 1924
4½ dec.	log tan x	0(·0001)·04(·001)1·57	T, Δ	**MT.C. 1931**
4½ dec.	log cot x	0(·001)1·57	Δ	**MT.C. 1931**
4 dec.	log tan x	0(·01)3·2 (Not in earlier editions)	No Δ	Allen 1941
4 dec.	Both fns.	0(·01)1·6	No Δ	Carmichael & Smith 1931
4 dec.	log tan x	·10(·01)1·57	No Δ	Knott 1905

8·2. Tables with Argument in Degrees and Decimals

The use of degrees and decimals for general mathematical purposes is very desirable, but tables are not numerous. The following lists include only tables strictly employing degrees and decimals. Thus values at interval 0°·01 (36″) might be extracted from tables with interval 1″, but such tables are not listed here.

8·21. Log sin x (Degrees and Decimals)

14 dec.	0°·01	Δ	**Briggs & Gellibrand 1633**

Correct to 12 decimals in general. For 6 errors, see Bauschinger & Peters 1936.

10 dec.	0°·01	No Δ	Roe & Wingate 1633
8; 9; 10 d.	0(0°·001)0°·7; 0°·7(0°·001)2°·1; 2°(0°·001)90°	S, Δ	**Peters 1919*a***

Maximum error < 1 final unit. 10-decimal S to 2°·1 in Peters 1919*b*.

8 dec.	0(0°·01)90°	Δ	J. Newton 1658
8 dec.	0(0°·001)3°	S	J. Newton 1658
7 dec.	0°·001	S, Δ, PPd	**Peters 1921*a***
6 dec.	0°·001	S, Δ, PPd	**Peters 1921*b***
6 dec.	0°·01	Sp. to 6°	Oughtred 1657
5 dec.	0°·01	$-S, \Delta$, fPPd	**Bremiker 1937 (1872)**

5 d.	$0(0°{\cdot}001)5°$ Supplement to Bremiker	fPPd	Von Rohr 1933 (1900)
5 d.	$0°{\cdot}01$	IΔ	Dwight 1941
5 d.	$5°(0°{\cdot}01)90°$	S, fPPd	Silberstein 1923
5 d.	$0°{\cdot}01$		Westrick 1892
4½ d.	$0(0°{\cdot}01)4°(0°{\cdot}1)90°$	S, Δ	**MT.C. 1931**
4½ d.	$0(0°{\cdot}01)5°(0°{\cdot}1)85°(0°{\cdot}01)90°$	S, PPd	Desvallées 1920
4 d.	$0(0°{\cdot}01)8°(0°{\cdot}1)82°(0°{\cdot}01)90°$	S, Δ, fPPd	**Bremiker 1907 (1874)**
4 d.	$0(0°{\cdot}01)10°(0°{\cdot}1)90°$	PPMD after 10°	Carey & Grace 1927
4 d.	$0(0°{\cdot}01)5°(0°{\cdot}1)85°(0°{\cdot}01)90°$	Δ, PPd	Granville 1908

8·22. Log tan x (Degrees and Decimals)

10 d.	$0°{\cdot}01$	Δ	**Briggs & Gellibrand 1633**
10 d.	$0°{\cdot}01$	No Δ	Roe & Wingate 1633
10 d.	$2°(0°{\cdot}001)88°$	Δ	**Peters 1919*a***
9 d.	$0°{\cdot}7(0°{\cdot}001)2°{\cdot}1$; $87°{\cdot}9(0°{\cdot}001)89°{\cdot}3$	Δ	**Peters 1919*a***
8 d.	$0(0°{\cdot}001)0°{\cdot}7$; $89°{\cdot}3(0°{\cdot}001)90°$	T, Δ	**Peters 1919*a***

Maximum error < 1 final unit. 10-decimal T to $2°{\cdot}1$ in Peters 1919*b*.

8 d.	$0(0°{\cdot}01)90°$	Δ	J. Newton 1658
8 d.	$0(0°{\cdot}001)3°$	T	J. Newton 1658
7 d.	$0°{\cdot}001$	T, Δ, PPd	**Peters 1921*a***
6 d.	$0°{\cdot}001$	T, Δ, PPd	**Peters 1921*b***
6 d.	$0°{\cdot}01$	Sp. to 6°	Oughtred 1657
5 d.	$0°{\cdot}01$	$-T, \Delta$, fPPd	**Bremiker 1937 (1872)**
5 d.	$0(0°{\cdot}001)5°$ Supplement to Bremiker	fPPd	Von Rohr 1933 (1900)
5 d.	$0°{\cdot}01$	IΔ	Dwight 1941
5 d.	$5°(0°{\cdot}01)85°$	$\pm T$, fPPd	Silberstein 1923
5 d.	$0°{\cdot}01$		Westrick 1892
4½ d.	$0(0°{\cdot}01)4°(0°{\cdot}1)90°$	T, Δ	**MT.C. 1931**
4½ d.	$0(0°{\cdot}01)5°(0°{\cdot}1)85°(0°{\cdot}01)90°$	T, PPd	Desvallées 1920
4 d.	$0(0°{\cdot}01)8°(0°{\cdot}1)82°(0°{\cdot}01)90°$	T, Δ, fPPd	**Bremiker 1907 (1874)**
4 d.	$0(0°{\cdot}01)10°(0°{\cdot}1)$ $80°(0°{\cdot}01)90°$	PPMD, 10° to 80°	Carey & Grace 1927
4 d.	$0(0°{\cdot}01)5°(0°{\cdot}1)85°(0°{\cdot}01)90°$	Δ, PPd	Granville 1908

8·3. Tables with Centesimal Argument

By centesimal argument is meant primarily one in either decimals of the quadrant or grades and decimals, a grade (1^g) being a hundredth of a right angle. For example $18° = 0^q{\cdot}20 = 20^g$. Thus Art. 8·31 contains substantially tables of $\log\sin\frac{1}{2}\pi x$. We have also included Davis's small table of $\log\sin \pi x$. The argument has always been expressed in grades; e.g. a table of $\log\sin\frac{1}{2}\pi x$ for $x = 0(\cdot01)1$ would be described as one of $\log\sin x$ for $x = 0(1^g)100^g$. Functions of $2\pi x$ are, however, considered separately in Art. 8·34.

8·31. Log sin x (Grades)

17 dec.	1^g	S; v^9, 50^g to 100^g	**Andoyer 1911**
15 dec.	1^g	No Δ	Jordan–Eggert 1939 (416)
15 dec.	$0(0^g{\cdot}01)50^g$; $50^g(0^g{\cdot}01)100^g$	Δ; Δ^3	Sang (MS. 1848–90)
14 dec.	$0^g{\cdot}1$	No Δ	Callet 1795
14 dec. (12 cor.)	$0^g{\cdot}001$	S	Prony (MS. 1794)

11 dec.	$0(0^{g}{\cdot}001)0^{g}{\cdot}1(0^{g}{\cdot}1)99^{g}{\cdot}9(0^{g}{\cdot}001)100^{g}$	No Δ	Borda & Delambre 1801
10 dec.	2^{g}	No Δ	Davis 1933 (189)
8 dec.	$0^{g}{\cdot}001$	S, PPd	**Service Géog. 1891**

For a few corrections given by Mendizábal Tamborrel see *M.T.A.C.*, **1**, 85, 1943.

7 dec.	$0(0^{g}{\cdot}0001)3^{g}(0^{g}{\cdot}001)100^{g}$	S, (f?)PPd	**Peters 1940*b***
7 dec.	$0(0^{g}{\cdot}001)3^{g}(0^{g}{\cdot}01)97^{g}(0^{g}{\cdot}001)100^{g}$	Δ	Hobert & Ideler 1799
7 dec.	$0(0^{g}{\cdot}001)3^{g}(0^{g}{\cdot}01)97^{g}(0^{g}{\cdot}001)100^{g}$	fPPd	Borda & Delambre 1801
7 dec.	$0^{g}{\cdot}01$	S, Δ	Callet 1795
6 dec.	$0(0^{g}{\cdot}001)20^{g}(0^{g}{\cdot}01)80^{g}(0^{g}{\cdot}001)100^{g}$	S, fPPd	**Jordan–Eggert 1939**
5 dec.	$0(0^{g}{\cdot}001)3^{g}{\cdot}5(0^{g}{\cdot}01)100^{g}$	S, Δ, fPPd	F. G. Gauss 1937
5 dec.	$0(0^{g}{\cdot}001)1^{g}{\cdot}2(0^{g}{\cdot}01)100^{g}$	PP(d)	Wijdenes 1937
5 dec.	$0^{g}{\cdot}01$	S, Δ, PPd	**Service Géog. 1914**
5 dec.	$0^{g}{\cdot}01$		Gravelius 1886
4 dec.	$0(0^{g}{\cdot}01)3^{g}(0^{g}{\cdot}1)100^{g}$	Δ	Hoüel 1901

8·32. Log tan x (Grades)

17 dec.	$0(1^{g})50^{g}$	No Δ	**Andoyer 1911**
15 dec.	$0(0^{g}{\cdot}01)50^{g}$	Δ	Sang (MS. 1848–90)
14 dec.	$0^{g}{\cdot}001$	T	Prony (MS. 1794)

Correct, like the logarithmic sines, to only about 12 decimals.

11 dec.	$0(0^{g}{\cdot}001)0^{g}{\cdot}1(0^{g}{\cdot}1)99^{g}{\cdot}9(0^{g}{\cdot}001)100^{g}$	No Δ	Borda & Delambre 1801
8 dec.	$0^{g}{\cdot}001$	T, PPd	**Service Géog. 1891**

For three corrections given by Mendizábal Tamborrel see *M.T.A.C.*, **1**, 85, 1943.

7 dec.	$0(0^{g}{\cdot}0001)3^{g}(0^{g}{\cdot}001)100^{g}$	T, (f?)PPd	**Peters 1940*b***
7 dec.	$0(0^{g}{\cdot}001)3^{g}(0^{g}{\cdot}01)97^{g}(0^{g}{\cdot}001)100^{g}$	Δ	Hobert & Ideler 1799
7 dec.	$0(0^{g}{\cdot}001)3^{g}(0^{g}{\cdot}01)97^{g}(0^{g}{\cdot}001)100^{g}$	fPPd	Borda & Delambre 1801
7 dec.	$0(0^{g}{\cdot}01)50^{g}$	T, Δ	Callet 1795
6 dec.	$0(0^{g}{\cdot}001)20^{g}(0^{g}{\cdot}01)80^{g}(0^{g}{\cdot}001)100^{g}$	T, fPPd	**Jordan–Eggert 1939**
5 dec.	$0(0^{g}{\cdot}001)3^{g}(0^{g}{\cdot}01)100^{g}$	T, Δ, fPPd	F. G. Gauss 1937
5 dec.	$0(0^{g}{\cdot}001)1^{g}{\cdot}2(0^{g}{\cdot}01)98^{g}{\cdot}8(0^{g}{\cdot}001)100^{g}$	PP(d)	Wijdenes 1937
5 dec.	$0^{g}{\cdot}01$	T, Δ, fPPd	**Service Géog. 1914**
5 dec.	$0^{g}{\cdot}01$		Gravelius 1886
4 dec.	$0^{g}{\cdot}1$	Δ	Hoüel 1901

8·33. Log sec x (Grades)

7 dec.	$0(0^{g}{\cdot}001)3^{g}(0^{g}{\cdot}01)97^{g}(0^{g}{\cdot}001)100^{g}$	fPPd	Borda & Delambre 1801
5 dec.	$0^{g}{\cdot}01$	Δ, PPd	Service Géog. 1914
4 dec.	$0^{g}{\cdot}1$	Δ	Hoüel 1901

8·34. Log sin $2\pi x$ and log tan $2\pi x$

Mendizábal Tamborrel 1891 takes as fundamental unit the "gone", which is equal to one revolution, and also employs microgones, centimiligones, etc. The extensive tables of trigonometrical logarithms may be described as giving the four functions log sin $2\pi x$, log cos $2\pi x$, log tan $2\pi x$ and log cot $2\pi x$ for $x = 0(\cdot000001)\cdot125000$, with S and T functions and PP(d). The tables give 8 decimals for

$$x = 0(\cdot000001)\cdot025000(\cdot000010)\cdot125000$$

and 7 decimals for the remaining arguments. The digits in the ninth to twelfth places of log sin $2\pi x$ for $x = 0(\cdot 0001)\cdot 2500$ are also given, so that 12-decimal values may be written down for these arguments.

Other tables with argument in gones or "cirs" (centigones) are unimportant; see *M.T.A.C.*, **1**, 40–41, 1943.

8·4. Tables with Argument in Degrees, Minutes, etc.

Working tables with argument in degrees, minutes, etc. are exceedingly numerous, and our lists are highly selective in respect of tables with fewer than 8 decimals. See also the nautical tables mentioned in Art. 8·5.

8·41. Log sin x (Degrees, Minutes, etc.)

22 dec.	$0(1'')10'(10')89°\,50'(1'')90°$	Δ^3 after $89°\,50'$	**Witt 1922**
17 dec.	$54'$	v^9, $45°(54')90°$	**Andoyer 1911**
15 dec.	$9'$	S; $v^7(10'')$, $45°(18')90°$	**Andoyer 1911**
14 dec.	$10''$	S, Δ	**Andoyer 1911**
14 dec.	$15'$	No Δ	Legendre 1816, 1826 (T. 3)
10 dec.	$0(1'')2°(10'')88°(1'')90°$	Δ, 2° to 88°	Vega 1794
10 dec.	$10''$	Δ	Vlacq 1633
10 dec.	$1'$	Δ	Vlacq 1628
10 dec.	$10'$	No Δ	Börgen 1908
8 dec.	$1''$	S, IΔ	**Bauschinger & Peters 1936 (1911)**
7 dec.	$1''$	S, PPd	**Peters 1911*b***
7 dec.	$1''$	PPMD	Shortrede 1844, 1849*b*

Other 7-decimal $1''$ tables are Robert (MS. 1784), Taylor 1792 and Bagay 1829.

7 dec.	$0(1'')6°(10'')84°(1'')90°$	S, Δ, PPd	**Bruhns 1922 (1870)**
7 dec.	$0(1'')5°(10'')90°$	S, Δ, PPd	Vega–Bremiker 1856
7 dec.	$0(1'')5°(10'')90°$	S, Δ, PPd	Dupuis 1862
7 dec.	$0(1'')1'\,40''(10'')90°$	S, Δ, PPd	Schrön 1860

Other 7-decimal $10''$ tables are Gardiner 1742 and Callet 1795.

7 dec.	$1'$	Δ	Pryde 1930
6 dec.	$0(1'')1°;\ 0(3'')90°$	No Δ; PP_{60}	Jahn 1838
6 dec.	$0(1'')5°(10'')90°$	S, Δ, PPd	**Bremiker 1924 (1852)**
6 dec.	$1'$	$S, \Delta/60$	Allen 1941; Ives 1934; G. W. Jones 1893
5 dec.	$10''$	PP	Gernerth 1866
5 dec.	$0(0'\cdot 1)6°(1')90°$	S, Δ, fPPd	**Becker 1928 (1882)**
5 dec.	$0(1'')1°(10'')6°(1')90°$	S, Δ, fPPd	F. G. Gauss 1870
5 dec.	$0(1'')1°(10'')4°(1')90°$	Δ, fPPd	Carmichael & Smith 1931
5 dec.	$0(1'')1°(10'')3°(1')90°$	Δ for $1''$	Chappell 1915
5 dec.	$1'$	S, Δ, PPd	Hedrick 1920; Service Géog. 1914
5 dec.	$1'$	Δ, PP	Hoüel 1921
5 dec.	$1'$	PPd	Beetle 1933
4½ dec.	$1'$	S, Δ	**MT.C. 1931**

8·42. Log tan x (Degrees, Minutes, etc.)

Exactly as for log sin x in Art. 8·41 (replacing S by T), except:

22 dec.	$0(1'')10'(10')45°$	T	**Witt 1922**
17 dec.	$0(54')45°$		**Andoyer 1911**
15 dec.	$0(9')45°$	T	**Andoyer 1911**
14 dec.	$10''$	T, Δ	**Andoyer 1911**
6 dec.	$1'$	$\pm T, \Delta/60$	Allen 1941; Ives 1934
5 dec.	$1'$	T, Δ, PPd	F. G. Gauss 1870
5 dec.	$1'$	Δ for $1''$	Chappell 1915

8·43. Log sec x (Degrees, Minutes, etc.)

11 dec.	$0(1'')10'$	No Δ	Börgen 1908
10 dec.	$1'$	Δ	Vlacq 1628
7 dec.	$1'$	Δ	Pryde 1930
6 dec.	$1'$	Δ	Domke 1892
5 dec.	$30''$	No Δ	Hughes-Comrie 1938
5 dec.	$1'$	Δ, PPd	Service Géog. 1914; Albrecht 1884
5 dec.	$1'$	Δ, PP	Hoüel 1921
5 dec.	$1'$	Δ for $1''$	Chappell 1915

See also Arts. 8·5 and 8·74.

8·431. $\frac{1}{2}$ log sec x (Degrees, Minutes, etc.)

6 dec.	$0(1')79°\,49'$	No Δ	Aquino 1924 (2*)

8·5. Tables with Argument in Time

These are required for astronomical and navigational purposes. $24^h = 360°$, $1^h = 15°$, $1^m = 15'$, $1^s = 15''$. Except for Airy 1838*b*, the following give log sin x and log tan x. Log sec x also is given in the nautical tables (Burton, Caillet, Davis, Inman, Norie and Raper) and in Istituto Idrografico 1916; these also print the argument in arc (degrees, minutes and seconds). Any proportional parts given in the nautical tables and applying to the arc argument are disregarded here. Shortrede has arc as primary argument, but is given on account of its small interval.

7 dec.	$1''$ (Time argument also given to $0^s{\cdot}01$)	PPMD	Shortrede 1844, 1849*b*
7 dec.	1^s	S, T, Δ, PPd	Herz 1885
6 dec.	1^s	PPMD	Caillet 1916
6 dec.	1^s	No Δ	Inman 1891
6 dec.	2^s		Norie 1894, 1904
6 dec.	2^s		Raper 1920
5 dec.	1^s	Δ, PPd	**Ist. Idrog. 1916**
5 dec.	$0(0^s{\cdot}1)8^m(1^s)6^h$	S, T, fPPd	**Peters 1912**
5 dec.	2^s		Norie 1938

5 dec.	4^g (Only 4 dec. within 5° of an infinity)		Inman 1906, 1940
5 dec.	4^g (Only 3 or 4 dec. where Δ for 5 dec. > 100)	Δ	Burton 1941
5 dec.	4^g (Only 3 or 4 dec. within 5° of an infinity)	(Δ)	Davis 1905
5 dec.	$0(10^s)24^h$ (Log sin x, log cos x, only, with signs)	No Δ	Airy 1838*b*
4½ dec.	$0(1^s)20^m(10^s)6^h$	S, T, Δ	**Comrie 1931**
4½ dec.	$0(1^s)20^m(10^s)6^h$	fPPd	Desvallées 1920

8·6. Auxiliary Functions S and T, All Arguments

These are defined by

$$S = \log \frac{\sin x^u}{x} \quad \text{and} \quad T = \log \frac{\tan x^u}{x}$$

where u is any angular unit. They are auxiliary functions to help in dealing with small angles. We have taken no account of trivial differences in the characteristics of the logarithms.

Radians: log (sin x/x) and log (tan x/x)

4 d. crit.	To ·08715 for S and ·06161 for T	**MT.C. 1931** (4)

Degrees and Decimals: log (sin $x°/x$) and log (tan $x°/x$)

10 dec.	0(0°·001)2°·1		Δ	**Peters 1919*b***
8 dec.	0(0°·001)0°·3			Peters 1919*a*
8 dec.	0(0°·01)3°		Δ	J. Newton 1658
7 dec.	0(0°·001)1°			**Peters 1921*a***
6 dec.	0(0°·01)5°·1			Witkowski 1904
6 dec.	0(0°·01)5°			Silberstein 1923
6 dec.	0(0°·001)0°·4			**Peters 1921*b***
6 dec.	0(0°·1)4°	($-S$ and $-T$ tabulated)		**Bremiker 1937**
5 dec.	0°·1(0°·2)0°·9			Desvallées 1920
4 dec.	0(0°·1)10°			**Bremiker 1907**
4 d. crit.	To 4°·890 for S and 3°·602 for T			**MT.C. 1931** (4)

Five-decimal critical tables with log sin $x°$ and log tan $x°$ replacing x as argument are given in Witkowski 1904 and Silberstein 1923.

Grades: log (sin x^g/x) and log (tan x^g/x)

17 dec.	$0(1^g)50^g$	S only	v^7	**Andoyer 1911**
14 dec.	$0(0^g{\cdot}001)5^g$			Prony (MS. 1794)
8 dec.	$0(0^g{\cdot}01)5^g$			Service Géog. 1891
7 dec.	$0(0^g{\cdot}001)3^g$			**Peters 1940*b***
7 dec.	$0(0^g{\cdot}01)5^g$			Callet 1795
6 dec.	$0(0^g{\cdot}01)2^g{\cdot}5$			**Jordan–Eggert 1939**
5 dec.	$0(0^g{\cdot}02)4^g$			Service Géog. 1914 Potin 1925 (224)
5 dec.	$0(0^g{\cdot}01)2^g$			F. G. Gauss 1937

Gones: log (sin $2\pi x/x$) and log (tan $2\pi x/x$)

8 dec.	$x = 0(\cdot000001)\cdot012499$	Tamborrel 1891

More precisely, Tamborrel uses microgones, and the characteristics of S and T are $\bar{6}$.

Minutes of Arc: log (sin x'/x) and log (tan x'/x)

6 dec.	$0(5')1^\circ\,40'$	Becker 1928
5 d. crit.	To $3^\circ\,1'$ for S and $3^\circ\,4'$ for T	Hedrick 1920
4 d. crit.	To $5^\circ\,6'{\cdot}6$ for S and $3^\circ\,26'{\cdot}7$ for T	**MT.C. 1931** (4)
4 dec.	$0(1^\circ)10^\circ$ ($\pm S$ and $\pm T$ tabulated)	Plummer 1941

Seconds of Arc: log (sin x''/x) and log (tan x''/x)

22 dec.	$0(1'')10'$ T only		Δ^3	**Witt 1922**
15 dec.	$0(9')3^\circ$	$v^5(10'')$, $0(18')3^\circ$, for S only		**Andoyer 1911**
14 dec.	$0(10'')3^\circ$		Δ	
10 dec.	$0(10'')4^\circ\,30'$		Δ	U.S. C. & G.S. 1934
8 dec.	$0(1'')5^\circ$			**Bauschinger & Peters 1936**
8 dec.	$0(5'')1'\,40''(10'')3^\circ$		Δ	Schrön 1860
8 dec.	$0(50'')16'\,40''(10'')2^\circ\,46'\,30''$			Bruhns 1922
7 dec.	$0(1'')2^\circ$			**Peters 1911*b***
7 dec.	$0(50'')16'\,40''(10'')2^\circ\,46'\,30''$			Vega–Bremiker 1856

There is also an inverse table (page 575).

6 dec.	$0(50'')16'\,40''(10'')2^\circ\,46'\,30''$		**Bremiker 1924**
6 dec.	$0(1')5^\circ$		G. W. Jones 1893
6 dec.	$0(1')2^\circ$	(S and $\pm T$ tabulated)	Allen 1941 Ives 1934
5 dec.	$0(1')6^\circ$		Komendantov 1932
5 dec.	$0(1')4^\circ$		Service Géog. 1914 Potin 1925 (222)
5 dec.	$0(1')3^\circ$	($\pm S$ and $\pm T$ tabulated)	F. G. Gauss 1870

Albrecht 1908 (305) tabulates $S = \log(\sin x''/x)$ and $T = \log(\tan x''/x)$ to 7 decimals with first differences for $\log x = 2{\cdot}2({\cdot}01)3{\cdot}2({\cdot}001)4{\cdot}16$.

Seconds of Time: log (sin x^s/x) and log (tan x^s/x)

7 dec.	$0(10^s)20^m$	Δ	Herz 1885
5 dec.	$0(4^s)24^m$		Komendantov 1932
5 dec.	$0(10^s)20^m$		Peters 1912
4 d. crit.	To $1628^s{\cdot}8$ [$= 6^\circ\,47'$] for S and $1188^s{\cdot}8$ [$= 4^\circ\,57'$] for T		**Comrie 1931** (32)

8·7. Logarithmic Versines, Haversines, etc., All Arguments

8·71. Log vers $x = \log(1 - \cos x) = \log 2 + 2 \log \sin \frac{1}{2}x$

Vers x stands for the versed sine of x. For natural values see Art. 7·71.

7 dec.	$0(1')180^\circ$	No Δ	Douglas 1809
7 dec.	$0(15'')135^\circ = 0(1^s)9^h$	Δ	Farley 1856
7 dec.	$0(1')90^\circ$	No Δ	Hutton 1811
5 dec.	$20^s(5^s)6^h\,59^m\,55^s$	PP_5MD	Norie 1894, 1904 (T. 29)
5 dec.	$0(10^s)12^h$	No Δ	Bowditch 1874 (T. 23)

The function tabulated by Norie and Bowditch is $5 + \log(1 - \cos x)$.

3; 4 d.	$0(1')12^\circ$; $12^\circ(1')180^\circ$	No Δ	Hall 1920

8·715. Log $(2 \sin^2 \frac{1}{2}x \operatorname{cosec} 1'')$

For natural values see Art. 7·715.

5; 6 dec.	$0(1^s)30^m$; $30^m(1^s)2^h$	Δ, PPd	**Albrecht 1908** (208)
5; 6 dec.	$0(1^s)30^m$; $30^m(1^s)1^h$	No Δ; Δ, PPd	Ambronn-Domke 1909 (84)
5 dec.	$0(1^s)42^m$	No Δ	C. F. W. Peters 1871
5 dec.	$0(1^s)30^m$	No Δ	Chauvenet 1863

8·716. Log $(2 \sin^4 \frac{1}{2}x \operatorname{cosec} 1'')$

For natural values see Art. 7·716.

5 dec.	$10^m(10^s)42^m$	No Δ	C. F. W. Peters 1871
4 dec.	$20^m(10^s)2^h$	Δ	Albrecht 1908 (232)
4 dec.	$20^m(10^s)1^h$	Δ	Ambronn-Domke 1909 (94)
4 dec.	$0(1^m)12^m(10^s)30^m$	No Δ	Chauvenet 1863

8·72. Log hav x = log $\frac{1}{2}(1 - \cos x)$ = 2 log sin $\frac{1}{2}x$

Hav x = haversine of x = half versed sine of x. The function is used mainly for nautical purposes. For natural values see Art. 7·72.

5; 6; 7 d.	$0(1^s)45^m$; $45^m(1^s)2^h\ 25^m$; $2^h\ 25^m(1^s)12^h$	Δ, PP_{100}	**Hannyngton 1876**

Argument $0(15'')180°$ also printed, but PP_{100} are for $\cdot01^s(\cdot01^s)\cdot69^s$.

6 dec.	$0(15'')180° = 0(1^s)12^h$	$PP_{15}MD$	Norie 1904 (T. 31)
6 dec.	$0(15'')135°(1')180° = 0(1^s)9^h(4^s)12^h$	No Δ	Inman 1891 (T. 34)
5; 6 d.	$0(15'')10°\ 59'\ 45''$; $11°(15'')180°$; time also	$PP_{15}MD$	Raper 1920 (T. 69)
5 dec.	$0(15'')120°(30'')135°(1')180°$; time also	No Δ	Norie 1938
5 dec.	$0(5^s)8^h\ 59^m\ 55^s$	PP_5MD	Norie 1894 (T. 31, 32)
5 dec.	$0(30'')15°(15'')120°(30'')135°(1')180°$; time also	No Δ	Davis 1905; Inman 1906, 1940; Bowditch 1925
4; 5 d.	$0(30'')1°$; $1°(30'')120°(1')180°$; time also	No Δ	Burton 1941

8·73. $\frac{1}{2}$ log hav x = log sin $\frac{1}{2}x$

It will be seen that this is merely a table of logarithmic sines with the arguments doubled.

6 dec.	$0(15'')15°(30'')60°(1')180°$; time also	No Δ	Inman 1891 (T. 33); Inman 1906 (400)

Inman 1940 (408) has the same arguments but only 5 decimals.

8·74. $-\frac{1}{2}$ log hav x = log cosec $\frac{1}{2}x$

6 dec.	$0(1')180°$ (Hour Angle, pp. 10*–36*)	No Δ	Aquino 1924

SECTION 9

INVERSE CIRCULAR FUNCTIONS. TRIGONOMETRICAL FUNCTIONS OF TWO OR THREE ARGUMENTS (INCLUDING PRODUCTS, SOLUTIONS OF PLANE AND SPHERICAL TRIANGLES, ETC.). SEXAGESIMAL INTERPOLATION TABLES

9·0. Introduction

Except in the case of inverse circular functions, many of the tables mentioned in this section occur in connection with navigation or nautical astronomy. We have not thought it right to exclude a field in which many millions of values of a purely mathematical character have been tabulated (as for instance in the triple-entry tables of Arts. 9·7 ff.). We have however confined ourselves almost entirely to navigational works in the English language, as we have encountered only a few continental ones. Additional references will be found in C. W. Wirtz, *Enc. d. Math. Wiss.*, VI 2, 1. Hälfte, 80–162, 1905.

The main subdivisions of the present section are as follows:

9·0 Introduction
9·1 Inverse Circular Functions
9·2 Mainly Multiples of Trigonometrical Functions
9·3 Tables concerning Right-Angled Plane Triangles
9·4 Tables concerning General Plane Triangles
9·5 Products of Trigonometrical Functions
9·6 Tables concerning Right-Angled (or Right-Sided) Spherical Triangles
9·7 Tables concerning General Spherical Triangles
9·8 Sexagesimal Interpolation Tables

9·1. Inverse Circular Functions

Tables of most of these are not particularly numerous. We consider separately the cases where the functions are tabulated in radians, in degrees and decimals, in grades, in degrees, minutes, etc. and in time.

Tables of inverse logarithmic trigonometrical functions (e.g. $\sin^{-1} x$ with argument $\log x$) are given in Schubert 1897, while A. N. Lowan, *M.T.A.C.*, **1**, 94, 1943, also mentions such tables by J. M. Peirce.

For inverse trigonometrical functions with complex argument see Art. 12·8.

9·11. $\sin^{-1} x$ and $\cos^{-1} x$ (Radians)

The New York W.P.A. manuscript and Davis 1941 give $\sin^{-1} x$ only; the others give both functions. On errors in Hayashi see *M.T.A.C.*, **1**, 196, 1944.

12 dec.	0(·001)·99(·00001)1	δ^4	New York W.P.A. (MS.)
	See A. N. Lowan, *M.T.A.C.*, **1**, 94, 1943.		
20; 10; 7 dec.	0(·00001)·001; ·001(·0001)·0999; ·1(·001)1	No Δ	Hayashi 1926
6 dec.	0(·001)1	Δ	R. A. Davis 1941

5 dec.	0(·001)·89(·0001)·999(var.)1		Schubert 1897
5 dec.	0(·01)1, also $\pi/4$	No Δ	Hayashi 1930*b*
5 dec.	0(·01)1	No Δ	Prévost 1933 (134*)
$4\frac{1}{2}$ dec.	0(·001)·99(·0001)1	Δ	**MT.C. 1931**
4 dec.	0(·001)1	IΔ	Dwight 1941

9·12. Tan^{-1} *x* and cot^{-1} *x* (Radians)

MT.C. 1931 and Schubert 1897 give $\tan^{-1} x$ and $\cot^{-1} x$; the remainder give $\tan^{-1} x$ only.

15; 23 d.	0(·0001)·01; ·01(·01)·1(·1)1		Miller (MS.)
12 dec.	0(·001)7(·01)50(·1)300(1)2000(10)10^4	δ^2	**New York W.P.A. 1942*d***
11 dec.	0(·01)1	v^4	Roman 1935
20; 10 d.	0(·00001)·001; ·001(·0001)·0999	No Δ	Hayashi 1926
8; 7 d.	·1(·001)2·999; 3(·01)10(·1)20(1)50		

On errors see Comrie 1938.

9 dec.	1(1)100	No Δ	Spence 1809 (63)
7 dec.	0(·01)5(·1)20(1)164	δ^2	**Comrie 1938**
5? dec.	0(·001)1·9(·01)6(·1)20(var.)3000		Schubert 1897
5 dec.	0(·01)10, also multiples of $\pi/4$	No Δ	Hayashi 1930*b*

On errors see New York W.P.A. 1942*d* (xxiv).

$4\frac{1}{2}$ dec.	0(·001)1·5(·01)5(·1)15(1)60	Δ	MT.C. 1931
4 dec.	0(·001)1(·01)4(·1)20(1)50	IΔ	Dwight 1941
4 dec.	0(·001)1	No Δ	Miller 1939

9·13. Sec^{-1} *x* and cosec^{-1} *x* (Radians)

$4\frac{1}{2}$ dec.	1(·0001)1·01(·001)1·5(·01)5(·1)15(1)60	Δ	MT.C. 1931

9·15. Tan^{-1} *x* and cot^{-1} *x* (Degrees and Decimals)

Miller and Seeley give $\tan^{-1} x$ and $\cot^{-1} x$, Jahnke & Emde $\tan^{-1} x$ only.

0°·00001	0(·01)1	Δ^3	Miller (MS.)
0°·0001	0(·001)·099	No Δ	Jahnke–Emde 1933 (17)
0°·001	·1(·01)1		
0°·001	1(·001)3(·01)10(·5)30(10)50	No Δ	Seeley 1936
0°·001	0(·001)1	Δ	Miller (MS.)
0°·01	0(·001)1	No Δ	Miller 1939

9·172. Tan^{-1} *x* (Grades)

0^g·01	0(·001)1·02	No Δ	Potin 1925 (210)

9·181. Sin^{-1} *x* (Degrees, Minutes and Seconds)

0″·000001	0(·01)1			Miller (MS.)
1″	0(·001)1		Δ	Kepler 1624
1′	0(·01)1	$\cos^{-1} x$ also given	No Δ	Prévost 1933 (134*)

See also Art. 9·33.

9·182. $\tan^{-1} x$ (Degrees, Minutes and Seconds)

0″·001	0(·00001)·01(·0001)1		Fergusson 1912

Errors greater than 10 final units are frequent.

0′·1; 1′	0(·01)·34; ·35(·01)1·02	No Δ	Trautwine 1937 (256)
1′	0(·001)1·02	No Δ	Potin 1925 (210)

Peurbach gave a sexagesimal table of $\tan^{-1} x$ for $x = 0(\frac{1}{1200})1$. See Cantor, *Geschichte der Math.*, **2**, 185, 1900 and De Morgan 1861, column 984.

9·19. $\cos^{-1} x$ (Time)

See the item Prussian Survey 1912 in Part II.

9·2. Mainly Multiples of Trigonometrical Functions

We deal here with some double-entry tables involving one angular and one non-angular argument. Tables of multiples of trigonometrical functions are chiefly concerned, but Art. 9·24 deals with tables of a different character.

9·21. Multiples of Sines and Cosines (Traverse Tables, etc.)

Many of the following are contained in nautical and surveying works under the name of "traverse tables". Such tables give values of $N \cos x$ and $N \sin x$ for various integral values of N and various values of x. The two quantities $N \cos x$ and $N \sin x$ are often called "difference of latitude" and "departure" (from the meridian) respectively, as they represent distances traversed in the north-south and east-west directions by a ship which travels a distance N miles (not too large) on a course inclined at an angle x to the meridian. The arrangement is usually semi-quadrantal, but we have indexed all items as tables of $N \sin x$, where x takes values throughout the quadrant.

Interpolation in N may obviously be performed by using the table itself.

We lack detailed information about the item C. W. Scott 1905 in Part II.

4 dec.	$N = 1(1)100$, $x = 0(1')90°$	**Gurden 1906**
2 dec.	$N = 1(1)10(10)100(100)1000(1000)10{,}000$, $x = 0(1')90°$	Crellin 1881
5 dec.	$N = 1(1)10$, $x = 0(1')90°$	Boileau 1839
5 dec.	$N = 1(1)9$, $x = 0(15')90°$	Woodward 1906
5 dec.	$N = 1(1)9$, $x = 0(1°)90°$	Lambert 1770, 1798; Potin 1925 (403)
4 dec.	$N = 1(1)10$, $x = 0(1')90°$	Louis & Caunt 1901
4 dec.	$N = 1(1)10$, $x = 0(1°)90°$	Hutton 1811
3 dec.	$N = 1(1)100$, $x = 0(2')90°$	Shortrede 1864 (Sang)
3 dec.	$N = 1(1)10$, $x = 0(15')90°$, quadrantal	C. M. L. Scott 1934
2 dec.	$N = 1(1)500$, $x = 0(10')90°$	Massaloup 1847

Also 1(1)109 times sin 0(1′)15°. On arrangement see B.A. 1873.

1 dec.	$N = 1(1)720$, $x = 0(1°)90°$	Pinto 1933
1 dec.	$N = 1(1)600$, $x = 0(1°)90°$, also time arg.	Raper 1920
1 dec.	$N = 1(1)600$, $x = 0(1°)90°$	Burton 1941; Inman 1906, 1940; Norie 1904, 1938; Bowditch 1925

1 dec.	$N = 1(1)400(100)1000$, $x = 0(30')15°(1°)75°(30')90°$	Blackburne 1923
1 dec.	$N = 1(1)300$, $x = 0(1°)90°$	Pryde 1930

The following tables have centesimal angular argument.

6 dec.	$N = 1$	$x = 0(0^g{\cdot}01)100^g$, also sexages. arg.	Clouth 1906
4 dec.	$N = 2(1)9$		
? dec.	$N = 1(1)100$,	$x = 0(1^g)100^g$	Sang (MS. 1848–90)

See Sang 1884*a* (543).

9·215. $N\cos^2 x$ and $\frac{1}{2}N\sin 2x$

Closely connected with traverse tables are the tacheometric tables giving $\frac{1}{2}N(1+\cos 2x)$ and $\frac{1}{2}N\sin 2x$. Such tables are fairly common.

$N\cos^2 x$	1 dec.	$N = 10(1)250$,	$x = 0(1')30°$	W. Jordan 1908
$\frac{1}{2}N\sin 2x$	2 dec.	$N = 10(1)250$,	$x = 0(2')30°$	
Both fns.	5 dec.	$N = 1(1)4$	$x = 0(0^g{\cdot}02)31^g$	Jadanza–Hammer 1909
	4 dec.	$N = 5(1)9$		
Both fns.	4 dec.	$N = 1(1)10$,	$x = 0(1')30°$	Louis & Caunt 1919

9·22. $N\tan x$

1 dec.	$N = 51(1)100$	$x = 0(1^g)50^g$	Stumpff 1939 (86)
0 dec.	$N = 102(2)500$		

9·23. $N\sec x$

Values of $N\sec x$ occur in tables giving difference of longitude for (small) departure from the meridian in a given latitude, since

$$\delta(\text{longitude}) = \text{departure} \times \sec(\text{latitude})$$

4–5 d.	$N = 1(1)9$,	$x = 0(1°)90°$		Potin 1925 (405)
2 dec.	$N = 1(1)10$,	$x = 4°(2°)14°(1°)69°$	PP	Raper 1920 (T. 3)
2 dec.	$N = \frac{1}{2}$, $1(1)10$,	$x = 4°, 8°(2°)14°(1°)70°$		Inman 1891 (T. 4)
1 dec.	$N = 1(1)100$,	$x = 4°, 8°(2°)14°(1°)59°$		Inman 1906 (88); Inman 1940 (104)
1 dec.	$N = 1(1)60$,	$x = 4°, 8°(2°)14°(1°)40°(30')59°$		Norie 1938 (94)
1 dec.	$N = 51(1)100$	$x = 0(1^g)50^g$		Stumpff 1939 (86)
0 dec.	$N = 102(2)500$			

According to Mehmke 1900 (150), Ulffers 1870 tabulates $N\operatorname{cosec} x$ for $N = 10(10)90$ and x at interval $0^g{\cdot}01$.

9·24. $\operatorname{Cot}^{-1}(k\cos x)$

This is the inclination to the east-west direction of a short displacement in latitude x for which d(longitude)/d(latitude) $= k$. It may thus be of use in connection with spherical polar co-ordinates.

$0°{\cdot}1$	$k = 0(\cdot01)1(\cdot02)2(\cdot1)5(1)20(\text{var.})800$; $x = 0(2°)10°(1°)65°$	Gingrich 1931 (48)
$0°{\cdot}1$	$k = 0(\cdot01)\cdot6(\cdot02)1{\cdot}2(\cdot04)2{\cdot}4(\cdot05)2{\cdot}8(\cdot1)5(\text{var.})800$; $x = 0(5°)10°(4°)18°(2°)30°(1°)83°$	Norie 1938 (515, 529)

$0^\circ\cdot 1 \quad \left\{\begin{array}{l} k = 0(\cdot 01)\cdot 1(\cdot 02)\cdot 5(\cdot 03)\cdot 8(\cdot 04)1(\cdot 05)1\cdot 5(\cdot 1)2\cdot 4(\cdot 2) \\ \quad 3(\text{var.})30 \\ x = 0,\ 10^\circ(5^\circ)30^\circ(4^\circ)42^\circ(3^\circ)60^\circ(2^\circ)68^\circ \end{array}\right\}$ Norie 1894, 1904 (T. 28)

Blackburne 1914 (T. 4) and Pinto 1933 (T. 14) are other tables to $0^\circ\cdot 1$.

9·3. Tables concerning Right-Angled Plane Triangles

We have to deal essentially with three sides and one angle, since the two angles other than the right angle are complementary and hence simply connected numerically.

9·31. Given an Angle and a Side

When the given side is the hypotenuse, traverse tables are of use if, for example, the angle is an integral number of degrees and the hypotenuse a two-figure one. For the various other possible combinations of angle and hypotenuse, see Art. 9·21. Traverse tables rarely give more than 4-figure accuracy. It is usually easy to interpolate (by means of the table itself) for a 4-figure hypotenuse, but awkward to interpolate in angle.

The ordinary traverse table, as contained in Pryde 1930 and in most nautical tables, exhibits hypotenuse, base and perpendicular (usually called distance, difference of latitude, and departure) in three parallel columns. The hypotenuse proceeds at unit interval, or other simple interval, and is a tabular argument, but it is easy to use the table inversely, i.e. to enter it with given base or perpendicular, using interpolation. Thus when the angle is simple, and a non-hypotenuse side is given, the other sides may be read off, to about 4-figure accuracy. As before, interpolation in angle is usually less convenient.

9·32. Given Two Sides

If the base and perpendicular are given, use may be made of tables for the conversion from rectangular to polar co-ordinates, such as **Miller 1939**, Seeley 1936 and Jahnke & Emde 1933 (17). These tabulate such quantities as $\sqrt{1+x^2}$ and $\tan^{-1} x$. If the hypotenuse is one of the given sides, tables of such quantities as $\sqrt{1-x^2}$ and $\sin^{-1} x$ may be useful. Even traverse tables may be useful for rough work.

Details of tabulations of *natural* values of such quantities as $\sqrt{1 \pm x^2}$ will be found in Arts. 2·64–2·6425. Inverse trigonometrical functions are treated in Arts. 9·1–9·19. We note a few tabulations of *logarithms* of such quantities as $\sqrt{1 \pm x^2}$.

Comrie 1938 tabulates $\log(1+x^2)$ to 7 decimals with δ^2 for $x = 0(\cdot 01)5(\cdot 1)15$.

Fergusson 1912 contains tables of $\log\sin A$, $\log\cos A$ and $\log\sec A$, where $x = \tan A$, for $x = 0(\cdot 00001)\cdot 01(\cdot 0001)1$, to 7 decimals, with PPMD. The functions tabulated are $\log \dfrac{x}{\sqrt{1+x^2}}$, $\log \dfrac{1}{\sqrt{1+x^2}}$ and $\log \sqrt{1+x^2}$ respectively.

The same work also contains tables of $\log\tan\frac{1}{2}A$ and $\log(1-\cos A)$, where $x = \tan A$, for $x = \cdot 010(\cdot 001)1$, to 7 decimals, with PPMD. The functions tabulated are $\log \dfrac{\sqrt{1+x^2}-1}{x}$ and $\log\left\{1 - \dfrac{1}{\sqrt{1+x^2}}\right\}$ respectively.

It also tabulates $\tan\frac{1}{2}A = \dfrac{\sqrt{1+x^2}-1}{x}$ for $x = \cdot 010(\cdot 001)1$, mainly to 6 figures, as mentioned in Art. 2·6425.

See also Art. 9·182.

9·33. Other Tables concerning Right-Angled Plane Triangles

Values of $\sin^{-1}(m/n)$ are given to the nearest degree as "track errors" in Hughes–Comrie 1938 (168) for $m = 2(1)15(5)45$ and $n = 25(5)150(10)290$. C. M. L. Scott 1934 gives $\sin^{-1}(I/D)$ to 1″ for

$$I = 0(\cdot 1)1(1)10 \quad \text{and} \quad D = 500(1)1000(2)1500(5)2000(10)5000$$

Stumpff 1939 (81) gives $\sqrt{a^2+b^2}$ to 1 decimal and $\tan^{-1}(a/b)$ to the nearest grade for $a = 1(1)50$ and $b = 1(1)50$. Gauss 1938 (92) gives $\sqrt{a^2+b^2}$ to 2 decimals for $a = 0(\cdot 02)1$ and $b = 0(\cdot 02)2\cdot 38$. See also Turner 1913 (37).

There are various tables of right-angled triangles with integral sides (see Lehmer 1941), and of these the following give also the angles.

Bretschneider 1841 gives to 0″·1 the angles of all such right-angled triangles of which the sides are $a = 2mn$, $b = m^2-n^2$, and $c = m^2+n^2$, where $m = 2(1)25$ and $n < m$; the cases in which a, b and c have a common factor are excluded, i.e. the triangles are "primitive". Bretschneider reports a somewhat similar table in Schulze 1778 as inaccurate, and lists the errors. See also Ligowski 1893 (39).

Sang 1864 gives the sides and (to 0″·01) the smallest angle of all right-angled triangles with integral sides and hypotenuse not exceeding 1105. See also Art. 9·44.

9·4. Tables concerning General Plane Triangles

There are a number of double-entry and triple-entry tables which may be of some help in working with 2 or 3 figures only.

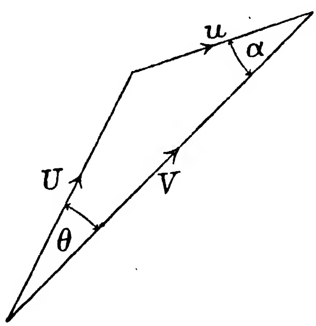
9·41. Given Two Sides and the Included Angle. Vector Addition

Tables of r and θ with arguments k and α, where $re^{i\theta} = 1+ke^{i\alpha}$, are given as follows:

r to 4 dec. θ to $0^g\cdot 001$ or $0^g\cdot 01$	$k = 0(\cdot 1)1$ $\alpha = 0(5^g)200^g$	Jahnke–Emde 1933
r to 1 dec. θ to 1°	$k = 0(\cdot 1)4(\cdot 2)6(\cdot 5)10$ $\alpha = 0(10^\circ)360^\circ$	Doodson & Warburg 1936 (T. 3)

9·42. Given Two Sides and a Non-included Angle

There are a number of tables of speed and course correction for sea and air navigators. We mention a few as typical.

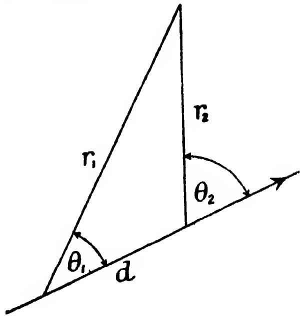


With the notation of the diagram, Hoar 1918 gives a triple-entry table showing V to 1 decimal and θ to 1° (or to 15′ if $\theta < 1°$) when $U = 3(1)30$, $u = 0(\cdot 5)8$ and $\alpha = 0(10°)180°$, except that θ is not given when $u > U$.

Some tables have $180° - \alpha$ instead of α as the angular argument.

Among double-entry tables, Hughes–Comrie 1938 gives in Table 8 V/U to 2 decimals for $u/U = 0(\cdot 01)\cdot 5$ and $180° - \alpha = 0(10°)180°$, and in Table 7 θ to 1° for $u/U = 0(\cdot 02)\cdot 5$ and $180° - \alpha = 0(10°)180°$. Norie 1938 (102) and *Brown's N.A.* 1935 (174) give V/U to 2 decimals and θ to 0°·1 for $u/U = 0(\cdot 05)\cdot 7$ and $\alpha = 0(10°)180°$.

9·43. Given Two Angles and One Side



If a ship observes an object to have bearings at angles θ_1 and θ_2 to the direction of the ship's motion at two instants, between which the ship has travelled a distance d, the distance r_2 of the object at the second instant is given by $\frac{r_2}{d} = \frac{\sin \theta_1}{\sin (\theta_2 - \theta_1)}$. The following "bow-angle tables" tabulate the function $\sin \theta_1 / \sin (\theta_2 - \theta_1)$.

2 dec.	$\theta_1 = 20°(2°)150°,\ \theta_2 - \theta_1 = 10°(2°)160°$	Bowditch 1925 (T. 5B)
2 dec.	$\theta_1 = 20°(2°)90°,\ \theta_2 - \theta_1 = 10°(2°)90°,\ \theta_2 < 160°$	Norie 1938 (110)
2 dec.	$\theta_1 = 20°(2°)90°,\ \theta_2 - \theta_1 = 20°(2°)90°,\ \theta_2 < 160°$	Inman 1906 (4); Inman 1940 (4)
2 dec.	$\theta_1 = 20°(1°)90°,\ \theta_2 = \overline{\theta_1 + 10°}(1°)120°$	Jauget 1906

9·44. Rational Triangles

Sang 1864 gives the sides and (to 0″·1) the smallest angle of all triangles with (i) one angle equal to 120°, (ii) sides integral, (iii) the largest side less than 1000. See also Ligowski 1893 (39). Rational right-angled triangles have been considered in Art. 9·33.

9·5. Products of Trigonometrical Functions

We may take $\sin x$, $\tan x$ and $\sec x$ as the three basic functions, since $\cos x$, $\cot x$ and $\operatorname{cosec} x$ respectively differ from them trivially. There are thus six products of two functions, namely $\sin x \sin y$, $\sin x \tan y$, $\sin x \sec y$, $\tan x \tan y$, $\tan x \sec y$ and $\sec x \sec y$. Double-entry tables of all six products have been formed for various purposes. We consider each product in turn. The products are often multiplied by 10 or 100 in tabulation, but we have treated a table of $100 \tan x \tan y$ to 1 decimal as a table of $\tan x \tan y$ to 3 decimals, and so on.

9·51. Sin x sin y

This is the least important case, since $2 \sin x \sin y = \cos (x-y) - \cos (x+y)$ may be used.

7 dec.	$x = 0(1°)90°,\ y = 0(1°)90°$ First differences are given in one argument.	Otho 1596
4 dec.	$x = 90° - \text{Lat.} = 25°(1°)90°,$ $y = 90° - \text{Az.} = 0(\tfrac{1}{2}°)90°$	Lecky 1923 (T. G)
2 dec.	$x = 90° - L = 20°(5°)90°,\ y = Z' = 0(2°)90°$	Aquino 1924 (170)

9·52. Sin x tan y

4 dec.	$x = \text{Lat.} = 0(1°)65°,\ y = \text{Alt.} = 0(10')50°$	Lecky 1923 (T. E)
2 dec.	$x = \text{Az.} = 0(2°)90°$ $y = \text{Lat.} = 0(10°)30°(2°)60°$	Littlehales 1923 (on correction slip to T. 3)
2 dec.	$x = 90° - \text{Lat.} = 30°(1°)42°(2°)72°,\ 75°(5°)90°$ $y = \text{Az.} = 0(2°)20°(1°)70°$	Blackburne 1923 (T. 75)
2 dec.	$x = 90° - \text{Lat.} = 30°(2°)88°(1°)90°$ $y = \text{Az.} = 0(1°)8°(2°)80°(1°)90°$	Bowditch 1925 (T. 48)

9·53. Sin x sec y

4 dec.	$x = \text{Dec.} = 0(1^\circ)65^\circ,\ y = \text{Alt.} = 0(10')50^\circ$	Lecky 1923 (T. F)
3 dec.	$\left\{\begin{array}{l} x = z = -10^\circ(1^\circ)+50^\circ \\ x+y = 90^\circ - z' = 20^\circ(2^\circ)30^\circ(1^\circ)70^\circ \end{array}\right\}$	Albrecht 1908 (196)

For tables to 2 decimals see Art. 9·43.

9·54. Tan x tan y

3 d. or 4 f.	$\left\{\begin{array}{l} x = \text{Lat.} = 0(1^\circ)65^\circ \\ y = 90^\circ - \text{H.A.} = 0(1^\circ)30^\circ(\tfrac{1}{2}^\circ)75^\circ(\tfrac{1}{4}^\circ)90^\circ \end{array}\right\}$ The argument y is also given in time.	Lecky 1923 (T. A)
3 dec.	$x = 0(1^\circ)80^\circ,\ y = 0(1^\circ)80^\circ$	Raper 1920 (T. 5)
3 d. or 3 f.	$\left\{\begin{array}{l} x = \text{Lat.} = 0(1^\circ)70^\circ \\ y = 90^\circ - \text{H.A.} = 0(1^\circ)90^\circ \end{array}\right\}$	Myerscough & Hamilton 1939 (column X)
2–3 d.	$\left\{\begin{array}{l} x = \text{Lat.} = 0(1^\circ)65^\circ \\ y = 90^\circ - \text{H.A.} = 0(1^\circ)90^\circ \end{array}\right\}$	Gingrich 1931 (col. X)

In the following tables y is measured in time. The function is usually described as $A = \tan \text{Lat.} \cot \text{H.A.}$, the printed arguments being Lat. and H.A. In each case $x = \text{Lat.}$ and $y = 6^{\text{h}} - \text{H.A.}$

About 3 f.	$x = 0(1^\circ)60^\circ,\ y = 0(1^{\text{m}})6^{\text{h}}$	Blackburne 1923 (T. 17)
3 d. or 3 f.	$x = 0(1^\circ)82^\circ,\ y = 0(4^{\text{m}})4^{\text{h}}(2^{\text{m}})6^{\text{h}}$	Burton 1941 (p. 232 and supplement, p. 4)
3 d. or 4 f.	$x = 0(1^\circ)60^\circ,\ y = 0(4^{\text{m}})4^{\text{h}}(2^{\text{m}})5^{\text{h}}\,30^{\text{m}}(1^{\text{m}})6^{\text{h}}$	Lecky 1892 (T. A)
2 d. or 3 f.	$\left\{\begin{array}{l} x = 0(\text{var.})10^\circ(1^\circ)83^\circ \\ y = 0(10^{\text{m}})2^{\text{h}}(4^{\text{m}})4^{\text{h}}(\text{var.})6^{\text{h}} \end{array}\right\}$	Norie 1938 (492, 512)

9·55. Tan x sec y

(i) In the following tables the function tabulated is mainly cot Az. sec Lat., the printed arguments being azimuth and latitude. The function gives d(Long.)/d(Lat.), and is often denoted by C.

3 d. or 4 f.	$x = 90^\circ - \text{Az.} = 0(\tfrac{1}{2}^\circ)90^\circ,\ y = \text{Lat.} = 0(1^\circ)65^\circ$	Lecky 1923 (T. C)
3 d. or 4 f.	$x = 90^\circ - \text{Az.} = 0(1^\circ)90^\circ,\ y = \text{Lat.} = 0(1^\circ)60^\circ$	Lecky 1892 (T. C)
2 d. or 3 f.	$\left\{\begin{array}{l} x = 90^\circ - \text{Az.} = 15^\circ(1^\circ)90^\circ,\ y = \text{Lat.} = 0(1^\circ)65^\circ \\ x = 90^\circ - \text{Az.} = 0(1^\circ)90^\circ,\ y = \text{Lat.} = 65^\circ(1^\circ)83^\circ \end{array}\right\}$	Norie 1938 (207, 212)
About 3 f.	$x = 90^\circ - \text{Az.} = 0(1^\circ)85^\circ,\ y = \text{Lat.} = 0(1^\circ)82^\circ$	Burton 1941 (p. 248 and supplement, p. 20)
About 3 f.	$x = 0(1^\circ)70^\circ,\ y = 0(1^\circ)90^\circ$	Myerscough & Hamilton 1939 (column Y)

(ii) In the following tables the function tabulated is tan Dec. cosec H.A., often denoted by B. The printed arguments are Dec. and H.A., the latter usually being in time, with interval sometimes less than $4^{\text{m}} = 1^\circ$. In each case $x = \text{Dec.}$ and $y = 6^{\text{h}} - \text{H.A.}$ or $90^\circ - \text{H.A.}$

3 d. or 4 f.	$x = 0(1^\circ)65^\circ$, $y = 0(1^\circ)30^\circ(\frac{1}{2}^\circ)75^\circ(\frac{1}{4}^\circ)90^\circ$	Lecky 1923 (T. B)
3 d. or 4 f.	$\begin{cases} x = 0(1^\circ)29^\circ \\ y = 0(4^m)4^h(2^m)5^h\ 30^m(1^m)6^h \end{cases}$	Lecky 1892 (T. B)
About 3 f.	$\begin{cases} x = 0(1^\circ)60^\circ, & y = 0(1^m)6^h \\ x = 60^\circ(\frac{1}{2}^\circ)90^\circ, & y = 0(2^m)6^h \end{cases}$	Blackburne 1923 (T. 17, 19)
3 d. or 3 f.	$x = 0(1^\circ)63^\circ$, $y = 0(4^m)4^h(2^m)6^h$	Burton 1941 (p. 233 and supplement, p. 5)
2 d. or 3 f.	$\begin{cases} x = 0(\text{var.})\,10^\circ(1^\circ)68^\circ \\ y = 0(10^m)2^h(4^m)4^h(\text{var.})6^h \end{cases}$	Norie 1938 (493)
2–3 dec.	$x = 0(1^\circ)65^\circ$, $y = 0(1^\circ)90^\circ$	Gingrich 1931 (col. *Y*)

9·56. Sec x sec y

3 dec.	$x = 0(1^\circ)80^\circ$, $y = 0(1^\circ)80^\circ$	Raper 1920 (T. 5)
4 fig.	$x = \text{Lat.} = 0(1^\circ)65^\circ$, $y = 90^\circ - \text{Az.} = 0(1^\circ)90^\circ$	Lecky 1923 (T. D)
4 fig.	$x = \text{Lat.} = 0(1^\circ)60^\circ$, $y = 90^\circ - \text{Az.} = 0(1^\circ)90^\circ$	Lecky 1892 (T. D)

9·565. Log (sec x sec y)

3 dec.	$x = \text{Lat.} = 0(1^\circ)90^\circ$, $y = 90^\circ - h = 0(1^\circ)90^\circ$	Hughes–Comrie 1938 (T. 1, quantity *D*)
3 dec.	$x = L = 0(1^\circ)65^\circ$, $y = 90^\circ - t = 0(1^\circ)90^\circ$	Dreisonstok 1935

Table 1, quantity *C*. Table 1A, contained in the third and sixth editions only, is an extension for $x = 66^\circ(1^\circ)90^\circ$. For further information and errata, see C. H. Smiley, *M.T.A.C.*, **1**, 79 and 87, 1943.

9·57. Sin² ½x sin² ½y

In calculating zenith distance by the formula

$$\sin^2 \tfrac{1}{2}z = \sin^2 \tfrac{1}{2}(\phi - \delta)\cos^2 \tfrac{1}{2}t + \cos^2 \tfrac{1}{2}(\phi + \delta)\sin^2 \tfrac{1}{2}t$$

both terms on the right may be taken from a single table, which is given in Soeken 1914 at interval 1° in $\phi \mp \delta$ and 1^m in t, with $\Delta/60$ in the first argument.

9·6. Tables concerning Right-Angled (or Right-Sided) Spherical Triangles

The alternative in the heading refers to the fact that the supplements of the angles and the supplements of the sides of a spherical triangle form the sides and angles respectively of the polar triangle; thus the polar of a right-sided triangle is a right-angled triangle, and the latter will be taken as the standard figure.

We shall suppose that the angle C is 90°. The formulæ are then:

$$\begin{cases} \sin a = \sin c \sin A \\ \sin b = \sin c \sin B \end{cases} \qquad \begin{cases} \tan b = \tan c \cos A \\ \tan a = \tan c \cos B \end{cases}$$

$$\cos c = \cos a \cos b \qquad \begin{cases} \tan a = \sin b \tan A \\ \tan b = \sin a \tan B \end{cases}$$

$$\begin{cases} \cos A = \cos a \sin B \\ \cos B = \cos b \sin A \end{cases} \qquad \cos c = \cot A \cot B$$

As Napier's rules emphasize, if we regard cos, cot and cosec as sin, tan and sec of the complementary arguments, the above formulæ can all be put in one or other of two forms, namely

$$\sin z = \sin x \sin y \qquad \sin z = \tan x \tan y$$

Thus whatever the two given elements and the one to be calculated, the latter is given essentially by one of four forms, which we may write as

$$\begin{array}{ll} \sin^{-1}(\sin x \sin y) & \qquad \sin^{-1}(\tan x \tan y) \\ \sin^{-1}(\sin x \sec y) & \qquad \tan^{-1}(\sin x \tan y) \end{array}$$

Tables of these functions are considered in Arts. 9·61 to 9·64, as well as a related function in Art. 9·612.

That tables ostensibly giving different information about spherical triangles may be substantially equivalent, i.e. derivable from one another by the mere taking of complements or supplements of some or all of the arguments and the function, is an important point; in Hughes–Comrie 1938 (xxxi) Comrie alludes to a case in which several table-makers have apparently missed it.

In each case we have stated the relation between the notation of the article and the notation (or one of the notations) of the author, in order to facilitate recognition of the quantities referred to.

On the use of some of the tables in solving the general spherical triangle, see Art. 9·7.

We lack adequate detailed information about tables by Aquino (1943), Souillagouët, Bertin, and Newton & Pinto which we have included in Part II.

9·61. $z = \sin^{-1}(\sin x \sin y)$

0′·1	$z =$ Dec.	$\left\{\begin{array}{l} x = 90^\circ - a = 0(1^\circ)90^\circ \\ y = b = 0(1^\circ)90^\circ \end{array}\right\}$	Aquino 1927
1′	$z = d$	$\left\{\begin{array}{l} x = 90^\circ - a = 0(1^\circ)6^\circ(30')90^\circ \\ y = b = 0(1^\circ)90^\circ \end{array}\right\}$	Aquino 1924
1′	$z = h$	$\left\{\begin{array}{l} x = 90^\circ - D = 0(0^\circ{\cdot}5)34^\circ(1^\circ)90^\circ \\ y = 90^\circ - V' = 0(1^\circ)90^\circ \end{array}\right\}$	Pierce 1931 (T. 2)
1′	$z = 90^\circ -$ Course	$\left\{\begin{array}{l} x = \text{Lat. of vertex} = 0(1^\circ)90^\circ \\ y = \text{Long. from vertex} = 0(1^\circ)90^\circ \end{array}\right\}$	Towson 1912
1′		$x = 90^\circ - b = 0(1^\circ)90^\circ,\ y = 90^\circ - a = 0(1^\circ)90^\circ$	Kelvin 1876 Tillman 1936 (T. 9)
1′	$z =$ Co-hyp.	$\left\{\begin{array}{l} x = 90^\circ - b = 0(1^\circ)90^\circ \\ y = 90^\circ - a = 32^\circ(1^\circ)39^\circ \end{array}\right\}$	Kelvin 1871
0°·1	$z = 90^\circ -$ Z.D.	$\left\{\begin{array}{l} x = 90^\circ - \text{Dec.} = 0(1^\circ)90^\circ \\ y = 90^\circ - \text{H.A.} = 0(4^m)6^h \end{array}\right\}$	Goodwin 1921

A table for $x = 0(1^\circ)90^\circ$, $y = 0(1^\circ)90^\circ$ is given in Otho 1596. The results are stated in degrees, minutes and seconds, but not always to the nearest second; for example, they may be rounded to the nearest multiple of 10″. First differences are given in one argument.

9·612. $w = \log\sqrt{1 - \sin^2 x \sin^2 y}$

Smart and Shearme 1924 gives $\log\cos z$, the others $\log\sec z$, where $\sin z = \sin x \sin y$. The tabulations have been made for nautical purposes, but may be useful in connection with elliptic integrals. In each case $x = 90^\circ -$ Lat. and $y =$ H.A.

5 dec.	$x = 30^\circ(15')90^\circ,\ y = 0(1^m)6^h$	Pinto 1933 (T. 12)
5–6 dec.	$w = -B,\ x = 0(1^\circ)90^\circ,\quad y = 0(1^\circ)90^\circ$	Ageton 1942

5 dec.	$w = -A$, $x = 0(1^\circ)90^\circ$, $y = 0(1^\circ)90^\circ$		Hughes–Comrie 1938 (T. 1)
5 dec.	$w = V$, $x = 23^\circ(1^\circ)90^\circ$, $y = 0(4^m)6^h$		Smart & Shearme 1924
5 dec.	$w = -A$, $x = 25^\circ(1^\circ)90^\circ$, $y = 0(1^\circ)90^\circ$		Dreisonstok 1935 (T. 1)

$x = 0(1^\circ)90^\circ$ in the third and sixth editions only. For further information and errata, see C. H. Smiley, *M.T.A.C.*, **1**, 79 and 87, 1943.

5–6 dec.	$w = -A$, $x = 25^\circ(1^\circ)90^\circ$, $y = 0(1^\circ)90^\circ$	Weems 1927

Also mainly to 5 decimals, with 1° interval in each argument, but mostly omitting high latitudes (i.e. the smaller values of x):

Gingrich 1931 (Table A, column A); Myerscough & Hamilton 1939 (column Q); Norie 1938 (p. 334, quantity A); and Ogura 1924 (Table A, column A).

9·62. $z = \sin^{-1}(\sin x \sec y)$

(i) The following are tables of "altitudes on the prime vertical". The formula is sin Alt. = sin Dec. cosec Lat., so that z = Alt., x = Dec. and $y = 90^\circ$ − Lat. It is necessary that $x+y < 90^\circ$, and the values of x and y given below are to be understood as paired to satisfy this condition.

1′	$x = 0(2^\circ)50^\circ$, $y = 20^\circ(2^\circ)40^\circ(1^\circ)90^\circ$	Norie 1894, 1904 (T. 46); Norie 1938 (540); Inman 1891 (T. 28)
0°·1	$x = 0(1^\circ)40^\circ$, $y = 20^\circ(1^\circ)90^\circ$; $x = 41^\circ(1^\circ)48^\circ$, $y = 27^\circ(1^\circ)90^\circ$	Raper 1920 (T. 29)
0°·1	$x = 0(1^\circ)34^\circ$, $y = 20^\circ(2^\circ)40^\circ(1^\circ)90^\circ$	Inman 1906 (132); Inman 1940 (133)

(ii) The following are tables of "amplitudes", or azimuths of rising or setting (measured from east or west), the triangle involved being right-sided. The formula is sin Amp. = sin Dec. sec Lat., so that z = Amp., x = Dec. and y = Lat., and, as before, $x+y < 90^\circ$.

1′	$x = 0(1^\circ)23^\circ$, also $23^\circ\ 28'$, $y = 0(1^\circ)66^\circ$	Pryde 1930 (414)
1′	$x = 0(1^\circ)23^\circ, 23^\circ\ 30', 23^\circ\ 45'$, $y = 0(1^\circ)64^\circ$	Inman 1891 (T. 37)
1′	$x = 0(1^\circ)21^\circ(30')23^\circ, 23^\circ\ 28'$, $y = 1^\circ(2^\circ)9^\circ(1^\circ)66^\circ$	Norie 1894, 1904 (T. 42); Norie 1938 (534)
0°·1	$x = 0(0^\circ{\cdot}5)30^\circ$, $y = 0(\text{var.})77^\circ$	Bowditch 1925 (T. 39)
0°·1	$x = 0(1^\circ)34^\circ$, $y = 0(1^\circ)64^\circ$	Inman 1906 (137); Inman 1940 (138)
0°·1	$x = 0(1^\circ)30^\circ$, $y = 0(5^\circ)10^\circ(2^\circ)24^\circ(1^\circ)74^\circ$	Blackburne 1923 (T. 76)

(iii) We may also notice:

0′·1	$z = t'$, $x = D = 0(1^\circ \text{ or } 0^\circ{\cdot}5)90^\circ$, $y = \text{Dec.} = 0(0^\circ{\cdot}1)28^\circ{\cdot}9$	Pierce 1931 (T. 1)
0′·1	$z = V$, $x = \text{Dec.} = 0(0^\circ{\cdot}1)28^\circ{\cdot}9$, $y = D = 0(1^\circ \text{ or } 0^\circ{\cdot}5)90^\circ$	

9·63. $z = \sin^{-1}(\tan x \tan y)$

(i) The following are tables of "hour angle on the prime vertical". The formula is cos H.A. = tan Dec. cot Lat., so that $z = 90^\circ$ − H.A., x = Dec. and $y = 90^\circ$ − Lat. It is necessary that $x+y < 90^\circ$, and the values of x and y given below are to be

understood as paired to satisfy this condition. The value of the H.A. is given in time ($1^m = 15'$).

1^m	$x = 0(2°)50°, \quad y = 20°(2°)40°(1°)90°$	Norie 1894, 1904 (T. 45); Norie 1938 (538); Inman 1891 (T. 27)
1^m	$\left\{\begin{array}{l} x = 0(1°)40°, \quad y = 20°(1°)90° \\ x = 41°(1°)48°, y = 27°(1°)90° \end{array}\right\}$	Raper 1920 (T. 29)
1^m	$x = 0(1°)34°, \quad y = 20°(2°)40°(1°)90°$	Inman 1906 (132); Inman 1940 (133)
1^m	$x = 0(2°)32°, \quad y = 20°(1°)88°$	*Brown's N.A.* 1935 (160)

(ii) The following are tables of "semi-diurnal arcs", giving hour angle of rising or setting, the triangle involved being right-sided. The formula is cos H.A. = −tan Dec. tan Lat., so that z = H.A. − 90°, x = Dec. and y = Lat., and, as before, $x + y < 90°$. The H.A. is given in time. (Inman tabulates H.A. − 90° = z.)

1^m	$x = 0(1°)34°, y = 0(1°)64°$	Inman 1906 (137); Inman 1940 (138)
1^m	$\left\{\begin{array}{l} x = 0(1°)38°, \text{ also } 23° 30' \\ y = 0(5°)10°(2°)24°(1°)74° \end{array}\right\}$	Blackburne 1923 (T. 78)
1^m	$x = 0(1°)23°$, also $23° 28'$, $y = 0(1°)60°$	Pryde 1930 (416)

Other tables to 1^m in Inman 1891 (T. 37), Norie 1894, 1904 (T. 43), Norie 1938 (536) and Raper 1920 (T. 26).

9·64. $z = \tan^{-1}(\sin x \tan y)$

$0'.1$	$z = 90° - p$	$\left\{\begin{array}{l} x = 6^h - \text{H.A.} = 0(1^m)6^h \\ y = 90° - \text{Lat.} = 30°(15')90° \end{array}\right\}$	Pinto 1933 (T. 12)
$0'.1$	$z = 90° - K$	$\left\{\begin{array}{l} x = 90° - h \quad = 0(1°)90° \\ y = 90° - \text{Lat.} = 0(1°)90° \end{array}\right\}$	Hughes–Comrie 1938 (T. 1)
$0'.1$	$z = 90° - K$	$\left\{\begin{array}{l} x = 90° - t \ = 0(1°)90° \\ y = 90° - L = 0(1°)90° \end{array}\right\}$	Ageton 1942
$0'.1$	$z = 90° - \text{H.A.}$	$\left\{\begin{array}{l} x = 90° - b = 0(1°)90° \\ y = 90° - a = 0(1°)90° \end{array}\right\}$	Aquino 1927
$0'.1$	$z = 90° - a$	$\left\{\begin{array}{l} x = a = 0(1°)90° \\ y = b = 0(1°)90° \end{array}\right\}$	Aquino 1927
$0'.1$	$z = U$	$\left\{\begin{array}{l} x = 6^h - \text{H.A.} = 0(4^m)6^h \\ y = 90° - \text{Lat.} = 23°(1°)90° \end{array}\right\}$	Smart & Shearme 1924
$0'.1$	$z = b$	$\left\{\begin{array}{l} x = 90° - t \ = 0(1°)90° \\ y = 90° - L = 25°(1°)90° \end{array}\right\}$	Dreisonstok 1935
		$y = 0(1°)90°$ in the third and sixth editions only.	
$0'.1$	$z = 90° - K$	$\left\{\begin{array}{l} x = 90° - \text{H.A.} = 0(1°)90° \\ y = 90° - \text{Lat.} \ = 25°(1°)90° \end{array}\right\}$	Weems 1927 (T. A); Ogura 1924 (T. A)
$1'$	$z = 90° - t$	$\left\{\begin{array}{l} x = 90° - b = 0(1°)90° \\ y = 90° - a = 0(1°)6°(30')90° \end{array}\right\}$	Aquino 1924
$1'$	z = Az.	$\left\{\begin{array}{l} x = \text{H.A.} = 0(4^m)6^h \\ y = 90° - \text{Dec.} = 0(1°)90° \end{array}\right\}$	Goodwin 1921
$1'$	$90° - z$ tabulated	$\left\{\begin{array}{l} x = b = 0(1°)90° \\ y = 90° - a = 0(1°)90° \end{array}\right\}$	Kelvin 1876; Tillman 1936 (T. 9)

$1'$	$z = 90^\circ - A$	$\begin{cases} x = b = 0(1^\circ)90^\circ \\ y = 90^\circ - a = 32^\circ(1^\circ)39^\circ \end{cases}$	Kelvin 1871
$1'$	$z = \text{Lat.}$	$\begin{cases} x = 90^\circ - \text{Long. from vertex} = 0(1^\circ)90^\circ \\ y = \text{Lat. of vertex} = 0(1^\circ)90^\circ \end{cases}$	Towson 1912
$1'$	$z = \text{Dist.}$	$\begin{cases} x = 90^\circ - \text{Lat. of vertex} = 0(1^\circ)90^\circ \\ y = \text{Long. from vertex} = 0(1^\circ)90^\circ \end{cases}$	Towson 1912

In this last table z is given in minutes, not in degrees and minutes. Towson's tables contain errors of several final units.

$0^\circ{\cdot}1$	$z = 90^\circ - Z_1$	$\begin{cases} x = \text{Lat.} = 0(1^\circ)90^\circ \\ y = h = 0(1^\circ)90^\circ \end{cases}$	Hughes–Comrie 1938 (T. 1)
$0^\circ{\cdot}1$	$z = 90^\circ - Z'$	$\begin{cases} x = L = 0(1^\circ)90^\circ \\ y = t = 0(1^\circ)90^\circ \end{cases}$	Ageton 1942
$0^\circ{\cdot}1$	$z = 90^\circ - Z'$	$\begin{cases} x = L = 0(1^\circ)65^\circ \\ y = t = 0(1^\circ)90^\circ \end{cases}$	Dreisonstok 1935

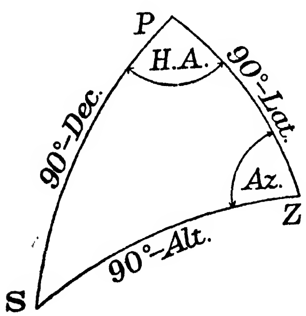
$x = 0(1^\circ)90^\circ$ in the third and sixth editions only.

Other tables, wholly or partly to $1'$, in:

Gingrich 1931 (Table A, quantity $90^\circ - K$); Myerscough & Hamilton 1939 (column P); and Norie 1938 (p. 334, quantity $90^\circ - K$).

9·7. Tables concerning General Spherical Triangles

As we have not met with such tables except in connection with the navigational or astronomical triangle, it will be convenient to describe the various tables in terms of this triangle. In the diagram, P indicates the pole, Z the observer's zenith and S the direction of the sun (or star, etc.). The sides and relevant angles are then as shown.



The fundamental calculation in modern navigation is for the determination of altitude and (less accurately) azimuth from values of latitude, declination and hour angle; i.e. essentially the determination of the third side and an angle, given two sides and the included angle. By the polar property (see Art. 9·6) the case when two angles and the intermediate side are given is essentially the same.

Using the various formulæ of spherical trigonometry, the calculation may be made by ordinary single-entry trigonometrical tables, including haversine tables (see Sections 7 and 8). It may be facilitated by double-entry tables; e.g. by tables giving PN, log sec ZN, etc. with arguments Lat. and H.A., where N is the foot of the (great circle) perpendicular from Z on PS. Various tables of this kind have been mentioned in Arts. 9·6–9·64. On their use, see such works as Hughes–Comrie 1938; W. M. Smart, *Spherical Astronomy*, Cambridge 1936, Chap. XIII; and the *Admiralty Navigation Manual*, London 1938.

It is however possible to tabulate altitude and azimuth directly in triple-entry tables with arguments Lat., Dec. and H.A. Such tables, if they are to be reasonably complete and general, are necessarily bulky.

It should be added that in the navigational case no serious difficulties of interpolation arise. It is possible to work from an "assumed position" chosen so that Lat. and H.A. are, say, integral multiples of 1°. The double-entry tables may then be used without interpolation, and the triple-entry ones with interpolation

in Dec. only. The tables are less easy to use in cases where this simplification is not possible.

Another calculation which may arise in navigation is the determination of H.A., or sometimes of Az., given Lat., Dec. and Alt.; i.e. essentially the determination of an angle, given three sides. By the polar property, the determination of a side, given three angles, is essentially the same. Raper's tables of $\tan x \tan y$ and $\sec x \sec y$ (see Arts. 9·54 and 9·56) are meant for use with the formula

$$\cos \text{H.A.} = \sin \text{Alt.} \sec \text{Lat.} \sec \text{Dec.} - \tan \text{Lat.} \tan \text{Dec.}$$

Extensive triple-entry tables exist.

We shall group the various triple-entry tables as follows: 9·71, tables of altitude; 9·72, tables of azimuth; 9·73, tables of altitude and azimuth; 9·74, tables of hour angle (or of azimuth, if altitude is an argument).

Tables are normally arranged for use in both hemispheres. Lat. and Dec. may be of the same sign ("same name") or of opposite signs ("contrary name"). For ease of statement, however, we have described all tables as if they referred to north latitude, and positive or negative declination.

9·71. Altitude Tables

Ball 1927 gives Alt. to $0'{\cdot}1$ for Lat. $= 31°(1°)60°$, $\pm$Dec. $= 0(1°)24°$ and H.A. $= 0(4^m)6^h$. Alt. is given even when negative, and for H.A. $= 6^h(4^m)12^h$ it is only necessary to consider $-$Dec., $12^h -$H.A. and $-$Alt., since

$$\sin \text{Alt.} = \sin \text{Lat.} \sin \text{Dec.} + \cos \text{Lat.} \cos \text{Dec.} \cos \text{H.A.}$$

The companion volumes, Ball 1918 for Lat. $= 0(1°)30°$ and $\pm$Dec. $= 0(1°)24°$, and Ball 1919 for Lat. $= 24°(1°)60°$ and $\pm$Dec. $= 24°(1°)60°$, are similar.

9·72. Azimuth Tables

We refer here to time-azimuths, the arguments being Lat., Dec. and H.A. It will be noticed that some tables give azimuth more accurately than the combined altitude and azimuth tables of Art. 9·73. For tables of altitude-azimuths, in which Alt. replaces H.A. as argument, see Art. 9·74.

$1'$ Lat. $= 30°(1°)60°$, $\pm$Dec. $= 0(1°)23°$ — Burdwood-Brown 1926

This is a completed version of Burdwood 1866. For positive Dec., H.A. $= 0(4^m) \ldots$ so long as Alt. is positive. For negative Dec., H.A. $= 0(4^m)6^h$, and Az. is printed in bold type when Alt. is negative.

$1'$ Lat. $= 30°(1°)60°$, $\pm$Dec. $= 0(1°)23°$ — Burdwood 1866

H.A. at 4^m interval so long as Alt. lies between 0 and 60°.

$1'$ Lat. $= 0(1°)30°$, $\pm$Dec. $= 0(1°)23°$ — Davis & Davis 1887

H.A. $= 0(4^m) \ldots$ so long as Alt. is positive.

? Lat. $= 0(1°)65°$, $\pm$Dec. $= 0(1°)23°$ — Cugle 1935, 1936

H.A. at 2^m interval. We have no information apart from the title of the work.

$1'$ Lat. $= 61°(1°)72°$, $\pm$Dec. $= 0(1°)23°$ — Softley 1915

H.A. $= 0(10^m) \ldots$ so long as Alt. is positive.

$1'$ Lat. $= 0(1°)60°$, Dec. $= -24°(1°)-30°$ — Goodwin 1896 (T. B)

H.A. $= 0(4^m)1^h$.

$0^\circ{\cdot}1$	Lat. $= 0(1^\circ)30^\circ$, $\pm$Dec. $= 0(1^\circ)24^\circ$	Davis & Davis 1921
$0^\circ{\cdot}1$	Lat. $= 30^\circ(1^\circ)64^\circ$, $\pm$Dec. $= 0(1^\circ)24^\circ$	Burdwood–Davis 1931

Both the above have H.A. at 4^m or 2^m interval so long as Alt. is positive.

$0^\circ{\cdot}1$	Lat. $= 0(1^\circ)34^\circ(1^\circ$ or $2^\circ)60^\circ$, $\pm$Dec. $= 23^\circ(1^\circ)64^\circ$ H.A. at 10^m or 5^m interval.	Davis 1931
$0^\circ{\cdot}1$	Lat. $= 61^\circ, 62^\circ(2^\circ)78^\circ$, $\pm$Dec. $= 0(2^\circ)32^\circ$ H.A. $= 0(10^m)$ ... so long as Alt. is positive.	Davis 1916
$0^\circ{\cdot}1$	Lat. $= 0(1^\circ)50^\circ(1^\circ$ or $2^\circ)64^\circ$, $\pm$Dec. $= 0(1^\circ$ or $2^\circ)34^\circ$ H.A. at various intervals. Mainly for high-altitude or circum-meridian cases.	Davis 1915

9·73. Altitude and Azimuth Tables

Davis 1918 gives Alt. to $1'$ and Az. to $0^\circ{\cdot}1$ for

$$\text{Lat.} = 30^\circ(1^\circ)64^\circ \quad \text{and} \quad \pm\text{Dec.} = 0(1^\circ)24^\circ$$

Alt. and Az. are given for alternate H.A. arguments. These are usually at 4^m intervals, so that each tabulated function is given at 8^m intervals in H.A., but sometimes the intervals in H.A. are reduced. The H.A. continues as long as Alt. > 0.

Davis 1921 deals with Lat. $= 0(1^\circ)30^\circ$ and $\pm$Dec. $= 0(1^\circ)24^\circ$, Davis 1922 with Lat. $= 30^\circ(1^\circ)64^\circ$ and $\pm$Dec. $= 23^\circ(1^\circ)64^\circ$. These companion volumes are arranged in the same way as Davis 1918.

H.O. 214 gives Alt. to $0'{\cdot}1$ and Az. to $0^\circ{\cdot}1$. Each ten-degree band of Lat. (0 to 9°, 10° to 19°, ..., 70° to 79°) is contained in a separate volume. The interval in Lat. is 1°. H.A. $= 0(1^\circ)$... as long as Alt. $> 5^\circ$. The remaining argument is $\pm$ Dec. $= 0(30')29^\circ(\text{irreg.})74^\circ\,30'$; all declinations above 29° are multiples of $30'$, but only declinations approximating to those of navigational stars are included. For interpolation, d(Alt.)/d(Dec.) and d(Alt.)/d(H.A.) are given, both to 2 decimals.

The *Astronomical Navigation Tables* (see under Nautical Almanac Office in Part II) are for official use only, and are not generally available; this also applies to the American reprint (H.O. 218). Each five-degree band of Lat. (0° to 4°, 5° to 9°, ...) is contained in a separate volume, the interval in Lat. being 1°. The values of $\pm$Dec. are $0(1^\circ)28^\circ$. The H.A. argument is at interval 1°, and covers at least the range for which $10^\circ <$ Alt. $< 80^\circ$. The functions tabulated are Alt. to $1'$ with first differences in Dec. only, and Az. to 1° without differences. But the altitude is corrected for refraction, so that only Az. is a purely mathematical quantity.

9·74. Hour-Angle Tables

We include here azimuth tables in which altitude is an argument.

Littlehales 1923, in which the main table occupies 847 pages, tabulates H.A. to 1^s and Az. to $0^\circ{\cdot}1$, along with variations due to $1'$ change in Dec., these being given to $0^s{\cdot}1$ and $0^\circ{\cdot}001$ respectively. The arguments are Lat. $= 0(1^\circ)60^\circ$, $\pm$Dec. $= 0(1^\circ)27^\circ$, and Alt. In general, Alt. goes from 5° at intervals of 1° up to the maximum possible Alt., which is $90^\circ -$ Lat. $+$ Dec., but in some cases where it has seemed advisable the lower limit has been reduced below 5° and the interval below 1° (e.g. intervals of $30'$, $20'$ and $10'$ occur).

Littlehales 1925 is a companion volume, which we have not seen.

Blackburne 1914 gives H.A. to $0^s{\cdot}1$, along with variations to $0^s{\cdot}01$ due to $1'$ change in each of the three arguments, which are Lat. $= 0(1^\circ)30^\circ$, $\pm$Dec. $= 0(1^\circ)23^\circ$, and Alt. When Dec. > 0, Alt. $= 0$, $10^\circ(2^\circ)24^\circ$ to 32° $(1^\circ)50^\circ$ to 60°, i.e. the change of

interval occurs somewhere between 24° and 32°, and the end between 50° and 60°, according to circumstances. When Dec. < 0, so that the maximum Alt. possible is small, more low values of Alt. are given; e.g. when Lat. = 30° and Dec. = −18°(1°) −23°, Alt. = 0(2°)4°(1°)31°.

Davis 1899 has arguments Lat. = 0(1°)50°, ±Dec. = 0(1°)23° and Alt. at 1° intervals. It gives H.A. to 1^s, with variations due to changes in each of the three arguments.

Goodwin 1896 (T. A) tabulates Az. to 1′ with arguments Lat. = 0 to 60°, ±Dec. = 24° to 30°, and Alt. The azimuth is not given after the altitude has come within 2° of the meridian altitude.

Extensive tables, both giving H.A. to 1^s, are contained in Lalande 1793 and Lynn 1827.

9·8. Sexagesimal Interpolation Tables

The sexagesimal measurement of angle and time can cause inconvenience in interpolation unless special tables are provided. We mention a few such tables, first natural and then logarithmic. On old tables (particularly the extensive ones of Bernoulli and Taylor) see B.A. 1873 (40). For interpolation in the decimal system see Section 23.

9·81. $xy/60$

1 dec.	$x = 1(1)60,\ y = 0(\cdot 5)60$	Ball 1927 (T. 3)
Nearest integer	$x = 1(1)60,\ y = 0(1)160$	Bowditch 1874 (T. 30)
0″·1	$x = 1(1)60,\ y = 0(1'')10'$	Lynn 1827 (244)

9·82. $xy/120$

1 dec.	$x = 0(1)60,\ y = 0(1)36$	*Brown's N.A.* 1935 (150)
Nearest integer	$x = 1(1)60,\ y = 4(4)360$	*N.A.* (abr.) 1936

$x/120 = 1^m(1^m)60^m/2^h = 12^m(12^m)12^h/24^h$, both printed in each of the above tables. The tables are contained in other recent annual issues of *Brown's N.A.* and the abridged *N.A.*

9·83. $xy/24^h$

0″·1	$x = 0(30^m)30^h,\ y = 0(30'')30'\,30''$	Raper 1920 (T. 21)

$\frac{1}{2}x$ is also printed. See also Art. 9·82.

9·84. Non-Linear Interpolation

Raper 1920 (T. 25) tabulates to 4 decimals values of $\frac{1}{2}n(1-n)$ for

$$n = \frac{0(12^m)24^h}{24^h} = \frac{0(6^m)12^h}{12^h} = \frac{0(1^m\,30^s)3^h}{3^h} = \frac{0(30^s)1^h}{1^h}$$

Lynn 1827 (284) tabulates the second-difference correction $\frac{1}{2}n(n-1)\Delta^2$ to 0″·1 for $n = \dfrac{1(1)60}{60}$ and $\Delta^2 = 1''(1'')100''$.

For non-linear interpolation with printed variations, see the *Nautical Almanac* for 1931 (728–729), or the issues for 1932–34.

Old tables of some interest are given in early issues of the Berlin *Astronomisches Jahrbuch*. For example, the volume for 1783 (published 1780) gives both Gregory–Newton coefficients and Stirling coefficients for $n = \frac{0(10^m)24^h}{24^h}$; the former originally appeared in the volume for 1776 (published 1774). The volume for 1778 (published 1776) gives Gregory–Newton coefficients for $n = \frac{0(10'')1^\circ}{1^\circ}$. All three tables give 5-decimal coefficients of differences up to and including the fifth.

9·85. Log $\frac{60}{x}$

3 dec.	$x \doteq 0(\cdot 1)60$	Ball 1927 (T. 4)

9·86. Log $\frac{1^\circ}{x''}$ = log $\frac{3600}{x}$ (Logistic Logarithms)

4 dec.	$x'' = 0(1'')88' = 0(1'')5280''$	Hutton 1811
4 dec.	$x'' = 0(1'')80'$	Gardiner 1742

9·87. Log $\frac{24^h}{x}$ (Diurnal Proportional Logarithms)

5 dec.	$x = 0(1^m)24^h$	Pryde 1930 (396)
4 dec.	$x = 0(1^m)24^h$	Norie 1894, 1904 (T. 33); Raper 1920 (T. 21A)

9·88. Log $\frac{x}{3^h}$ (Ternary Proportional Logarithms)

5 dec.	$x = 0(0^s\!\cdot\!1)\,1^m(1^s)3^h$	Inman 1891 (T. 42)
5 dec.	$x = 0(1^s)3^h$	Pryde 1930 (398)
4 dec.	$x = 0(1^s)3^h$	Norie 1894, 1904 (T. 34); Raper 1920 (T. 74)

9·89. Logarithms for Non-Linear Interpolation

Sawitsch 1851 contains two tables reprinted from a work by Wrangel. The first table (pp. 425–427) gives logarithms of Bessel coefficients of odd and mean even differences, for every 5 minutes of an interval of 12 hours. The logarithm of the coefficient of the first difference is given to 7 decimals, and those of the coefficients of the second to fifth differences to 5 decimals. No differences are given. The second table (pp. 434–436) gives, again for every 5 minutes of a 12-hour interval and without differences, 5-decimal logarithms of the derivatives with respect to n of the second to fifth Bessel coefficients, for finding the derivative of a tabulated function anywhere in the interval (see Art. 23·4). The first table (only) is reproduced from Sawitsch in Loomis 1873 (T. XXIV, pp. 394–396). See also Parkhurst 1889 (172).

Raper 1920 (T. 25) tabulates log $\frac{1}{2}n(1-n)$ to 5 decimals for the values of n mentioned in Art. 9·84 above.

SECTION 10

NATURAL AND LOGARITHMIC VALUES OF EXPONENTIAL AND HYPERBOLIC FUNCTIONS

10·0. Introduction

The arrangement of this section will be evident from the short summary given below. It is convenient in many cases to consider separately the ascending exponential e^x and the descending exponential e^{-x}, where x is positive.

The tabulation of exponential and hyperbolic functions is more modern than that of, for instance, trigonometrical functions, but fairly adequate tables are now available. We have, however, not met with any tables of natural logarithms of hyperbolic functions.

For $e^{\pm x^2}$ and similar functions, see Section 15. For $e^x/\sqrt{2\pi x}$, etc., see Art. 18·59. For the Poisson distribution, see Art. 23·86.

10·1. Natural Values of e^x

We consider first fundamental tables, mostly to 9 or more figures, and then other tables, mostly to 8 or fewer figures. Radix tables of e^x are considered in Art. 10·15.

On the value of e, see Art. 5·2, where a few integral powers of e to very many figures are mentioned.

10·11. Fundamental Tables of e^x

Extended tables of e^x are somewhat difficult to arrange in order of precision, as often neither the number of decimals nor the number of figures is fixed. In this article we therefore adopt chronological order. The New York W.P.A. tables are now the chief source of information. On errors in some of the earlier tables, see Van Orstrand 1921, Holtappel 1938 and Glaisher 1883*a*.

28–29; 32–33 f.	1(1)24; 25, 30 and 60		Schulze 1778

Reproduced in Glaisher 1883*a*. Last few figures uncertain.

20 dec.	1(1)10	No Δ	**Bretschneider 1843**
9 fig.	0(·001)·1(·01)2(·1)10(1)500	No Δ	**Glaisher 1883*a***
24 dec. 4–15; 10–15 dec. 1 dec.	10(1)20 5(·2)6·8; 7(·2)20 0(·1)15	No Δ	Gram 1884
12; **16** dec.	0(·001)2; 0(·1)3	No Δ	**Newman 1889**

20–22 fig.	0(·1)32	No Δ	Van Orstrand 1913
33; 42 fig.	0(·1)50; 1(1)100	No Δ	**Van Orstrand 1921**
32 fig.	1(1)32	No Δ	**Peters & Stein 1922**
8–9 fig.	0(·02)16	No Δ	Watson 1922
20 dec.	0(·00001)·001		
10 dec.	·001(·0001)·1(·001)3(·01)10	No Δ	Hayashi 1926
15 d.; 33 f.	10(·1)20(1)41; 42(1)50		
5; 10 dec.	0(·01)10; 0(·0001)·01 and 10(1)40	No Δ	Hayashi 1930*b*
9 fig.	0(·01)·5; 0(·5)20	δ^2; no Δ	**B.A. 6, 1937**
10 dec.	0(·001)10		
17; 18 dec.	0(·001)1; 0(·1)10	No Δ	**Holtappel 1938**
22–20 dec.	1(1)24		
18 dec.	0(·000001)·0001(·0001)1		
15 dec.	1(·0001)2·5(·001)5	No Δ	**New York W.P.A. 1939*b***
12 d.; 19 f.	5(·01)10; 1(1)100		

10·12. Other Tables of e^x

6 dec.; 9 fig.	0(·001)3(·01)6·90; 6·91(·01)15 and 1(1)50	No Δ	**Becker & Van Orstrand 1924**
6; 5 dec.	0(·0001)·1(·001)3(·01)5·99; 6(·01)6·3(·1)10	No Δ	Hayashi 1921
8 dec.; 9 fig.	1(1)10; 11(1)100		
6; 5 dec.	0(·005)·1(·01)3(·05)4(·1)7; 7(·1)10	Δ or Δ^2	Sakamoto 1934
7 fig.	0(·01)10	No Δ	Vega 1783, 1797 Vega–Hülsse 1849

On last-figure errors, see Glaisher 1883*a*.

7 fig.	0(·01)10	No Δ	Köhler 1847
3 d. (7–8 f.)	8(·001)9·809 (See Art. 10·41)	IΔ	Burrau 1907
6 fig.	0(·01)2(·1)6 (Errors > ½ final unit)	No Δ	Dale 1903
5 fig.	0(·01)5·5(·1)10	No Δ	Fowle 1933
5 fig.	0(·01)3(·05)4(·1)6(·25)7(·5)10	No Δ	Carmichael & Smith 1931
4½ d.; 4½ f.	0(·001)1; 1(·01)6	Δ	**MT.C. 1931**
4; 3 dec.	0(·001)1·499; 1·5(·001)5	IΔ	Dwight 1941

10·15. Radix Tables of e^x

These are analogous to the antilogarithmic radixes of Art. 6·5, where the base is 10. In some cases the values are intended to be complementary to those already listed in Arts. 10·11 and 10·12.

62 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 1(1)10$	**Van Orstrand 1921**
20 dec.	$x = m.10^{-n}$, $m = 1(1)999$, $n = 6(3)15$	**Holtappel 1938**
17 dec.	$x = 0(·001)·999$	
24 dec.	$x = 0(·001)·099$	Holtappel 1938
18 dec.	$x = 0(·1)9·9$	

24–25 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 1(1)3$	Holtappel 1938
20–23 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 6(1)10$	Hayashi 1926 (8)
18 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 7(1)10$	New York W.P.A. 1939*b*
8–12 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 0(1)4$	Hayashi 1921 (174)
8 dec.; 9 fig.	$x = 1(1)10$; $11(1)100$	

10·2. Natural Values of e^{-x}

As in the case of e^x, we consider first the major tables, then other ordinary tables, and finally radix tables.

10·21. Fundamental Tables of e^{-x}

As in Art. 10·11, we adopt chronological order. The New York W.P.A. tables are again the chief source of information.

20 dec.	1(1)10	No Δ	**Bretschneider 1843**
12; 14 dec.	0(·001)15·349; 15·35(·002)17·3(·005)27·635	No Δ	**Newman 1883**
18 d. (about 16 correct)	0(·1)37		

Errors in 18-dec. table at 3·5, 26·1, 26·4, 26·9 (Van Orstrand).

9 fig.	0(·001)·1(·01)2(·1)10(1)500	No Δ	**Glaisher 1883*a***
30 dec.	0(·5)1(1)10	No Δ	Burgess 1898
9 dec.	0(·001)1(·01)2 (From Newman)	No Δ	Miller & Rosebrugh 1903
20 dec.	0(·1)32	No Δ	Van Orstrand 1913
33; 38; 43; 48 dec.	0(·1)14·9; 15(·1)24·9; 25(·1)35; 35(·1)50	No Δ	**Van Orstrand 1921**
52–62 dec.	1(1)100		
32 dec.	1(1)32	No Δ	**Peters & Stein 1922**
20 dec.	0(·00001)·001	No Δ	Hayashi 1926
10 dec.	·001(·0001)·1(·001)3(·01)10		
15 dec.	10(·1)20(1)36		
8 dec.; 8–9 f.	0(·01)·5; 0(·5)20	δ^2; no Δ	**B.A. 6, 1937**
10 dec.	0(·001)10	No Δ	**Holtappel 1938**
22; 16 dec.	1(1)10; 11(1)24		
18 dec.; 19 f.	0(·000001)·0001(·0001)2·4999; 1(1)100	No Δ	**New York W.P.A. 1939*b***

10·22. Other Tables of e^{-x}

7 dec.; 9 fig.	0(·001)3(·01)15; 0(1)50	No Δ	**Becker & Van Orstrand 1924**
7 dec.	0(·0001)·1(·001)3(·01)5·99	No Δ	Hayashi 1921
6 dec.	6(·01)6·3(·1)6·9		
7; 8 dec.	7(·1)9·2; 9·3(·1)10		
9 fig.	1(1)100		
10; 5 dec.	0(·0001)·01 and 10(1)24; 0(·01)10	No Δ	Hayashi 1930*b*
6 dec.	0(·005)·1(·01)3(·05)4(·1)10	Δ to 2·9	Sakamoto 1934

6 dec.	0(·01)5·5(·1)10	No Δ	Fowle 1933
5; 6; 7 d.	0(var.)·01(·01)2·3; 2·31(·01)6·9; 6·91(·01)10	No Δ	Gruner 1906
5 dec.	0(·01)3(·05)4(·1)6(·25)7(·5)10	No Δ	Carmichael & Smith 1931
5 dec.	0(·01)2(·1)6 (Errors > ½ final unit)	No Δ	Dale 1903
4 (or 5) f.	0(·01)7	IΔ	Jahnke–Emde 1933
4½; 5½ d.	0(·001)1(·01)2; 2(·01)6	Δ	**MT.C. 1931**
4 dec.	0(·001)·1(·01)3; 3(·1)8·9	PPMD; no Δ	Kaye & Laby 1941

10·25. Radix Tables of e^{-x}

See Arts. 10·21 and 10·22 for other values by the same authors.

63 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 1(1)10$	**Van Orstrand 1921**
18 dec.	$x = m.10^{\pm n}$, $m = 1(1)9$, $n = 7(1)10$	New York W.P.A. 1939*b*
18 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 1(1)3$	Holtappel 1938
20–28 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 6(1)10$	Hayashi 1926 (8)
8–12 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 0(1)4$	Hayashi 1921 (174)
9 fig.	$x = 1(1)100$	

10·3. Other Quantities (Natural and Logarithmic) connected with $e^{\pm x}$

10·31. $\text{Log}_{10}(e^x)$

For integral multiples of $\log_{10} e$, see Art. 5·331. The tables mentioned below differ only in the position of the decimal point in the tabulated function, but may be more convenient in general computation. The tables may of course be used to interpolate themselves.

10 dec.	$x = 0(\cdot 001)\cdot 1(\cdot 01)2(\cdot 1)10(1)500$	**Glaisher 1883*a***
10 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 0(1)5$; $x = 10(10)100(100)500$	Peirce 1910
7 dec.	$x = 0(\cdot 001)3(\cdot 01)15$	**Becker & Van Orstrand 1924**
7 dec.	$x = 0(\cdot 01)10$	Vega 1797; Vega–Hülsse 1849
	$x = \cdot 46$, for 17 read 77, Glaisher.	
5 dec.	$x = 0(\cdot 01)5\cdot 5(\cdot 1)10$	Fowle 1933
5 dec.	$x = 0(\cdot 01)3(\cdot 05)4(\cdot 1)6(\cdot 25)7(\cdot 5)10$	Carmichael & Smith 1931
5 dec.	$x = 0(\cdot 0001)\cdot 01(\cdot 01)1(1)50$	**MT.C. 1931** (T. 11)
5 dec.	$x = 0(\cdot 01)\cdot 1(\cdot 1)10$	Dale 1903

10·32. $\text{Log}_{10}(e^{-x})$

The following tables give the above function, which is $-\log_{10}(e^x)$, with positive mantissa:

10 dec.	$x = 0(\cdot 001)\cdot 1(\cdot 01)2(\cdot 1)10(1)500$	**Glaisher 1883*a***
10 dec.	$x = m.10^{-n}$, $m = 1(1)9$, $n = 0(1)5$; $x = 10(10)100(100)500$	Peirce 1910
7 dec.	$x = 0(\cdot 0001)\cdot 001(\cdot 001)\cdot 01(\cdot 01)10$	Gruner 1906

10·33

We group together here a variety of expressions involving $e^{\pm x}$.

5–6 fig.	$\frac{1}{2}e^x$	$x = 0(\cdot 01)10$	No Δ	Meyer & Deckert 1924
7–9 fig.	$\frac{1}{2}e^x$	$x = 11(1)20$		
3 dec.	$\frac{1}{2}e^x$	$x = 4(\cdot 01)10$	No Δ	Kennelly 1921
2; 1 d.	$\frac{1}{2}e^x$	$x = 6(\cdot 01)7\cdot 5;$ $7\cdot 5(\cdot 01)10$	No Δ	Hall 1905
7 dec.	$\mathrm{Log}_{10}(\frac{1}{2}e^x)$	$x = 4(\cdot 01)10$	No Δ	Kennelly 1921
5–6 fig.	$\frac{1}{2}e^{-x}$	$x = 0(\cdot 01)8\cdot 09$	No Δ	Meyer & Deckert 1924
4 dec.	$e^x - 1$	$x = 0(\cdot 001)\cdot 499$	1Δ	Hawelka 1931
7 dec.	$1 - e^{-x}$	$x = 0(\cdot 1)16\cdot 9$	δ^{2n}	K. Pearson 1922

Table 1 or 2, $p = 0$.

4 dec.	$1 - e^{-x}$	$x = 0(\cdot 001)\cdot 499$	1Δ	Hawelka 1931
7 dec.	$\mathrm{Log}_{10}\left(\frac{e^x - 1}{x}\right)$	$x = 0(\cdot 01)10\cdot 4$	Δ, $\frac{1}{4}\Delta^2$	Makeham 1870
9 dec.	$(1 - e^{-x})/x$	$x = 0(\cdot 001)\cdot 1$	No Δ	Miller & Rosebrugh 1903
6 dec.	$(1 - e^{-x})/x$	$x = 0(\cdot 05)5$	No Δ	G. F. Hardy 1918

Other functions connected with annuities are also tabulated.

8 dec.	$\mathrm{Log}_{10}\left(\frac{1 - e^{-x}}{x}\right)$	$x = 0(\cdot 1)1\cdot 5$	δ^4	K. Pearson 1922

Table 3, $p = 0$.

5 dec.	$x/(1 - e^{-x})$	$x = 0(\cdot 01)10$	Δ	Steffensen 1938
27 dec.	$\frac{2}{\sqrt{\pi}}e^{-x}$	$x = 0(\cdot 5)1(1)10$	No Δ	Burgess 1898

10·34. e^{-x}/x

9 dec.	$\cdot 1(\cdot 001)1(\cdot 01)2$	(See Art. 10·33)	No Δ	Miller & Rosebrugh 1903
5–6 fig.	$20(\cdot 02)50$		Δ^2	Akahira 1929

10·35. $\int_x^\infty xe^{-x}dx = (x + 1)e^{-x}$

9 dec.	$0(\cdot 001)1(\cdot 01)2$	No Δ	Miller & Rosebrugh 1903

Elderton 1903*a* and Pearson 1930 (37) give $-\log_{10}\{(1 + x)e^{-x}\}$ to 6 decimals without differences for $x = -1(\cdot 05)+2\cdot 5$.

10·36. $\int_x^\infty x^2e^{-x}dx = (x^2 + 2x + 2)e^{-x}$

9 dec.	$0(\cdot 001)1(\cdot 01)2$	No Δ	Miller & Rosebrugh 1903

For $\int_x^\infty x^{-1}e^{-x}dx$ and $\int_x^\infty x^{-2}e^{-x}dx$, see Section 13. For general incomplete gamma function, see Section 14.

10·37. Planck's Radiation Function

Jahnke & Emde 1933 (46) tabulates $y = \frac{1}{x^5(e^{1/x} - 1)}$ to 4–5 figures without differences for $x = \cdot 06(\cdot 001)\cdot 17(\cdot 002)\cdot 18(\cdot 005)1(\cdot 01)1\cdot 95$. y has a maximum y_m of about

21·2 at $x = x_m$ = about ·201, and Fabry 1927 tabulates y/y_m to 4 figures for x/x_m = ·1 (var.) 50. Most tables relating to Planck's formula are not purely mathematical in form, but involve experimentally determined physical constants. For such tables, and further references, see Moon 1937, Lowan & Blanch 1940 and New York W.P.A. 1941*f*. For radiation integrals, see Art. 22·6.

10·38. Einstein Functions

Sherman and Ewell 1942 gives for x = 0(·005)3(·01)8(·05)15 to 6 decimals without differences the three functions

$$\frac{x}{e^x - 1}, \qquad \frac{x^2 e^x}{(e^x - 1)^2}, \qquad -\log_e(1 - e^{-x})$$

Smaller tables of the same three functions multiplied by three times the experimentally determined gas constant R are given in Nernst 1924, MacDougall 1926 and Landolt & Börnstein 1927 (Erster Ergbd., pp. 702–704).

For Debye functions, see Art. 22·6.

10·4. Natural Values of Hyperbolic Functions of x, etc.

Some functions of $x/\sqrt{2}$ are dealt with in Art. 10·49.

10·41. Sinh x and cosh x

20 dec.	1(1)10	No Δ	**Bretschneider 1843**
15 dec.	0(·1)10	No Δ	**B.A. 1926, B.A. 1, 1931 (Airey)**
20 dec.	0(·00001)·001		
10 dec.	·001(·0001)·1(·001)3(·01)10	No Δ	Hayashi 1926
15; 11–15 d.	10(·1)20(1)39; 40(1)50		
9 dec.	0(·0001)2(·1)10	No Δ	**New York W.P.A. 1940*a***
7 dec.	0(·0001)·025, and 0(·01)2·5		Campbell 1929
7 dec.	0(·01)4	No Δ	Blakesley 1890

On last-figure errors, see Archibald, *M.T.A.C.*, 1, 45, 1943.

6 dec.; 6 fig.	0(·0001)·2(·001)2; 2(·01)8?		Forti 1892

See Becker & Van Orstrand 1924 and Archibald, *M.T.A.C.*, 1, 45, 1943, which differ.

6; 5; 4 dec.	0(·01)2; 2(·01)5; 5(·01)8	Δ	Ligowski 1890
6; 5; 4 dec.	0(·01)2; 2(·01)2·5(·1)5; 5(·1)7·5	No Δ	Kennelly 1921

From Ligowski.

5 dec.	0(·0001)·1(·001)3(·01)6·3(·1)10	No Δ	Hayashi 1921
5 dec.	0(·01)10	No Δ	Hayashi 1930*b*
5 dec.	0(·005)·1(·01)3(·05)4(·1)10	Δ	Sakamoto 1934
5; 4 dec.	0(·0001)·1(·001)3; 3(·01)6	v	**Becker & Van Orstrand 1924**
5; 4; 3 dec.	0(·001)2·009; 2(·001)5·009; 5(·001)8·009	IΔ	Burrau 1907
5; 4 dec.	0(·001)1·509; 1·5(·001)3·009	PPMD	Meyer & Deckert 1924
5; 4 dec.	0(·01)3; 3(·1)5	No Δ	Fowle 1933
5 d.; 6 f.	0(·01)2; 2(·1)6	No Δ	Dale 1903

4½ d.; 4½–5½ f.	0(·001)1(·01)1·5; 1·5(·01)6(·1)10	Δ	**MT.C. 1931**
4 d.; 5 f.	0(·001)3; 3(·01)8		Campbell 1929
4; 3 dec.	0(·001)1·999; 2(·001)3	IΔ	Dwight 1941
4; 3 dec.	0(·01)4; 4(·01)6	No Δ	Hall 1905
4 dec.	0(·01)3·99	No Δ	Silberstein 1923
4 d.; 5 f.	0(·01)3·39; 3·4(·01)5·09	No Δ	Potin 1925 (445)
4 d.; 5 f.	0(·01)2·99; 3(·05)4(·1)6(·25)7(·5)10	No Δ	Carmichael & Smith 1931
4 dec.	0(·01)1 (Reproduced in Peirce 1910)	No Δ	Byerly 1893
4 dec.	0(·02)3·98	No Δ	McMahon 1906
4 d.; 5 f.	0(·01)2·5(·1)2·9; 3(·1)7·5	No Δ	Russell 1916

10·42. Tanh *x*

20; 10 d.	0(·00001)·001; ·001(·0001)·0999	No Δ	Hayashi 1926
8; 10 d.	·1(·001)3; 3(·01)10(·1)12·3		
12 dec.	0(·01)1	No Δ	Newman 1892 (101)
10 dec.	0(·1)10	No Δ	**New York W.P.A. 1943*b***
8 fig.	0(·0001)2		
7 dec.	0(·01)3 (See Art. 21·58)	δ^2	Milne-Thomson 1932
6 dec.	0(·0001)·2(·001)2(·01)8?	Δ	Forti 1892

See Becker & Van Orstrand 1924 and New York W.P.A. 1943*b* (xxvii), which differ.

5 dec.	0(·0001)·1(·001)3(·01)6·3(·1)10	No Δ	Hayashi 1921
5 dec.	0(·0001)·1(·001)3(·01)6	*v*	**Becker & Van Orstrand 1924**
5 dec.	0(·001)1·9(·01)3·09	PPMD	Meyer & Deckert 1924
5 dec.	0(·01)6·45	No Δ	Hayashi 1930*b*
5 dec.	0(·01)3(·1)6·5 (See Art. 21·52)	Δ	Milne-Thomson 1931*b*
5 dec.	0(·01)3(·1)5	No Δ	Fowle 1933
5 dec.	0(·01)2·5(·1)7·5	No Δ	Kennelly 1921
5 dec.	0(·01)2	No Δ	Ligowski 1890
4½ dec.	0(·001)1(·01)6	Δ	**MT.C. 1931**
4; 5; 6 d.	0(·001)·999; 1(·001)2·499; 2·5(·001)3	IΔ	Dwight 1941
4 dec.	0(·001)1·8(·01)3(·1)4·9		Campbell 1929
4 dec.	0(·01)2·5(·1)5·4	No Δ	Russell 1916
4; 5 dec.	0(·01)2; 2(·01)6	No Δ	Hall 1905
4; 5 dec.	0(·02)2·98; 3(·02)3·98	No Δ	McMahon 1906
4 dec.	0(·01)3(·1)5	No Δ	Potin 1925 (447)

10·43. Coth *x*

10 dec.	0(·1)10	No Δ	**New York W.P.A. 1943*b***
8 d.; 8 f.	0(·0001)·1; ·1(·0001)2	(δ^2)	
5 fig.	0(·0001)·1(·001)3(·01)5·3	*v*	**Becker & Van Orstrand 1924**
5 fig.	0(·01)3(·1)5	No Δ	Fowle 1933
4–6 fig.	0(·01)2·5(·1)7·5	No Δ	Kennelly 1921
4½–5½ fig.	0(·001)1(·01)6	Δ	**MT.C. 1931**

10·44. Sech x and cosech x

5 dec.	0(·01)4(·1)12·9	(sech x only, see Art. 21·52)	Δ	Milne-Thomson 1931*b*
3–5 fig.	0(·01)2·5(·1)7·5		No Δ	Kennelly 1921

10·46. $\dfrac{\sinh x}{x}$

7; 5 dec. 4; 3 dec.	0(·01)·25; ·26(·01)·75 ·76(·01)2(·1)4; 4·1(·1)6	No Δ	Kempe 1927 (Burls)
5 dec.	0(·01)7	No Δ	Hayashi 1930*b* (48)
5; 4 dec.	0(·001)1; 1(·001)3		Campbell 1929

10·461. $\dfrac{\cosh x}{x}$

5 dec.	0(·01)7	No Δ	Hayashi 1930*b* (48)
4 f.; 4 d.	0(·001)·5; ·5(·001)3		Campbell 1929

10·462. $\dfrac{\cosh x - 1}{x}$

5; 4 dec.	0(·001)1; 1(·001)3	Campbell 1929

10·47. Tables concerning Catenaries

For a symmetrical portion of a common catenary, we have $x = cu$, $y = c \cosh u$, span = chord = $2cu$, sag = subtense = $c(\cosh u - 1)$ and arc = $2c \sinh u$.

Pearson 1930 (62) gives

$$\frac{\text{sag}}{\text{chord}} \quad \text{to 3 decimals for} \quad \frac{\text{arc} - \text{chord}}{\text{chord}} = \cdot 06(\cdot 001)1\cdot 17(\cdot 01)2\cdot 5$$

The differences are very small and are not tabulated.

Prohaska 1933 places in 6 parallel columns values of the 6 functions

$$f_1 = \frac{\sinh u}{u}, \quad K_1 = \frac{1}{2u}, \quad K_2 = \tfrac{1}{2}\coth u, \quad K_3 = \tfrac{1}{2}\operatorname{cosech} u,$$

$$K_4 = \tfrac{1}{2}(\coth u - \operatorname{cosech} u), \quad K_5 = \tfrac{1}{2}\left(\coth u - \frac{1}{u}\right)$$

None of these quantities proceeds at equal intervals, and inspection shows that the true argument is $u = \cdot 04(\cdot 01)1\cdot 96(\cdot 02)2\cdot 18$, though this is not printed. The quantities are given to 4 decimals, except that the first 16 values of f_1 have 5 decimals. Prohaska also gives a 4-decimal table of $\tanh^{-1} x$ for $x = 0(\cdot 01)1$. In both tables the last digit is uncertain, and no differences are provided.

Double-entry tables relating to the catenary are given in C. G. Watson 1935.

10·48

Pritchard 1938 gives tables of the three functions

$$\frac{6(1 - 2x \operatorname{cosech} 2x)}{(2x)^2}, \quad \frac{3(2x \coth 2x - 1)}{(2x)^2} \quad \text{and} \quad \frac{3(x - \tanh x)}{x^3}$$

to 4 decimals for $x = 0(\cdot 01)1\cdot 6$. See also Art. 7·68.

10·49. Sinh $\frac{x}{\sqrt{2}}$ and cosh $\frac{x}{\sqrt{2}}$

12 dec.	0(·1)20	No Δ	Airey 1935*d*

Some basic values to 18 and 20 decimals. See also Arts. 5·23 and 7·9.

10·5. Logarithmic Values of Hyperbolic Functions of *x*

Gudermann 1832*b–f* and Pernot & Woods 1918 are the standard sources for common logarithms of hyperbolic functions in the ranges 2 to 12 and 0 to 2 respectively.

10·51. Log sinh *x* and log cosh *x*

For auxiliary function for log sinh *x* when *x* is small, see Art. 10·54.

12 dec.	0(·001)2·018 (log cosh to 2·032)	Δ^3 or Δ^4	**Pernot & Woods 1918**
9; 10 dec.	2(·001)5; 5(·01)12	Δ	**Gudermann 1832*b–f***
7 dec.	0(·005)·6, log cosh only	v, Δ^2	Pernot 1915
6 dec.	0(·001)2	Δ	Pernot 1918
6 dec.	0(·0001)·2(·001)2(·01)8?		Forti 1892

Errors of several final units.

5 dec.	0(·0001)·1(·001)3(·01)6	v	**Becker & Van Orstrand 1924**
5 dec.	0(·001)3·01(·01)6·09	PPMD	Meyer & Deckert 1924
5; 6; 7 d.	0(·001)2; 2(·01)5; 5(·01)8	Δ	Ligowski 1890
10 dec.	8(·01)9(1)12	No Δ	Ligowski 1890
5 dec.	0(·01)5·09	No Δ	Schorr 1916
5 dec.	0(·01)3(·05)4(·1)6(·25)7(·5)10	No Δ	Carmichael & Smith 1931
5 dec.	0(·01)3(·1)5	No Δ	Fowle 1933
4½ dec.	0(·001)1(·01)6(·1)10	Δ	**MT.C. 1931**
4 dec.	0(·01)5·09	IΔ	Hütte 1920 (32)
4 dec.	1(·01)3(·1)5(1)10	No Δ	Byerly 1893

Reproduced in Peirce 1910.

10·52. Log tanh *x* and log coth *x*

For auxiliary function for small *x*, see Art. 10·55. Becker & Van Orstrand 1924 and MT.C. 1931 give both log tanh *x* and log coth *x*, the remainder give log tanh *x* only.

12 dec.	0(·001)2·018	Δ^3 or Δ^4	**Pernot & Woods 1918**
9; 10 dec.	2(·001)5; 5(·01)12	Δ	**Gudermann 1832*b–f***

6 dec.	o(·001)2	Δ	Pernot 1918
6 dec.	o(·0001)·2(·001)2(·01)8?		Forti 1892
Errors of several final units.			
5 dec.	o(·0001)·1(·001)3(·01)6	v	**Becker & Van Orstrand 1924**
5 dec.	o(·001)1·9(·01)3·09	PPMD	Meyer & Deckert 1924
5; 6; 7 d.	o(·001)2; 2(·01)5; 5(·01)6	Δ	Ligowski 1890
7; 9; 10 d.	o(·01)2; 2(·1)4·9; 5(·1)12	No Δ	Ligowski 1890
5 dec.	o(·01)3(·1)5	No Δ	Fowle 1933
$4\frac{1}{2}$ dec.	o(·001)1(·01)6	Δ	**MT.C. 1931**
4 dec.	o(·01)2·39	IΔ	Hütte 1920 (34)

10·54. $Sh = \log \dfrac{\sinh x}{x}$

12 dec.	o(·001)·506	Δ^3	**Pernot & Woods 1918**
7 dec.	o(·005)·6	v, Δ^2	Pernot 1915
6 dec.	o(·001)·499	No Δ	Pernot 1918
4 dec. crit.	To x = ·08718		MT.C. 1931 (4)

10·55. $Th = \log \dfrac{\tanh x}{x} = -\log(x \coth x)$

12 dec.	o(·001)·506,	$-Th$ tabulated	Δ^3	**Pernot & Woods 1918**
6 dec.	o(·001)·499,	$-Th$,,	No Δ	Pernot 1918
4 dec. crit.	To x = ·06166,	$+Th$,,		MT.C. 1931 (5)

10·6. Exponential and Hyperbolic Functions of πx, etc.

Tables in Jahnke & Emde giving exponential and hyperbolic functions of $\pi x/2$ for x = o(·1)2 have been described as giving functions of πx for x = o(·05)1. Similarly in respect of $\pi x/4$ in Fowle. Some functions of $\frac{1}{2}x\sqrt{\pi}$ are dealt with in Art. 10·69.

10·61. $e^{\pi x}$ and $e^{-\pi x}$

For many-figure values of $e^{\pm\pi}$, $e^{\pm\pi/2}$, etc., see Art. 5·52.

25 dec. or 25 fig.	1·25(·25)4(1)10, also $1\frac{1}{8}(\frac{1}{8})3\frac{5}{8}$	No Δ	Van Orstrand 1921 (46)
20 dec.	o(·5)4		B.A. 1928
8–23 dec.	o(·01)·1(·1)10	No Δ	Hayashi 1930*a*
10; 13 dec.	o(·25)·50; ·75(·25)3	No Δ	Hayashi 1926 (79, 111, 12)
10 d. or 13+ f.	o(·01)·1(·1)5(·5)9, 10	No Δ	Sakamoto 1934
5–9 fig.	o(·01)10	No Δ	Hayashi 1933
5 dec.	o(·01)·1(·1)10, $1(\frac{1}{4})4$, $1(\frac{1}{8})4$	No Δ	Hayashi 1930*b* (62, 60)
5 fig.	o(·25)5	No Δ	Dale 1903
5 fig.	o(·25)5	No Δ	Fowle 1933
4–6 fig.	o(·1)4	No Δ	Morse 1936
4–5 fig.	o(·05)1	No Δ	Jahnke–Emde 1933 (60)

10·62. $\text{Log}_{10}(e^{\pi x})$ and $\log_{10}(e^{-\pi x})$

For $\pi \log_{10} e$, see Art. 5·511.

5 dec.	o(·25)5	No Δ	Fowle 1933

10·63. Sinh πx and cosh πx

20–24 fig.	0(·01)·1(·1)10	No Δ	Hayashi 1930*a*
15 dec.	0(·0001)·01	δ^2	B.A. **1**, 1931 (Comrie)
15 dec.	0(·01)4	No Δ	**B.A. 1928** (Airey)
14–16 fig.	0(·01)4	No Δ	**B.A. 1, 1931**
10; 13 dec.	0(·25)·5; ·75(·25)3	No Δ.	Hayashi 1926 (79, 111, 13)
10 d. or 13+ f.	0(·01)·1(·1)5(·5)9, 10	No Δ	Sakamoto 1934

On errors, see Archibald, *M.T.A.C.*, **1**, 46, 1943.

4–8 fig.	0(·01)10	No Δ	Hayashi 1933
5 dec.	0(·01)·1(·1)10, 1($\frac{1}{4}$)4, 1($\frac{1}{8}$)4	No Δ	Hayashi 1930*b* (62, 60)
About 5 fig.	0(·1)4	No Δ	Morse 1936
4–5 fig.	0(·05)1	No Δ	Jahnke–Emde 1933 (60)

10·64. Tanh πx

14 dec.	·75(·25)3	No Δ	Hayashi 1933 (66)
8; 13 dec.	0(·25)·5; ·75(·25)3	No Δ	Hayashi 1926 (79, 111, 13)
5 dec.	0(·01)·1(·1)2·1	No Δ	Hayashi 1930*b* (62)
4 dec.	0(·05)1	No Δ	Jahnke–Emde 1933 (60)

10·65. Coth πx

4–5 fig.	0(·05)1	No Δ	Jahnke–Emde 1933 (60)

10·66. Sech πx

14 dec.	·75(·25)3	No Δ	Hayashi 1933 (66)

10·69. Exp $(\pm\frac{1}{4}x\sqrt{\pi})$ and $\log_{10}$ exp $(\pm\frac{1}{4}x\sqrt{\pi})$

The 4 functions are given for x = 1(1)20 in Fowle 1933. The natural values are to 5 figures and the logarithms to 5 decimals, both without differences.

10·7. Exponential and Hyperbolic Functions of $\pi x/360$

10·71. $e^{\pi x/360}$ and $e^{-\pi x/360}$

23 dec.	0(1)360 (See also Art. 10·61)	No Δ	Van Orstrand 1921
10 dec.	0(1)360	No Δ	Hayashi 1926 (96)

10·73. Sinh $(\pi x/360)$ and cosh $(\pi x/360)$

10 dec.	0(1)360	No Δ	Hayashi 1926 (96)

SECTION 11

NATURAL LOGARITHMS OF NUMBERS AND OF TRIGONOMETRICAL FUNCTIONS. INVERSE HYPERBOLIC FUNCTIONS

11·0. Our work on this section (except for the articles on inverse hyperbolic functions) has been much facilitated by the existence of Henderson 1926, which should be consulted for further details.

Tables of natural logarithms may usually be relied on to contain the natural logarithms of a certain number of positive, and perhaps negative, integral powers of 10. These are multiples of $\log_e 10$, the reciprocal of the modulus, and tables of such multiples (often the first 100) are very common. Some of the more extensive are described in Art. 5·331. We have therefore not thought it necessary to index such tables here. They are normally brief compared with the main table or tables.

The older tables containing Napierian and similar logarithms have been described in Arts. 11·4–11·53 in modern terms, a certain amount of interpretation (insertion or transference of decimal point, etc.) being usually necessary. In these cases B.A. 1873, Knott 1915 and Henderson 1926 provide further information.

The arrangement of this section is as follows:

11·1 Natural Logarithms of Integers
11·2 Natural Logarithms of Decimal Numbers; $\log \pi x$
11·3 Radix Tables of Natural Logarithms
11·4 Natural Logarithms of Reciprocals, Square Roots, etc.
11·5 Natural Logarithms of Trigonometrical Functions
11·6 Inverse Hyperbolic Functions

11·1. Natural Logarithms of Integers

For many-figure natural logarithms of a few particular integers (for example 2, 3, 5, 7) see Art. 5·321.

We consider first fundamental tables to 16 or more decimals, and then other tables, which have 11 or fewer decimals.

11·11. Fundamental Tables of Natural Logarithms of Integers

Unless an unusual number of decimals is required, the New York W.P.A. tables are now the standard source.

Except for Thiele's important extension and Callet's values round a million, all the 48-decimal logarithms are substantially Wolfram's. For 13 errors affecting the first 20 decimals in Vega 1794, see the introduction to New York W.P.A. 1941*a* (or the other New York volumes of natural logarithms). Only 2 errors affect the first 10 decimals; at 1099, read 5 for 1 in the 5th place, and at 7853, read 8 for 7 in the 3rd place. On errors in Peters & Stein see Archibald, *M.T.A.C.*, **1**, 57, 1943.

155+ d.	11(primes)101, also 9901	**Uhler 1943**
137+ d.	2(1)10, various primes to 113	**Uhler 1942*b***
82 dec.	Primes to 127 (All but last due to Grimpen)	**Peters & Stein 1922**
48 dec.	1(1)5000(primes)10,000(1)10,014	**W. Thiele 1908**
48 dec.	1(1)2200(primes and a few others)10,009	**Vega 1794 (Wolfram)**

48 dec.	1(1)2200(primes and a few others, 6 blanks)10,009		Schulze 1778 (Wolfram)

The 6 missing logarithms were published in Schulze 1780.

48 dec.	1(1)146(primes)10,000 (From Vega 1794)		**Peters & Stein 1922**
48 dec.	1(1)100(primes)1097, from Wolfram; 999,980(1)1,000,021		Callet 1795
20 dec.	1(1)1200; 101,000(1)101,179	No Δ; Δ^3	Callet 1795
16 dec.	1(1)100,000	No Δ	**New York W.P.A. 1941*a, b***

11·12. Other Tables of Natural Logarithms of Integers

11 dec.	1(1)1000		Borda & Delambre 1801
10 dec.	1(1)1000(primes)10,333		Salomon 1827
8 dec.	1(1)9999	No Δ	Barlow 1814

For 25 errors, see Wackerbarth 1867*b*.

8 dec.	1(1)9999 (From Barlow)	No Δ	Rees 1819*a*
8 dec.	1(1)1000(primes)11,273		Hantschl 1827
8 dec.	1(1)1000(primes)10,000	No Δ	Vega 1783, 1797; Vega–Hülsse 1849; Köhler 1847
7 dec.	1(1)1200 (1811 and later eds.)	No Δ	Hutton 1811
7 dec.	1(1)1000 (See also Art. 11·2)	No Δ	Dase 1850
6 dec.	1(1)1218(primes)10,000	No Δ	G. W. Jones 1893
6 dec.	1(1)1000	Δ	J. Speidell 1622
5 dec.	1(1)10,800	No Δ	Potin 1925 (598)
5 dec.	1(1)1009; 1000(1)10,009	No Δ; PPd	Wackerbarth 1867*a*
5 dec.	1(1)10,000	IΔ, PP	Stegmann 1856
5 dec.	1(1)1000; 1000(primes)10,000	*v*; no Δ	**Becker & Van Orstrand 1924**
5 dec.	1(1)1100; 1(1)1500	No Δ	Hütte 1920; 1931
4½ dec.	1(1)500(10)1500(100)6000	Δ	MT.C. 1931 (T. 12)
4 dec.	1(1)1000	No Δ	Carey & Grace 1927
4 dec.	10(1)209	No Δ	Peirce 1910

11·2. Natural Logarithms of Decimal Numbers

For many purposes a table of natural logarithms of numbers comparable with unity (say less than 10) is more useful than one with integral arguments. An important gap has been filled by the recent New York W.P.A. tables.

16 dec.	0(·0001)10	No Δ	**New York W.P.A. 1941*c, d***
20; 10 d.	0(·00001)·001; ·001(·0001)·1(·001) 3(·01)10(·1)20(1)50	No Δ	Hayashi 1926
7 dec.	1000(·1)10,500 (See also Art. 11·12)	PP	**Dase 1850**
7 dec.	1(·01)10·59		Callet 1783
7 dec.	1(·01)10		1770 ed. of Gardiner 1742, Lambert 1770, Lambert–Felkel 1798, Hutton 1811 and Napier–Filipowski 1857. Henderson 1926 mentions other tables, but none outside the period 1770–1858.
6 dec.	0(·01)10(1)... (See Art. 11·12)	No Δ	G. W. Jones 1893
6 dec.	1(·01)10 (1845 and presumably later eds.)		Rühlmann 1837

5 dec.	1(·001)10·009	PPMD	Hunter 1921
5 dec.	0(·01)10, 0(·1)100, 0(1)1000	Δ	Carmichael & Smith 1931
3; 5 d.	0(·01)1; 1(·001)5·01(·01)10·09	No Δ; IΔ	Peirce 1910
5 dec.	1(·01)10(·1)100	PPMD	Fisher & Yates 1938
5 dec.	0(·01)10	No Δ	Hayashi 1930*b*

5-decimal tables for 1(·01)10 with PPMD(!) are common (Dale 1903 and Castle 1910).

4½ dec.	0(·001)1·5(·01)15(·1)50(1)... (See Art. 11·12)	Δ	MT.C. 1931 (T. 12)
4 dec.	1(·001)3(·01)10	IΔ	Dwight 1941
4 dec.	1(·001)2(·01)10	IΔ	Dwight 1934

4-decimal tables for 1(·01)10 are common (Plummer 1941 with PPMD, Hoüel 1901, Jahnke & Emde 1933 and Allen 1941 with IΔ).

11·25. Log$_e$ (πx)

10 dec.	½(½)3	No Δ	Hayashi 1926 (79, etc.)
5 dec.	½(½)3	No Δ	Hayashi 1930*b* (4, etc.)

11·3. Radix Tables of Natural Logarithms

These are analogous to the radix tables of common logarithms considered in Art. 6·2. We have included a few slightly special methods of radix character. For further information see Henderson 1926, also Ellis 1881 and Glaisher 1915. We have not distinguished between the positive and negative forms in which mantissæ of log $(1-m.10^{-n})$ may be given.

137 d.	2(1)10, etc. $\begin{cases} 1+m.10^{-n}, & m=1(1)9, \quad n=1(1)21 \\ 1-m.10^{-n}, & m=1(1)9, \quad n=1(1)69 \end{cases}$	**Uhler 1942*b***
48 d.	2(1)146(primes)9973; $1\pm m.10^{-n}$, $m=1(1)9$, $n=4(1)6$	**Peters & Stein 1922**

On errors, see Uhler, *M.T.A.C.*, **1**, 58, 1943. See also Callet 1795 (48 dec.) in Art. 11·11.

33 d.	2(1)99; $m.10^{-n}-\log(1+m.10^{-n})$ tabulated, $m=1(1)9$, $n=2(1)16$	Hoppe 1876
40 d.	2(1)10	**New York W.P.A. 1941*d***
25 d.	$1\pm m.10^{-n}$, $m=1(1)9$, $n=1(1)13$	**New York W.P.A. 1941*d***
20 d.	2(1)9; $1\pm m.10^{-n}$, $m=1(1)9$, $n=1(1)11$	Wace 1873
20 d.	2(1)9; $1+m.10^{-n}$, $m=1(1)9$, $n=1(1)11$	Hoüel 1901
20 d.	2(1)9; $1+m.10^{-n}$, $m=1(1)9$, $n=1(1)11$	Leonelli 1803
20 d.	2(1)9; $1+m.10^{-n}$, $m=1(1)9$, $n=1(1)10$	Steinhauser 1880

See also Callet 1795 (20 dec.) in Art. 11·11.

16 d.	2(1)9; $1+m.10^{-n}$, $m=1(1)9$, $n=1(1)10$	Schrön 1860
13 d.	See Henderson 1926 (179) for a description of a table by G. Atwood of date 1786	

11·4. Natural Logarithms of Reciprocals, Square Roots, etc.

In all the tables in Arts. 11·41–11·43 the mantissæ are positive.

11·41. Natural Cologarithms or $\log_e \frac{1}{x}$

7 dec.	0(var.)·001(·001)1		Δ	Kepler 1624
6 dec.	1(1)1000		Δ	J. Speidell 1622
5 dec.	0(·00001)·1?	(See Henderson 1926, p. 35)		Crüger 1634
4 dec.	·1(·001)1		IΔ	Allen 1941

11·42. Half-logarithms or $\log_e \sqrt{x}$

6 dec.	1(1)1000	Δ	J. Speidell 1622

11·43. $\text{Log}_e\left(\frac{1}{\sqrt{x}}\right)$

6 dec.	1(1)1000	Δ	J. Speidell 1622

11·48. $\frac{x}{\log_e x - 1}$

Nearest integer	$x = 0(50{,}000)9{,}000{,}000$	James Glaisher 1883

Other functions of the same kind are also tabulated, all in connection with the distribution of primes. We have not investigated what other tables of this kind may be found in the literature of the subject.

11·5. Natural Logarithms of Trigonometrical Functions

As in previous sections, we regard sin x, tan x and sec x as the main trigonometrical functions, the others being derived from them by taking the complement of the argument. We thus consider separately tables of $\log_e \sin x$, $\log_e \tan x$ and $\log_e \sec x$. All the tables mentioned are old, and may require the insertion or transference of decimal points in order to agree with the descriptions below. It may be noticed that the 8-decimal Napierian canon of Ursinus 1624 has never been equalled subsequently. Like other old works, it contains many errors. According to Cantor (*Geschichte der Math.*, **2**, 739, 1900) the Berlin Library possesses a copy corrected by the hand of Ursinus. For further details on tables mentioned below, and for information on a few other old tables, see Henderson 1926, especially pp. 22–38, as well as Knott 1915 and B.A. 1873.

11·51. $\text{Log}_e \sin x$

5 dec.	0(1′)90°	No Δ	J. Speidell 1619

Other tables of $\log_e \sin x$ and $\log_e \cos x$ may be formed from those of $\log_e \operatorname{cosec} x$ and $\log_e \sec x$ (see Art. 11·53) by taking arithmetical complements of tabular values.

11·52. $\text{Log}_e \tan x$

8 dec.	45°(10″)90°		Δ	Ursinus 1624
8 dec.	45°(1′)86°(10″)90°	(From Ursinus)	No Δ	Schulze 1778
7 dec.	45°(1′)90°		No Δ	Napier 1614
6 dec.	45°(1′)90°		No Δ	Napier–Wright 1618
5 dec.	45°(1′)90°		No Δ	Ursinus 1618
5 dec.	0(1′)90°		No Δ	J. Speidell 1619

It should be added that the tables of $\mathrm{gd}^{-1}x = \log_e \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ described in Arts. 12·21–12·24 may be used as tables of natural logarithms of tangents, and that in this way radian and centesimal arguments are also available.

11·53. $\text{Log}_e \sec x$

8 dec.	0(10″)90°		Δ	Ursinus 1624
8 dec.	0(10″)4°(1′)86°(10″)90°	(From Ursinus)	No Δ	Schulze 1778
7 dec.	0(1′)90°		No Δ	Napier 1614
7; 6 d.	0(1′)1°; 1°(1′)90°		No Δ	Napier–Wright 1618
5 dec.	0(1′)90°		No Δ	Ursinus 1618
5 dec.	0(1′)90°		No Δ	J. Speidell 1619

11·6. Inverse Hyperbolic Functions

Tables of these are recent and not numerous, but the New York W.P.A. workers had computations in progress in May 1942.

11·61. $\text{Sinh}^{-1} x = \log_e (x+\sqrt{1+x^2})$

20; 10 d. 7 dec.	0(·00001)·001; ·001(·0001)·0999 ·1(·001)3(·01)10(·1)20(1)50	No Δ	Hayashi 1926
5 dec.	0(·01)10	No Δ	Hayashi 1930*b*
4; 3 dec.	0(·001)1·5(·01)5; 5(·1)15(1)60	Δ	**MT.C. 1931**
4 dec.	0(·001)2(·01)10	IΔ	Dwight 1941

11·615. $\text{Log}_e (x+\sqrt{x^2+y^2})$

7 dec.	$x = 1(1)16$, $y = 1(1)16$	Wright 1930

This paper also contains other tables in connection with the potential and attraction of rectangular bodies.

11·617. $\int_0^\theta \sinh^4 \theta\, d\theta$

Chandrasekhar 1935 (225) and 1939 (361) tabulate

$$f(x) = 8\int_0^\theta \sinh^4 \theta\, d\theta = x(2x^2-3)\sqrt{x^2+1} + 3 \sinh^{-1} x$$

with argument $x = \sinh \theta$.

11·62. $\text{Cosh}^{-1} x = \log_e (x+\sqrt{x^2-1})$

8; 7 dec.	1(·001)1·414; 1·415(·001)3(·01)10(·1)20(1)50	No Δ	Hayashi 1926
5 dec.	1(·01)10	No Δ	Hayashi 1930*b*
4; 3 dec.	1(·0001)1·01(·001)1·5(·01)5; 5(·1)15(1)60	Δ	**MT.C. 1931**
4 dec.	1(·001)2(·01)10	IΔ	Dwight 1941

11·63. $\text{Tanh}^{-1} x = \frac{1}{2} \log_e \left(\frac{1+x}{1-x}\right)$

20; 10; 9 d.	0(·00001)·001; ·001(·0001)·0999; ·1(·001)1	No Δ	Hayashi 1926
7 dec.	0(·01)·9(·001)1	δ^4	Fisher 1921
5 dec.	0(·01)1	No Δ	Hayashi 1930*b*
4; 3 dec.	0(·001)·95; ·95(·001)·99(·0001)1	Δ	**MT.C. 1931**
4 dec.	0(·001)1	IΔ	Dwight 1941
4; 3 dec.	0(·01)·96; ·97(·01)1	No Δ	Hall 1905
4 dec.	0(·01)1 (See Art. 10·47)	No Δ	Prohaska 1933

11·64. $\text{Coth}^{-1} x = \frac{1}{2} \log_e \left(\frac{x+1}{x-1}\right)$

7 dec.	1(·001)1·15(·01)5(·1)15(1)60	Δ^2	Milne-Thomson 1930*a*
3; 4 dec.	1(·0001)1·01(·001)1·05; 1·05(·001)1·5(·01)5(·1)15(1)60	Δ	**MT.C. 1931**

11·65. $\text{Sech}^{-1} x = \log_e \frac{1+\sqrt{1-x^2}}{x}$

3; 4 dec.	0(·001)·1; ·1(·001)·99(·0001)1	Δ	**MT.C. 1931**

11·66. $\text{Cosech}^{-1} x = \log_e \frac{1+\sqrt{1+x^2}}{x}$

3; 4 dec.	0(·001)·1; ·1(·001)1·5(·01)5(·1)15(1)60	Δ	**MT.C. 1931**

11·67. $\text{Sinh}^{-1} (\pi x)$

7 dec.	$x = \frac{1}{4}(\frac{1}{4})3$	Hayashi 1926 (79, etc.)
5 dec.	$x = \frac{1}{4}(\frac{1}{4})3$	Hayashi 1930*b* (5, etc.)

11·68. $\text{Cosh}^{-1} (\pi x)$

7 dec.	$x = \frac{1}{2}(\frac{1}{4})3$	Hayashi 1926 (111, etc.)
5 dec.	$x = \frac{1}{2}(\frac{1}{4})3$	Hayashi 1930*b* (9, etc.)

11·69. $\text{Tanh}^{-1} \frac{\pi}{4}$

Hayashi 1926 (79) gives this as 1·05931 1169, but we find that the correct value is 1·05930 6171.

SECTION 12

THE GUDERMANNIAN, COMBINATIONS OF CIRCULAR AND HYPERBOLIC FUNCTIONS, CIRCULAR AND HYPERBOLIC FUNCTIONS OF A COMPLEX VARIABLE

12·0. Introduction

The later part of this section deals with circular and hyperbolic functions (and inverse functions) of a complex variable. Tables of these with two arguments have been constructed mainly for electrical use.

With some arguments a table of $\sin(x+iy)$ may differ trivially from one of $\sinh(x+iy)$, with other arguments it may differ trivially from one of $\cos(x+iy)$, and so on. Hawelka 1931 and Jahnke & Emde 1933 tabulate circular functions, others hyperbolic functions. We have thought it best in cases of this kind not to alter a function as tabulated by an author, and have been content to place corresponding circular and hyperbolic tables in adjacent articles.

Where authors have used decimals of a right angle, we have given information in terms of grades (1 right angle = 100 grades), as in our trigonometrical sections.

There are various cases, since both variable and function may be complex numbers in rectangular or polar form. Where the function is tabulated in polar form, the first item on any line gives the precision of the modulus, followed by that of the argument (phase).

We have not indexed the earlier tables by Kennelly now incorporated in Kennelly 1921 (see pages 211–212 of that work). We have no detailed information about U. Meyer 1921. On L. Cohen 1913 see Jahnke & Emde 1933 (78).

Diagrams and charts which may be useful in complex-variable work to 2 or 3 figures are given in Kennelly 1914, Hawelka 1931 and Jahnke & Emde 1933.

In August 1941 the New York W.P.A. workers reported as under consideration "tables of circular functions for complex arguments".

The arrangement of this section is as follows:

12·1. The Gudermannian

The gudermannian of a real quantity u is the angle ϕ lying between $\pm\frac{\pi}{2}$ and satisfying the equations

$$\sin\phi = \tanh u \qquad \operatorname{cosec}\phi = \coth u$$
$$\cos\phi = \operatorname{sech} u \qquad \sec\phi = \cosh u$$
$$\tan\phi = \sinh u \qquad \cot\phi = \operatorname{cosech} u$$

Thus a table of the gudermannian enables values of the hyperbolic functions to be found from trigonometrical tables.

We write

$$\phi = \operatorname{gd} u = \int_0^u \operatorname{sech} u \, du = 2 \tan^{-1} e^u - \tfrac{1}{2}\pi$$

$$u = \operatorname{gd}^{-1} \phi = \int_0^\phi \sec \phi \, d\phi = \log_e \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$$

Tables of the inverse gudermannian $u = \operatorname{gd}^{-1} \phi$ are considered in Art. 12·2.

The principal functional notations are:

$\phi = \operatorname{gd} u$ (Cayley and modern English)
$= \operatorname{amh} u$ = hyperbolic amplitude of u (Hoüel)
$= \mathfrak{Amp}\, u$ (modern German)
= transcendent angle (Lambert) = Lambert's angle
$= l\, u$ = Longitudinalzahl (Gudermann)

$u = \operatorname{gd}^{-1} \phi$
$= \mathfrak{L}\, \phi$ = Längezahl (Gudermann)
$= \operatorname{lam} \phi$ = lambertian of ϕ (Huntington, Kennelly 1929)

The amh and $\mathfrak{Amp}$ notations are suggested by the fact that $u = \int_0^\phi \sec \phi \, d\phi$, so that ϕ is the amplitude of u in the special case of elliptic functions when the modulus k is equal to unity.

Diagrams for complex argument are given in Kennelly 1929.

12·11. Gd u in Radians

7; 8 dec.	0(·001)3(·01)6; 6(·01)9·99	No Δ	Hayashi 1926 (48)
7 dec.	0(·001)3(·01)6	v	**Becker & Van Orstrand 1924**
6–8 dec.	0(var.)·35(·01)2·75, 2·8(·1)8·8	Δ	Sakamoto 1934
4½ dec.	0(·01)4·5(·1)9	Δ	**MT.C. 1931**
4 dec.	0(·02)1(·05)2(·1)6(·2)7(·5)9	No Δ	McMahon 1906

12·12. Gd u in Degrees and Decimals

0°·0001; 0°·001	0(·01)·09; ·1(·01)1	No Δ	Byerly 1893
Reproduced in Peirce 1910.			
0°·001	0(·02)1(·05)2(·1)3·9	No Δ	McMahon 1906

12·14. Gd u in Degrees, Minutes, etc.

0″·01; 0″·001	0(·001)3(·01)6; 6(·01)9·99	No Δ	Hayashi 1926 (48)
0″·0001; 0″·001 0″·01; 0″·001	0(·001)·05; ·05(·001)·3 ·3(·001)2; 2(·01)6	Δ	Ligowski 1890
0″·01	0(·001)3(·01)6	v	**Becker & Van Orstrand 1924**
0′·01 or less	0(var.)·35(·01)2·75, 2·8(·1)8·8	(PP)	Sakamoto 1934
1″	0(·01)7·49	No Δ	Hayashi 1930*b*
1″	0(·01)5(·05)8	No Δ	Jahnke–Emde 1909
Abridged from Forti 1892.			
1′	0(·01)3(·1)5	No Δ	Fowle 1933

12·15. Gd πu in Radians

7; 8 dec.	$u = \frac{1}{4}(\frac{1}{4})\frac{7}{4};\ 2(\frac{1}{4})3$	No Δ	Hayashi 1926 (79, etc.)

12·17. Gd πu in Grades

A table of $\frac{2}{\pi}\,\mathrm{gd}\,\frac{\pi u}{2}$ is given to 4 (or 5) decimals without differences for $u = 0(\cdot 1)2$ in Jahnke & Emde 1933 (60). This is equivalent to gd πu to $0^g\cdot 01$ (or $0^g\cdot 001$) for $u = 0(\cdot 05)1$.

12·18. Gd πu in Degrees, Minutes, etc.

$0''\cdot 01$; $0''\cdot 001$	$u = \frac{1}{4}(\frac{1}{4})\frac{7}{4};\ 2(\frac{1}{4})3$	No Δ	Hayashi 1926 (79, etc.)
$1''$	$u = \frac{1}{4}(\frac{1}{4})\frac{9}{4}$	No Δ	Hayashi 1930*b* (5, etc.)

12·2. The Inverse Gudermannian

For notation, see Art. 12·1.

12·21. $\mathrm{Gd}^{-1}\,\phi$ (ϕ in Radians)

$4\frac{1}{2}$; $3\frac{1}{2}$ dec.	$0(\cdot 01)1;\ 1(\cdot 01)1\cdot 47(\cdot 001)1\cdot 57$	Δ	MT.C. 1931
4 dec.	$0(\cdot 01)1\cdot 57$	No Δ	Hall 1905

12·22. $\mathrm{Gd}^{-1}\,\phi$ (ϕ in Degrees and Decimals)

12 dec.	$0(0°\cdot 5)90°$	Δ^5	**Legendre 1816, 1826** (T. 4)
9 dec.	$0(1°)90°$	No Δ	Legendre 1816, 1826 (T. 9)

Given in column headed $F(90°)$ in Legendre's great table of elliptic integrals. Reproduced in Potin 1925, Legendre–Emde 1931 and Legendre–Pearson 1934, whereas the 12-decimal table is not.

7 dec.	$0(1°)90°$, 7-dec. radian arg. also given	No Δ	Cayley 1885, Hobson 1891
5 dec.	$0(0°\cdot 5)90°$	No Δ	Dale 1903
5 dec.	$0(1°)90°$, 5-dec. radian arg. also given	No Δ	Greenhill 1892

Lambert 1768, 1770 and Lambert–Felkel 1798 give only the common (not the natural) logarithm of $\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$.

12·23. $\mathrm{Gd}^{-1}\,\phi$ (ϕ in Grades)

11 dec.	$88^g(0^g\cdot 01)100^g$ (See also Art. 12·25)	No Δ	**Gudermann 1832*g***
7 dec.	$0(0^g\cdot 01)100^g$, sexages. arg. also given	Δ	**Gudermann 1831*a*, *b*, 1832*a***
5 dec.	$0(0^g\cdot 01)100^g$	No Δ	Potin 1925 (496)
4; 3 dec.	$0(0^g\cdot 1)95^g;\ 95^g(0^g\cdot 1)100^g$	Δ	Hoüel 1901 (36)

12·24. Gd$^{-1}\phi$ (ϕ in Degrees, Minutes, etc.)

9 dec.	$0(1'' \text{ to } 30'')1^\circ 2' 30''$	Δ	Sakamoto 1934
6 dec.	$1^\circ(1' \text{ to } 4')88^\circ$; $88^\circ(2'' \text{ to } 1')90^\circ$	Δ; Δ^3	Sakamoto 1934

Proportional parts for both direct and inverse interpolation are given.

5 dec.	$0(1')90^\circ$	No Δ	Potin 1925 (450); Vassal 1872
5 dec.	$0(10')90^\circ$	No Δ	Hayashi 1930*b* (2)

12·25. Auxiliary Tables for gd$^{-1}\phi$ as ϕ Approaches a Right Angle

Gudermann 1832*g* tabulates $\text{gd}^{-1}\phi^g + \log_e(10{,}000 - 100\,\phi)$ to 11 decimals with Δ for $\phi^g = 88^g(0^g\cdot01)100^g$. The quantity $10{,}000 - 100\,\phi$ is the number of centesimal minutes in the complement of ϕ^g. See also Art. 12·23.

12·26. $\text{Gd}^{-1}\phi - \sin\phi = \int_0^\phi \frac{\sin^2\phi}{\cos\phi}\,d\phi$

6; 5 dec.	$0(10')20^\circ$; $20^\circ(10')35^\circ$	No Δ	Noda 1929

12·28. $\frac{10800'}{\pi}\,\text{gd}^{-1}\phi$ (ϕ in Degrees and Minutes)

This is the inverse gudermannian expressed in minutes of arc. It occurs in tables of "meridional parts" for a spherical earth, giving the distance of the parallel of latitude ϕ from the equator on a Mercator chart, in terms of the chart distance representing 1′ of longitude at the equator. Such tables date from a celebrated table to 0′·1 at interval 10′ published in 1599 by E. Wright in ignorance of its logarithmic character.

Modern tables sometimes give meridional parts for the actual spheroidal earth, which fall outside the scope of this *Index* (so Burton 1941, Hughes–Comrie 1938 and Norie 1938, p. 555).

0′·001	$0(1^\circ)90^\circ$, 7-dec. radian arg. also given	No Δ	Cayley 1885
0′·01	$0(1')90^\circ$	No Δ	Norie 1938 (151)
0′·01	$0(1')90^\circ$	No Δ	Inman 1891, 1906, 1940
0′·01	$0(1')90^\circ$ (From Inman)	No Δ	Becker & Van Orstrand 1924
0′·1	$0(1')87^\circ$; $87^\circ(0'\cdot1)90^\circ$	PPMD; no Δ	Guyou 1896 (82)
0′·1; 1′	$0(1')86^\circ 59'$; $87^\circ(1')90^\circ$	No Δ	Pryde 1930
1′	$0(1')90^\circ$	No Δ	Norie 1894, 1904
1′	$0(1')83^\circ 59'$	No Δ	Bowditch 1874
1′	$0(1')78^\circ 59'$	No Δ	Raper 1920

12·3. Cosh x cos x, etc.

Hayashi 1926 gives to 8 decimals the values of the four functions $\sinh x \sin x$, $\sinh x \cos x$, $\cosh x \sin x$ and $\cosh x \cos x$ for $x = 0(\cdot001)3(\cdot01)10$, without differences. This is equivalent to a tabulation of $\sinh(x+ix)$ and $\cosh(x+ix)$. See also the next article. For circular and hyperbolic functions of $x/\sqrt{2}$, see Arts. 7·9 and 10·49 respectively.

Hayashi 1930*b* gives values of the same four functions to 5 decimals for $x = 0(\cdot 01)10$, without differences.

Umansky 1932 gives to 4 or 5 decimals the values of the four functions

$$A = \cosh x \cos x \qquad B = \tfrac{1}{2}(\cosh x \sin x + \sinh x \cos x)$$
$$C = \tfrac{1}{2} \sinh x \sin x \qquad D = \tfrac{1}{4}(\cosh x \sin x - \sinh x \cos x)$$

for $x = 0(\cdot 001)\cdot 04(\cdot 01)6\cdot 3(\cdot 1)10$ (R. C. Archibald, *Scripta Math.*, **2**, 195, 1934).

See also Hohenemser & Prager 1933.

12·35. Polar Functions of Semi-Imaginaries

Kennelly 1921 (T. 6) gives moduli and arguments (phases) of all six hyperbolic functions of $z = re^{i\pi/4} = \dfrac{r+ir}{\sqrt{2}}$, for $r = 0(\cdot 1)6(\cdot 05)20\cdot 5$, without differences. The moduli of $\sinh z$ and $\cosh z$ are given to 5 figures, those of $\tanh z$, $\coth z$, $\operatorname{sech} z$ and $\operatorname{cosech} z$ to 4 or 5 figures. The phases are given to 1′ throughout.

Kennelly 1921 (T. 22) is a supplementary table for $r = 0(\cdot 01)\cdot 5$. It gives moduli to 5 decimals and phases to $0^\circ\cdot 0001$ of $\sinh z$, $\cosh z$, $\tanh z$, $\coth z$, $\operatorname{sech} z$, $\operatorname{cosech} z$, $\dfrac{\sinh z}{z}$ and $\dfrac{\tanh z}{z}$, without differences.

Perry 1910 gives moduli to 5–6 figures and phases to $0^\circ\cdot 01$ of $\sinh z$ and $\cosh z$ for $r = 0(\cdot 01)\cdot 1(\cdot 05)1$, without differences.

12·4. Rectangular Functions of a Rectangular Variable

The following tables give real and imaginary parts of functions of $z = x+iy$.

12·41. Sinh z

5 dec.	$x = 0(\cdot 05)4,\ y = 0(5^g)200^g$	No Δ	Kennelly 1921 (T. 7 and 13)
5 dec.	$x = \cdot 2(\cdot 2)1(1)3,\ y = 0(5^g)50^g(10^g)200^g$	No Δ	Sakamoto 1934
4 dec.	$x = 0(\cdot 1)3(\cdot 2)4(\cdot 5)7(1)10,\ y = 0(15^\circ)90^\circ$	No Δ	Johnstone 1911
4 (or 5) d.	$x = 0(\cdot 1)1\cdot 5,\ y = 0(\cdot 1)1\cdot 5,\ \tfrac{1}{2}\pi$	No Δ	McMahon 1906

12·42. Cosh z

5 dec.	$x = 0(\cdot 05)4,\ y = 0(5^g)200^g$	No Δ	Kennelly 1921 (T. 8 and 13)
5 dec.	$x = \cdot 2(\cdot 2)1(1)3,\ y = 0(5^g)50^g(10^g)200^g$	No Δ	Sakamoto 1934
4 dec.	$x = 0(\cdot 1)3(\cdot 2)4(\cdot 5)7(1)10,\ y = 0(15^\circ)90^\circ$	No Δ	Johnstone 1911
4 (or 5) d.	$x = 0(\cdot 1)1\cdot 5,\ y = 0(\cdot 1)1\cdot 5,\ \tfrac{1}{2}\pi$	No Δ	McMahon 1906

12·43. Tanh z

5 dec.	$x = 0(\cdot 05)4,\ y = 0(5^g)200^g$	No Δ	Kennelly 1921 (T. 9 and 13)
5 dec.	$x = \cdot 2(\cdot 2)1(1)3,\ y = 0(5^g)50^g(10^g)200^g$	No Δ	Sakamoto 1934

12·5. Polar Functions of a Rectangular Variable

The following tables give moduli and arguments (phases) of functions of $z = x + iy$.

12·51. Sinh z

5 dec., 0°·001 — $x = 0(\cdot05)4,\ y = 0(5^g)200^g$ — No Δ — Kennelly 1921 (T. 10 and 13)

12·515. Sin z

4; 3 dec., $0^g\cdot01$ — $\left\{\begin{array}{l} x = 0(2^g)100^g \\ y = 0(\cdot02)1 \\ 1/y = 1(\cdot02)\cdot34;\ \ \cdot32(\cdot02)\cdot20 \end{array}\right\}$ — Δ, Δ' — Hawelka 1931

12·52. Cosh z

5 dec., 0°·001 — $x = 0(\cdot05)4,\ y = 0(5^g)200^g$ — No Δ — Kennelly 1921 (T. 11 and 13)

12·53. Tanh z

5 dec., 0°·001 — $x = 0(\cdot05)4,\ y = 0(5^g)200^g$ — No Δ — Kennelly 1921 (T. 12 and 13)

12·535. Tan z

4 dec., $0^g\cdot01$ — $\left\{\begin{array}{l} x = 0(2^g)50^g \\ y = 0(\cdot02)1,\ \ 1/y = 1(\cdot02)\cdot20 \end{array}\right\}$ — Δ, Δ' — Hawelka 1931

12·6. Polar Functions of a Polar Variable

The following tables give moduli and arguments (phases) of functions of $z = re^{i\theta}$.

12·61. Sinh z

5 fig., 0°·001 — $r = 0(\cdot1)3,\ \ \theta = 45°(1°)90°$ — No Δ — Kennelly 1921 (T. 1)

6 dec., 0°·0001 — $r = 0(\cdot05)1,\ \theta = 0(5°)45°$ — No Δ — Kennelly 1921 (T. 17)

Modulus to only 5 decimals for $r < \cdot5,\ \theta = 45°$.

12·62. Cosh z

5 d.; 5 f., 0°·001 — $r = 0(\cdot1)1;\ \ 1\cdot1(\cdot1)3,\ \theta = 45°(1°)90°$ — No Δ — Kennelly 1921 (T. 2)

6 dec., 0°·0001 — $r = 0(\cdot05)1,\ \theta = 0(5°)45°$ — No Δ — Kennelly 1921 (T. 18)

Modulus to only 5 decimals for $r < \cdot5,\ \theta = 45°$.

12·63. Tanh z

5 d.; 5 f., 0°·001 — $r = 0(\cdot1)1;\ \ 1\cdot1(\cdot1)3,\ \theta = 45°(1°)90°$ — No Δ — Kennelly 1921 (T. 3)

6 dec., 0°·0001 — $r = 0(\cdot05)1,\ \theta = 0(5°)45°$ — No Δ — Kennelly 1921 (T. 19)

Modulus to only 5 decimals for $r < \cdot5,\ \theta = 45°$.

12·67. $\dfrac{\text{Sinh } z}{z}$

5 d.; 5 f., $0^\circ{\cdot}001$	$r = 0(\cdot 1)1;\ 1{\cdot}1(\cdot 1)3,\ \theta = 45^\circ(1^\circ)90^\circ$	No Δ	Kennelly 1921 (T. 4)
6 dec., $0^\circ{\cdot}0001$	$r = 0(\cdot 05)1,\ \theta = 0(5^\circ)45^\circ$	No Δ	Kennelly 1921 (T. 20)

Modulus to only 5 decimals for $r < \cdot 5,\ \theta = 45^\circ$.

12·68. $\dfrac{\text{Tanh } z}{z}$

5 d.; 5 f., $0^\circ{\cdot}001$	$r = 0(\cdot 1)3,\ \theta = 45^\circ(1^\circ)90^\circ$	No Δ	Kennelly 1921 (T. 5)
6 dec., $0^\circ{\cdot}0001$	$r = 0(\cdot 05)1,\ \theta = 0(5^\circ)45^\circ$	No Δ	Kennelly 1921 (T. 21)

Modulus to only 5 decimals for $r < \cdot 5,\ \theta = 45^\circ$.

12·8. Rectangular Inverse Functions of a Polar Variable

The following tables give real and imaginary parts, u and v, of functions of $z = re^{i\theta}$. According to New York W.P.A. 1942*d* (xxiii), the tables in Jahnke & Emde also occur in Emde 1924.

12·81. $\text{Sin}^{-1} z$

u to $0^g{\cdot}01$; v to 4 dec.	$r = 0(\cdot 02)1,\ 1/r = 1(\cdot 02)0$; $\theta = 0(2^g)100^g$	Δ, Δ'	Hawelka 1931
u to $0^r{\cdot}0001$; u to $0^\circ{\cdot}01$; v to 4 dec.	$r = 0(\cdot 2)2{\cdot}2,\ \theta = 0(15^\circ)90^\circ$	No Δ	Jahnke–Emde 1933

12·83. $\text{Tan}^{-1} z$

u to $0^g{\cdot}01$; v to 4 dec.	$r = 0(\cdot 02)1,\ \theta = 0(2^g)100^g$	Δ, Δ'	Hawelka 1931
u to $0^r{\cdot}0001$; u to $0^\circ{\cdot}01$; v to 4 dec.	$r = 0(\cdot 2)1,\ \theta = 0(15^\circ)90^\circ$	No Δ	Jahnke–Emde 1933
u to $0^r{\cdot}0001$; u to $0^\circ{\cdot}01$; $1/v$ to 4 dec.	$r = \cdot 8(\cdot 02)\cdot 98,\ \cdot 99,\ \cdot 999,\ 1-10^{-6},\ 1-10^{-9},\ 1$; $\theta = 75^\circ(5^\circ)90^\circ$	No Δ	Jahnke–Emde 1933

12·9. Miscellaneous

12·91. Coth x − cot x and cosech x − cosec x

4 dec. or 5 fig.	$x = 0(1^\circ)5^\circ(5^\circ)40^\circ(2^\circ)44^\circ(1^\circ)270^\circ$	No Δ	Darnley 1921

SECTION 13

EXPONENTIAL AND LOGARITHMIC INTEGRALS, SINE AND COSINE INTEGRALS, ETC.

13·0. Introduction

The fundamental notation of this section is as follows:

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,dt \qquad\qquad \mathrm{Si}(x) = \int_{0}^{x} \frac{\sin t}{t}\,dt$$

$$-\mathrm{Ei}(-x) = \int_{x}^{\infty} \frac{e^{-t}}{t}\,dt \qquad\qquad \mathrm{si}(x) = \int_{\infty}^{x} \frac{\sin t}{t}\,dt = \mathrm{Si}(x) - \frac{\pi}{2}$$

$$\mathrm{Li}(x) = \int_{0}^{x} \frac{dt}{\log_e t} = \mathrm{Ei}(\log_e x) \qquad\qquad \mathrm{Ci}(x) = \int_{\infty}^{x} \frac{\cos t}{t}\,dt$$

Other notations used by various authors are referred to in the appropriate articles. x is positive almost everywhere in the present section. The integrals for $\mathrm{Ei}(x)$ when $x > 0$ and for $\mathrm{Li}(x)$ when $x > 1$ are principal values; e.g.

$$\mathrm{Ei}(x) = \lim_{\epsilon \to +0} \left\{ \int_{-\infty}^{-\epsilon} \frac{e^t}{t}\,dt + \int_{\epsilon}^{x} \frac{e^t}{t}\,dt \right\}$$

The general arrangement of the section is as follows:

13·1. $\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,dt$

20 dec.	1(1)10	No Δ	Bretschneider 1843, 1861

Reprinted in Schlömilch 1848. On errors see Glaisher 1870.

20 dec.	10(1)20	No Δ	Gram 1884
18 dec.	0(·01)1	Δ^3	
11 dec.	1(·1)5; 6(1)15	Δ^3; no Δ	**Glaisher 1870**
43; 12 dec.	2; 20	—	
10–11 fig.	5(·1)15	No Δ	**B.A. 1927** (Airey)
	(For 0(·1)5, see Art. 13·11)	δ_m^{2n}	**B.A. 1, 1931**
10 dec.	0(·01)1(·1)7·5	No Δ	Bretschneider 1861
10 dec.	1(1)10	No Δ	Schlömilch 1846
	(From Bretschneider 1843)		
10; 9; 8 d.	5(·2)14; 14·2(·2)15; 15(·2)20	Δ	Gram 1884
10 fig.	0(·001)10	$I\delta^2$	**New York W.P.A. 1940*d***
10–11 fig.	10(·1)15, from B.A. 1	δ_m^4	
9 dec.	0(·0001)1·9999	$I\delta^2$	**New York W.P.A. 1940*c***

10 fig.	15(1)50	No Δ	**Coulson & Duncanson 1942**
9 dec.	0(·01)1(1)10	No Δ	Láska 1888–94
6 dec.	0(·01)1(·1)5(1)15	No Δ	Fowle 1933
5 d.; 5–6 fig.	0(·01)1(·1)3; 3·1(·1)5(1)15	No Δ	Dale 1903
4; 3; 2 dec.	0(·01)·05, ·1(·1)3·9; 4(·1)6·9; 7(1)10	No Δ	Hoüel 1901 (61)
4–6 fig.	0(·01)1(·1)5(1)15 Called $\overline{\mathrm{Ei}}(x)$	No Δ	Jahnke–Emde 1933, 1938
4–6 fig.	0(·01)·1(·05)1(·1)5(1)15	No Δ	Jahnke–Emde 1909

13·11. $\mathrm{Ei}(x) - \log_e x$

This is an auxiliary function for small x, since as x tends to zero, $\mathrm{Ei}(x) \sim \log_e x + \gamma$, where $\gamma = ·5772\ldots$ is Euler's constant.

11 dec.	0(·1)5	δ_m^6	B.A. **1**, 1931 (Fisher)
11 dec.	0(·01)·27	δ^4	Bickley (MS. 1940)
9 dec.	0(·0001)·01	$\mathrm{I}\delta^2$	New York W.P.A. 1940*c*

13·15. $\mathrm{Log}_{10}\,\mathrm{Ei}(x)$

6 dec.	4(·1)5(1)10, 20	Δ^2	Bellavitis 1874 (159)

For ·9(·05)1·3 and 1·2(·1)4, auxiliary functions are tabulated.

13·2. $-\mathrm{Ei}(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt$

This function is positive for positive x. Some authors tabulate $\mathrm{Ei}(-x)$, which is negative.

A few small tables bearing on $\mathrm{Ei}(-x)$ occur as particular cases in the tabulation of the incomplete gamma function (see Arts. 14·8 ff.), but these are insignificant compared with the tables dealing specifically with the exponential integral, and are not considered here.

20 dec.	1(1)10	No Δ	Bretschneider 1843, 1861

Reprinted in Schlömilch 1848. On errors see Glaisher 1870.

18+ dec. (8+ fig.)	0(·5)18(1)22, 24, 25, 26(2)42, 48	No Δ	Kotani, etc. 1938 + 1940

1940 paper gives a table containing errors and also gives errata in 1938 values.

18 dec.	0(·01)1	Δ^3	**Glaisher 1870**
11 dec.	1(·1)5; 6(1)15	Δ^3; no Δ	**Glaisher 1870**
43; 12 d.	2; 20	—	**Glaisher 1870**
10 fig.	15(1)50	No Δ	**Coulson & Duncanson 1942**
13–14 dec.	5(·1)15 (For 0(·1)5, see Art. 13·21)	No Δ	**B.A. 1927** (Airey)
		δ_m^{2n}	**B.A. 1, 1931**
10 dec.	0(·01)1(·1)7·5	No Δ	Bretschneider 1861
9 fig.	0(·001)10	$\mathrm{I}\delta^2$	**New York W.P.A. 1940*d***
14 dec.	10(·1)15, from B.A. **1**	δ_m^4	**New York W.P.A. 1940*d***
9 dec.	0(·0001)1·9999	$\mathrm{I}\delta^2$	**New York W.P.A. 1940*c***

9 dec.	·1(·001)1(·01)2 (For 0(·001)·1, see Art. 13·21)	No Δ	Miller & Rosebrugh 1903
9 dec.	0(·01)1(1)10	No Δ	Láska 1888–94
6 d.; 7 d.; 4–6 fig.	0(·01)1(·1)1·5; 1·6(·1)5; 6(1)15	No Δ	Fowle 1933
5–6 fig.	20(·02)50	Δ^2	Akahira 1929
5 dec.	0(·01)1(·1)5(1)10	No Δ	Dale 1903
4; 5; 6–8 dec.	0(·01)·05, ·1(·1)·9; 1(·1)6·9; 7(1)10	No Δ	Hoüel 1901 (61)
4 d.; 4 f.	0(·01)1(·1)1·5; 1·6(·1)5(1)15	No Δ	Jahnke-Emde 1933, 1938
4–5 fig.	0(·01)·1(·05)1(·1)5(1)15	No Δ	Jahnke-Emde 1909

13·21. $\mathrm{Ei}(-x) - \log_e x$

This is another auxiliary function for small x, as in Art. 13·11. It tends to γ as x tends to zero. New York W.P.A. 1940*c* tabulates $-\mathrm{Ei}(-x) + \log_e x$.

11 dec.	0(·1)5	δ_m^4	B.A. **1**, 1931 (Fisher)
11 dec.	0(·01)·27	δ^4	Bickley (MS. 1940)
9 dec.	0(·0001)·01	$\mathrm{I}\delta^2$	New York W.P.A. 1940*c*
9 dec.	0(·001)·1	No Δ	Miller & Rosebrugh 1903

13·25. $\displaystyle\int_x^\infty \frac{e^{-t}}{t^2}\,dt = \frac{e^{-x}}{x} + \mathrm{Ei}(-x)$

Any integral of the form $\displaystyle\int_x^\infty \frac{e^{-t}}{t^n}\,dt$, where n is a positive integer, may be made to depend on $\mathrm{Ei}(-x)$. When $n = 2$, the following direct tabulation has been made. See also Arts. 13·71 and 14·8 ff.

9 dec.	·1(·001)1(·01)2	No Δ	Miller & Rosebrugh 1903

For 0(·001)·1, see Art. 13·26.

13·26. $\displaystyle\int_x^\infty \frac{e^{-t}}{t^2}\,dt - \frac{1}{x} - \log_e x$

This is an auxiliary function for small x.

9 dec.	0(·001)·1	No Δ	Miller & Rosebrugh 1903

13·27. $-\mathrm{Ei}(-x) \times e^x(x + \log_e 2)$

Bellavitis 1874 (149) tabulates this for $x = 1(1)10(10)100$, naturally to 6 decimals without differences, and logarithmically to 6 decimals with Δ^4.

13·3. Hyperbolic Sine and Cosine Integrals

13·31. $\displaystyle\mathrm{Shi}(x) = \int_0^x \frac{\sinh t}{t}\,dt = \frac{\mathrm{Ei}(x) - \mathrm{Ei}(-x)}{2}$

20 dec.	1(1)10	No Δ	Bretschneider 1843, 1861
10 dec.	0(·01)1(·1)7·5	No Δ	Bretschneider 1861

At 4·9, for 66679 read 66687 (Glaisher 1870).

13·36. $\mathrm{Chi}(x) = \int_a^x \frac{\cosh t}{t}\,dt = \frac{\mathrm{Ei}(x) + \mathrm{Ei}(-x)}{2}$

In the above, a is such that $\mathrm{Ei}(a) + \mathrm{Ei}(-a) = 0$, or $a = \pm\, 0{\cdot}52382\ 257$.

20 dec.	1(1)10	No Δ	Bretschneider 1843, 1861
10 dec.	0(·01)1(·1)7·5	No Δ	Bretschneider 1861

13·4. $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$

For some tables of the integrand see Art. 7·63.

20 dec.	1(1)10	No Δ	Bretschneider 1843, 1861
Reprinted in Schlömilch 1848.			
18 dec.	0(·01)1	Δ^3	**Glaisher 1870**
11 dec.	1(·1)5; 6(1)15	Δ^3; no Δ	**Glaisher 1870**
7 dec.	20(5)100(10)200(100)1000(1000) 11,000, also $10^{5(1)8}$	No Δ	**Glaisher 1870**
43; 12 d.	2; 20	—	**Glaisher 1870**
11; 10 d.	0(·1)5; 5(·1)20(·2)40	δ_m^{2n}	**B.A. 1, 1931**
11 dec.	0(·01)·27	δ^4	Bickley (MS. 1941)
10 dec.	5(·1)20	No Δ	**B.A. 1927** (Airey)
10 dec.	20(·2)40	No Δ	**B.A. 1928** (Airey)
10 dec.	0(·01)1(·1)7·5	No Δ	Bretschneider 1861
At 4·9, for 56963 read 56955 (Glaisher 1870).			
10 dec.	0(·001)10	$\mathrm{I}\delta^2$	**New York W.P.A. 1940*d***
10 dec.	10(·1)20(·2)40, from B.A. 1	δ_m^{2n}	**New York W.P.A. 1940*d***
9 dec.	0(·0001)1·9999	$\mathrm{I}\delta^2$	**New York W.P.A. 1940*c***
10 dec.	10(·01)100	δ^2	**New York W.P.A. 1942*a***
10 dec.	1(1)10 (From Bretschneider 1843)	No Δ	Schlömilch 1846
9; 7 dec.	0(·01)1(1)10; $10^{2(1)5}$	No Δ	Láska 1888–94
5 dec.	0(·1)25, also 50	No Δ	Pedersen 1935
Reproduced in King 1937.			
5 dec.	16(1)60	No Δ	Rayleigh 1914
5 dec.	0(·01)1(·1)5(1)15	No Δ	Dale 1903
4 dec.	20(5)100(10)200(100)1000(1000) 10^4, 10^5, 10^6	No Δ	Dale 1903
4–5 fig.	0(·01)·17	No Δ	Jahnke–Emde 1933, 1938
4 dec.	·18(·01)1(·1)5(1)15(5)100(10) 200(100)10^3, also $10^{4(1)7}$	No Δ	Jahnke–Emde 1933, 1938
4–5 fig.	0(·01)·1	No Δ	Jahnke–Emde 1909
4 dec.	·15(·05)1(·1)5(1)15(5)100(10) 200(100)10^3, also $10^{4(1)7}$	No Δ	Jahnke–Emde 1909

13·41. $\mathrm{si}(x) = \int_\infty^x \frac{\sin t}{t}\,dt$

6 dec.	0(·01)50	Δ	Tani 1931

There is also a collection of formulæ relating to sine and cosine integrals.

13·45. $\int_0^x \frac{\sin \pi t}{\pi t} dt = \frac{1}{\pi} \mathrm{Si}(\pi x)$

7 dec.	0(·01)10	Δ	Jørgensen 1916

13·47. Si($n\pi$)

These are maxima and minima of Si(x). See also Art. 13·49.

15 dec.	n = 1(1)3 (See Art. 13·48)	New York W.P.A. 1940*d*

13·48. Si($n\pi \pm h$)

These are values of Si(x) near maxima and minima. New York W.P.A. 1940*d* gives them to 15 decimals with δ^2 and Iδ^4 for n = 1(1)3, h = 0(·0001)·01(·001)·05.

13·49. si($n\pi$) = Si($n\pi$) − $\frac{1}{2}\pi$

These are maxima and minima of si(x). See also Art. 13·47.

7 dec.	n = 1(1)40, also $n = p$ and p + 1, where p = 50(10)100(50)500(100)1000, 2000	Glaisher 1870
5 fig.	n = 1(1)15	Jahnke–Emde 1933, 1938

13·5. $\mathrm{Ci}(x) = \int_\infty^x \frac{\cos t}{t} dt$

20 dec.	1(1)10	No Δ	Bretschneider 1843, 1861
	Reprinted in Schlömilch 1848.		
18 dec.	0(·01)1	Δ^3	**Glaisher 1870**
11 dec.	1(·1)5; 6(1)15	Δ^3; no Δ	**Glaisher 1870**
7 dec.	20(5)100(10)200(100)1000(1000) 11,000, also $10^{5(1)8}$	No Δ	**Glaisher 1870**
43; 12 d.	2; 20	—	**Glaisher 1870**
10 dec.	5(·1)20(·2)40 (For 0(·1)5, see Art. 13·51)	δ_m^{2n}	**B.A. 1, 1931**
10 dec.	5(·1)20	No Δ	**B.A. 1927** (Airey)
10 dec.	20(·2)40	No Δ	**B.A. 1928** (Airey)
10 dec.	0(·01)1(·1)7·5	No Δ	Bretschneider 1861
10 dec.	0(·001)10	Iδ^2	**New York W.P.A. 1940*d***
10 dec.	10(·1)20(·2)40, from B.A. 1	δ_m^{2n}	**New York W.P.A. 1940*d***
9 dec.	0(·0001)1·9999	Iδ^2	**New York W.P.A. 1940*c***
10 dec.	10(·01)100	δ^2	**New York W.P.A. 1942*a***
10 dec.	1(1)10 (From Bretschneider 1843)	No Δ	Schlömilch 1846
9; 7 dec.	0(·01)1(1)10; $10^{2(1)5}$	No Δ	Láska 1888–94
	At x = 100, insert two zeros after decimal point.		
6 dec.	0(·0001)·05(·001)1(·01)50	Δ	**Tani 1931**
5 dec.	0(·01)1(·1)5(1)15	No Δ	Dale 1903
4 dec.	20(5)100(10)200(100)1000(1000) 10^4, 10^5, 10^6	No Δ	Dale 1903

4–5 fig.	0(·01)1(·1)5(1)15(5)100(10) 200(100)900	No Δ	Jahnke–Emde 1933, 1938
1–3 fig.	$10^{3(1)7}$		
About 4 f.	0(·01)·1(·05)1(·1)5(1)15(5)100(10) 200(100)1000, $10^{4(1)7}$	No Δ	Jahnke–Emde 1909

13·51. $\mathrm{Ci}(x) - \log_e x$

This is an auxiliary function for small x, since as x tends to zero, $\mathrm{Ci}(x) \sim \log_e x + \gamma$, where $\gamma = \cdot 5772\ldots$ is Euler's constant.

11 dec.	0(·1)5	δ^4_m	B.A. 1, 1931 (Fisher)
11 dec.	0(·01)·27	δ^4	Bickley (MS. 1941)
9 dec.	0(·0001)·01	$\mathrm{I}\delta^2$	New York W.P.A. 1940*c*

13·52. $\displaystyle\int_0^x \frac{1-\cos t}{t}\,dt = \log_e x + \gamma - \mathrm{Ci}(x)$

Pedersen calls this $S_1(x)$.

5 dec.	0(·1)25, also 50	No Δ	Pedersen 1935

Reproduced in King 1937.

13·57. $\mathrm{Ci}\left(\dfrac{2n-1}{2}\pi\right)$

These are maxima and minima of $\mathrm{Ci}(x)$.

15 dec.	$n = 1(1)3$ (See Art. 13·58)	New York W.P.A. 1940*d*
7 dec.	$n = 1(1)40$, also $n = p$ and $p+1$, where $p = 50(10)100(50)700$	Glaisher 1870
5 fig.	$n = 1(1)16$	Jahnke–Emde 1933, 1938

13·58. $\mathrm{Ci}\left(\dfrac{2n-1}{2}\pi \pm h\right)$

These are values of $\mathrm{Ci}(x)$ near maxima and minima. New York W.P.A. 1940*d* gives them to 15 decimals with δ^2 and $\mathrm{I}\delta^4$ for $n = 1(1)3$, $h = 0(\cdot 0001)\cdot 01(\cdot 001)\cdot 05$.

13·6. $\mathrm{Li}(x) = \displaystyle\int_0^x \frac{dt}{\log_e t}$

We consider separately tables of $\mathrm{Li}(x)$ for the smaller values of x and for the larger values which occur in connection with the distribution of primes.

13·61. $\mathrm{Li}(x)$ for Smaller Values of x

7 dec.	0(·01)1(·1)2(·5)3(1)20	No Δ.	Soldner 1809
8 fig.	22(2)40(5)80(10)160(20)320(40)640(80)1280		

Reprinted in De Morgan 1842*a* (662), where for argument 1220 read 1280. De Morgan also gives Li(·1) and Li(10) to 10 decimals, and the zero of $\mathrm{Li}(x)$ as 1·45136 92346 (from Soldner).

13·62. Li(x) for Larger Values of x

6; 5 dec.	$10^3, 10^4; 10^5(10^5)6\times10^5, 10^6$	No Δ	Bessel 1812*a*

Reprinted in James Glaisher 1883 (94), which see on accuracy (decimal part of Li(500,000) entirely wrong). 6-decimal values for $10^5(10^5)4\times10^5$ and for 10^6 were given by Bessel in 1810–11; see *Abhandlungen*, **2**, 327 and 329, 1876.

1 dec.	0(500,000)3,000,000	No Δ	Gauss 1849
0 dec.	0(50,000)10,000,000	No Δ	D. N. Lehmer 1914
0 dec.	0(50,000)9,000,000 (pp. 71, 84)	Δ (p. 74)	James Glaisher 1883
0 dec.	$10^7, 10^8$ (p. 89)	——	James Glaisher 1883
0 dec.	0(100,000)9,000,000, also $10^7, 10^8$	No Δ	B.A. 1883

From Glaisher.

Gauss 1863*a* gives $\mathrm{Li}(x+100{,}000)-\mathrm{Li}(x)$ to 2–5 decimals for $x = 1{,}000{,}000(100{,}000)2{,}900{,}000$, as well as Li(2,000,000) − Li(1,000,000) to 2 decimals and Li(3,000,000) − Li(2,000,000) to 3 decimals.

13·63. $\mathrm{Li}(x) - \log_e(1-x)$

This remains finite at $x = 1$.

10 dec.	0(·01)1	Δ^4	Legendre 1816 (448), 1826 (516)

Bretschneider 1837 gives as vulgar fractions, and also to 37 decimals, the coefficients of $x, x^2, \ldots x^{14}$ in the expansion of $\mathrm{Li}(1+x) - \log_e x$.

13·64. $\mathrm{Li}(x) - \log_e|\log_e x| - \log_e x$

This remains finite at $x = 1$.

6 dec.	·5(·01)2·8	Δ^2	Bellavitis 1874 (152)

For $x = 0(\cdot01)\cdot62$ Bellavitis tabulates (p. 150) $\dfrac{\mathrm{Li}(x)}{x}\log_e\left(\dfrac{x}{2}\right)$ to 6 decimals with Δ^3.

13·65. $\mathrm{Li}(10^x)$

We have not seen Stenberg's rare tables. Differences are apparently given. See J. Glaisher 1883 (91), Gram 1884 (267), New York W.P.A. 1940*c* (xxii), etc.

18 dec.	$x = -15(\cdot05)-10(\cdot02)-7(\cdot01)-\cdot5$	Stenberg 1861
15–13 dec.	$x = 0(\cdot01)3\cdot5$	Stenberg 1867
14 fig.	$x = 3\cdot5(\cdot01)10$	Stenberg 1871

13·66. $\mathrm{Li}(10^x) - \log_e\log_e(10^{-x})$

This remains finite at $x = 0$.

18 fig.	$x = -\cdot5(\cdot01)0$	Stenberg 1861

13·68. $\displaystyle\int_2^x \frac{dt}{(\log_e t)^2}$

This is equal to $\displaystyle\int_2^x \frac{dt}{\log_e t} - \frac{x}{\log_e x} + \frac{2}{\log_e 2}$, and is given to the nearest integer for $x = (1, 2, 4, 6, 8)\times 10^{2(1)4}$ and $10^5(10^5)8\times10^5$ in Sutton 1937 (34), in connection with the distribution of twin primes.

13·7. Integrals depending on Exponential, Sine and Cosine Integrals

We deal below mainly with Ser 1938, which tabulates two kinds of integral. The first depends on the exponential integral and the second on sine and cosine integrals.

13·71. Ser's Integrals of the First Kind

Ser 1938 uses the notation

$$g_n(y) = \int_0^\infty \frac{e^{-x}x^n dx}{(x+y)^{n+1}} \qquad \gamma_n(y) = \int_0^\infty \frac{e^{-x}y^n dx}{(x+y)^{n+1}}$$

These integrals depend on $\mathrm{Ei}(-y)$; compare Art. 13·25. We have in fact

$$g_0(y) = \gamma_0(y) = -e^y \mathrm{Ei}(-y)$$

and the remaining integrals may be evaluated by recurrence relations. Ser tabulates $g_n(y)$ to 4–8 decimals for $n = 0(1)6, y = 0(\cdot 01)\cdot 1(\cdot 05)\cdot 5(\cdot 1)5(\text{var.})20$. He also gives $g_n(1)$ and $\gamma_n(1)$ to 13–15 decimals for $n = 0(1)14$, and $g_0(y)$ to 12–15 decimals for $y = 1, 1\cdot 5, 2, 3$. His value of $g_0(1) = -e\mathrm{Ei}(-1)$ is ·59634 73623 23194, which may be compared with

$$e\int_1^\infty \frac{e^{-x}dx}{x} = \cdot 59634\ 73623\ 24$$

given in De Morgan 1842*a* (650). Our calculations confirm Ser's value. See also Glaisher 1875 (256).

13·72. Ser's Integrals of the Second Kind

Ser 1938 also considers the integrals

$$A_n(z) = \int_0^\infty \frac{e^{-x}x^{2n}dx}{(x^2+z^2)^{n+1}} \qquad \alpha_n(z) = \int_0^\infty \frac{e^{-x}z^{2n}dx}{(x^2+z^2)^{n+1}}$$

$$B_n(z) = \int_0^\infty \frac{e^{-x}x^{2n+1}dx}{(x^2+z^2)^{n+1}} \qquad \beta_n(z) = \int_0^\infty \frac{e^{-x}xz^{2n}dx}{(x^2+z^2)^{n+1}}$$

These depend on $\mathrm{si}(z)$ and $\mathrm{Ci}(z)$. We have in fact

$$A_0(z) = \alpha_0(z) = \frac{\sin z\, \mathrm{Ci}(z) - \cos z\, \mathrm{si}(z)}{z}$$

$$B_0(z) = \beta_0(z) = -\sin z\, \mathrm{si}\, z - \cos z\, \mathrm{Ci}(z)$$

and the remaining integrals may be evaluated by recurrence relations. Ser tabulates the four integrals to 8 decimals for $n = 0(1)10$, $z = 1, 2$. His values

$$A_0(1) = \alpha_0(1) = \cdot 62144\ 962 \qquad B_0(1) = \beta_0(1) = \cdot 34337\ 796$$

may be compared with

$$\int_0^\infty \frac{e^{-x}dx}{1+x^2} = \cdot 62144\ 96242\ 36 \qquad \int_0^\infty \frac{e^{-x}x dx}{1+x^2} = \cdot 34327\ldots$$

given in De Morgan 1842*a* (651). Our calculations confirm De Morgan's first value. His second is wrong in the fourth and later places, and we do not quote it in full. We find that it should be ·34337 79615 56.

13·73. Other Integrals

King 1937 tabulates to 4 figures without differences for $x = 0(\cdot 1)8$ the functions

$$F(x) = \int_0^x \frac{\sin t}{t} \sin(x-t)\,dt = -\tfrac{1}{2}\cos x\, S_1(2x) + \tfrac{1}{2}\sin x\, \mathrm{Si}(2x)$$

$$G(x) = \int_0^x \frac{\sin t}{t} \cos(x-t)\,dt = \tfrac{1}{2}\sin x\, S_1(2x) + \tfrac{1}{2}\cos x\, \mathrm{Si}(2x)$$

On S_1 see Art. 13·52. King also tabulates other functions depending on the sine and cosine integrals.

Lommel 1899 (505) tabulates to 4 decimals without differences for $x = 0(\cdot 05)3(\cdot 1)11(\cdot 5)15\cdot 5$ the two functions

$$\int_0^x \left(\frac{\sin t}{t}\right)^2 dt = \mathrm{Si}(2x) - \frac{\sin^2 x}{x} \quad \text{and} \quad \frac{1}{x}\int_0^x \left(\frac{\sin t}{t}\right)^2 dt$$

The first function is also given to 5 decimals for $x/\pi = \cdot 5(\cdot 5)2\cdot 5$.

See also Bieler 1921, B.A. 1928 (319) and Schwarz 1944.

SECTION 14

FACTORIAL OR GAMMA FUNCTION, PSI FUNCTION, POLYGAMMA FUNCTIONS, BETA FUNCTION, INCOMPLETE GAMMA AND BETA FUNCTIONS

14·0. Introduction

The factorial function and psi function used in this section are

$$x! \quad \text{and} \quad \psi(x) = \frac{d}{dx}(\log_e x!)$$

Whatever choice of notation is made, much alteration of printed arguments by unity is necessary if the various tables are to be indexed on a uniform basis.

The most common notations are:

$x! = \Gamma(1+x)$	Legendre
$= \Pi(x)$	Gauss
$\frac{d}{dx}(\log_e x!) = \psi(x)$	Gray, Mathews and MacRobert; this *Index*
$= \Psi(x)$	Gauss; Jahnke and Emde
$= \psi(1+x)$	Whittaker and Watson
$= \Psi(1+x)$	Davis
$= \mathsf{F}(x)$	Pairman

In common with Lodge, the British Association, and Gray, Mathews and MacRobert, we also use the function

$$\phi(x) = \psi(x) + \gamma = \sum_{r=1}^{\infty} \frac{x}{r(r+x)} = \sum_{r=1}^{\infty} \left(\frac{1}{r} - \frac{1}{r+x}\right)$$

where $\phi(0) = 0$, $\psi(0) = -\gamma$, and, if n is a positive integer,

$$\phi(n) = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \frac{1}{n}$$

For the definitions of the polygamma functions see Art. 14·5.

The main divisions of this section are as follows:

14·0 Introduction
14·1 Natural Values of $x!$, etc.
14·2 Logarithmic Values of $x!$, etc.
14·3 Various Quantities connected with $x!$ and its Derivatives
14·4 $\psi(x)$ and $\phi(x)$
14·5 Polygamma Functions
14·6 Gamma Function of Complex Argument. Generalized Gamma Function
14·7 Beta Function, etc.
14·8 Incomplete Gamma Function
14·9 Incomplete Beta Function

14·1. $x! = \Gamma(1+x)$

For integral x see Art 3·11.

18 dec.	$x = 0(\cdot 01)1$		Johnston (MS.)

Computed from the 20-decimal logarithms of Gauss (see Art. 14·2). It is hoped to include this table in a B.A. volume.

12 dec.	$x = 0(\cdot 01)1$	δ^4_m	**B.A. 1, 1931** (Fisher)

Computed from the 12-decimal logarithms of Legendre (see Art. 14·2).

12 dec.	$x = 0(\cdot 1)1$	No Δ	Jeffery 1864
10 dec.	$x = 0(\cdot 001)1$ (T. 2)	No Δ	Davis 1933
10 dec.	$x = 0(\cdot 0001)\cdot 1$ (T. 1)	No Δ	Davis 1933
10 fig.	$x = -11(\cdot 01)0$ (T. 5)	No Δ	Davis 1933

On last-figure errors detected by Comrie in Table 2, see *Scripta Math.*, 3, 363–364, 1935 and Davis 1935 (xii).

8 dec.	$x = -3(\cdot 01)0(\cdot 001)1(\cdot 01)3$	No Δ	Hayashi 1926
7 dec.	$x = -6(\cdot 01)-3,\ 3(\cdot 01)4$	No Δ	Hayashi 1926
7 d.; 5 f.	$x = 0(\cdot 001)1\cdot 05;\ -1(\cdot 1)+10$		Cassinis 1930
5 dec.	$x = -6(\cdot 01)0(\cdot 001)1(\cdot 01)4$	No Δ	Hayashi 1930*b*
5 dec.	$x = 0(\cdot 001)1$	IΔ	Dwight 1941
5 dec.	$x = 0(\cdot 01)1$	Δ	Dwight 1934
5 dec.	$x = 0(\cdot 01)1$	No Δ	Schorr 1916
$4\frac{1}{2}$ dec.	$x = 0(\cdot 01)1$	Δ	MT.C. 1931
$4\frac{1}{2}$ d.; $6\frac{1}{2}$ f.	$x = -\cdot 5(\cdot 5)+4\cdot 5;\ 5(\cdot 5)9$	No Δ	MT.C. 1931
4 d.; 4–5 f.	$x = 0(\cdot 01)2;\ 2(\cdot 01)4$	IΔ	Jahnke–Emde 1933, 1938
4 dec.	$x = 0(\cdot 01)1$	No Δ	Allen 1941
4 dec.	$x = 0(\cdot 05)1$	No Δ	Jahnke–Emde 1909 Silberstein 1923

14·15. $\dfrac{1}{x!} = \dfrac{1}{\Gamma(1+x)}$

For integral x see Art. 3·12.

12 dec.	$x = -2(\cdot 1)0$	No Δ	Davis 1933 (270)
5 fig.	$x = -3(\cdot 01)+3$	IΔ	Jahnke–Emde 1938

14·2. $\text{Log}_{10}(x!) = \log_{10}\Gamma(1+x)$

For integral x see Art. 3·14.

20 dec.	$x = 0(\cdot 01)1$	No Δ	Gauss 1813

Reprinted in Davis 1933 (T. 12) and Brunel 1886.

15 dec.	$x = -11(\cdot 01)0$	No Δ	Davis 1933
14 dec.	$x = -1(\frac{1}{12})+1$	No Δ	Legendre 1814 (37), 1826 (455) De Morgan 1842*a* (590) Schlömilch 1848 (**1**, 171)
12 dec.	$x = 0(\cdot 001)1$	Δ^3	**Legendre 1814** (86), **1826** (490)

Reproduced in facsimile in Legendre–Pearson 1921 and reprinted in Schlömilch 1848 and Davis 1933. Errors of one unit in last figure. At ·564, for 346 read 446 (Comrie), and at ·664, for 045 read 145 (Miller). This table supersedes the 7-decimal table in Legendre 1811 (302), which is inaccurate in the seventh place.

12 dec.	$x = 0(\cdot01)1$		De Morgan 1842*a* (587)

Abridged from Legendre, with enough information about Legendre's differences to enable Legendre's table to be reconstructed.

12 dec.	$x = 1(\cdot01)10;\ 10(\cdot1)100$	δ^4; no Δ	Davis 1933
10 dec.	$x = 0(\cdot0001)\cdot1$	Δ^2	Davis 1933
10 dec.	$x = 1(\cdot1)4(\cdot2)69$	δ^{2n}	E. S. Pearson 1922
9 dec.	$x = 0(\cdot01)1$	No Δ	Láska 1888–94 (290)
8; 13 d.	$x = -1(\cdot01)0;\ 0(\cdot00001)\cdot001$	No Δ	Hayashi 1926
10; 8 d.	$x = \cdot001(\cdot0001)\cdot1;\ \cdot1(\cdot001)1(\cdot01)2$		
8 dec.	$x = 9(\cdot1)11$	Δ^3	Bellavitis 1874
7 dec.	$x = 0(\cdot01)49\cdot99$	δ^2	Brownlee 1923 (Russell)
7 dec.	$x = -5(\cdot01)+15$, $-\log_{10}(x!)$ tabulated	Δ	Jørgensen 1916
7 dec.	$x = 0(\cdot001)1$	Δ	Duffell 1909; K. Pearson 1930 (58)
7 dec.	$x = 0(\cdot001)1$	No Δ	Fowle 1933; Glover 1930; Potin 1925; Carr 1886 (30); Bertrand 1870 (285); Legendre 1811 (302)
6 dec.	$x = 0(\cdot001)1$	No Δ	Elderton 1938; Williamson 1884; Davenport & Ekas 1936
5 dec.	$x = 0(\cdot001)1$	IΔ	Plummer 1940
5 dec.	$x = 0(\cdot01)1$	Δ	Dwight 1934
5 dec.	$x = 0(\cdot01)1$	IΔ	Hoüel 1901
5 dec.	$x = 0(\cdot01)1$	No Δ	Dale 1903; G. W. Jones 1893 (38); Hayashi 1930*b*
$4\frac{1}{2}$ dec.	$x = 0(\cdot01)1;\ -\cdot5(\cdot5)+9$	Δ; no Δ	MT.C. 1931
4 dec.	$x = 0(\cdot01)1$	No Δ	Jahnke-Emde 1909; Peirce 1910

14·21. $\mathrm{Log}_{10}\dfrac{\Gamma(1+x)}{\sqrt{2\pi}} = \log_{10}(x!) - 0\cdot39908\ 99342\ldots$

This is tabulated to 10 decimals for $x = 0(\cdot01)1\cdot05$ with Δ^3 in Bessel 1812*b*. The logarithms are negative and are given in entirely negative form, so that the digits of the mantissæ are those of $\log_{10}\dfrac{\sqrt{2\pi}}{x!}$.

14·22. $\mathrm{Log}_e(x!)$

10 dec.	$x = 0(\cdot005)1$	No Δ	B.A. 1916 (Watson)
8 dec.	$x = 9(\cdot1)10\cdot1$	Δ^3	Bellavitis 1874

14·25. $\int_0^x \log_{10}(x!)\,dx$

10 dec.	$x = 0(\cdot01)1$		No Δ	B.A. 1916 (Watson)
10 dec.	$x = 0(\cdot01)1$	From the above	δ^2_m	B.A. 1, 1931

14·3. Various Quantities connected with the Gamma Function and its Derivatives

14·31. $\frac{1}{n!}\left(\frac{d}{dx}\right)^n \Gamma(1+x)$

27+ dec.	$n = 1(1)4,\ x = 0(1)9$	No Δ	Miller (MS.)
12 dec.	$n = 1(1)20,\ x = 0(1)4$		
12 dec.	$n = 0, 1,\ x = 0(\cdot 1)1$	No Δ	Jeffery 1864
12; 10 dec.	$n = 2(1)20,\ x = 0;\ \cdot 1(\cdot 1)1$		

14·32. Main Minimum of $x! = \Gamma(1+x)$

It has been stated in Legendre 1814 (71) and 1826 (436), and quoted by various authors, that the main minimum occurs at $x_0 = 0{\cdot}46163\ 21451\ 105$, and that $\log_{10}(x_0)!$ is $\bar{1}{\cdot}94723\ 91743\ 9340$. Davis 1933 (278) however gives $x_0 = 0{\cdot}46163\ 21450$. Calculations to about 25 decimals by J. C. P. Miller give (retaining 15 decimals)

$$\begin{aligned} x_0 &= 0{\cdot}46163\ 21449\ 68362 \\ \log_{10}(x_0)! &= \bar{1}{\cdot}94723\ 91743\ 93385 \\ \log_e(x_0)! &= \bar{1}{\cdot}87851\ 37094\ 64150 \\ (x_0)! &= 0{\cdot}88560\ 31944\ 10889 \end{aligned}$$

The natural value of $(x_0)!$ is wrongly given as 0·88560 24 in Gauss 1813 (at any rate as reproduced in *Werke*, **3**, 147, 1866). The correct 7-decimal value is given in Bertrand 1870 (284), Carr 1886 (364) and Hayashi 1926 (273), 1930*b* (53, 155).

14·325. Negative Minima of $|x!| = |\Gamma(1+x)|$

The first 4 negative roots of $\Gamma'(1+x_n) = 0$ to 3 decimals and the corresponding stationary values, $\Gamma(1+x_n)$, to 9 decimals are given in Hayashi 1926 (273) and Hayashi 1930*b* (53, 155).

Values $x_1 = -1{\cdot}504072$, $x_2 = -2{\cdot}573587$ are given in Brunel 1886 (117), from Bourguet 1881.

Ginzel 1931 (292) quotes Hayashi's values in abridged form and (348) gives x_5, x_6 to 4 decimals and x_7, x_8, x_9 to 5 decimals.

14·33. $\Gamma(\frac{1}{3})$ and $\Gamma(\frac{2}{3})$

Airey 1916*a* (525) gives the values

$$\log_{10}\Gamma(\tfrac{1}{3}) = 0{\cdot}42796\ 27493\ 1426 \qquad \Gamma(\tfrac{1}{3}) = 2{\cdot}67893\ 85347\ 077$$
$$\log_{10}\Gamma(\tfrac{2}{3}) = 0{\cdot}13165\ 64916\ 8402 \qquad \Gamma(\tfrac{2}{3}) = 1{\cdot}35411\ 79394\ 264$$

The logarithms agree with Legendre's 14-decimal table of $\log\Gamma(x)$ at interval $\frac{1}{12}$ (see Art. 14·2). Calculations to 20 and more decimals by L. J. Comrie and J. C. P. Miller confirm all four values. We may also note that

$$\log_e\Gamma(\tfrac{1}{3}) = 0{\cdot}98542\ 06469\ 2777 \qquad \log_e\Gamma(\tfrac{2}{3}) = 0{\cdot}30315\ 02751\ 4752$$

See also Art. 5·831.

14·34. Coefficients in Power Series

Bourguet 1883 gives 16-decimal values of the coefficients in the power series for $1/\Gamma(x)$ and for five other functions involving $\Gamma(x)$. The values are reproduced in Davis 1933 (185–186), and five of the six sets are reproduced in Brunel 1886 (123–127). For a few corrections and additions see E. Isaacson and H. E. Salzer, *M.T.A.C.*, **1**, 124, 1943.

If $\log_e \Gamma(2+x) = C_1 x + C_2 x^2 - C_3 x^3 + C_4 x^4 - \ldots$, the coefficients C_n occur in several series. We have $C_1 = 1-\gamma$, where γ is Euler's constant (see Art. 5·41), while for $n > 2$ we have $C_n = (-)^n \psi^{(n-1)}(1)/n! = (S_n - 1)/n$, see Arts. 14·53 and 4·6111. Values of C_n for $n = 1(2)11$ are given to 10 decimals in Schlömilch 1879 (258) and Láska 1888–94 (272), and to 9 decimals in Jahnke & Emde 1933 (87), 1938 (10). All these have the error in C_3 mentioned in Art. 4·6113.

Values of MC_n to 12–14 decimals and of $\log_{10}(MC_n)$ to 10–14 decimals, where $M = \log_{10} e$ (see Art. 5·31), are given in Legendre 1814 (71), 1826 (437) for $n = 1(1)15$ and in Schlömilch 1848 (**1**, 171) for $n = 1(2)15$. Láska 1888–94 (289) gives MC_n and $\log_{10}(MC_n)$ to 10 decimals for $n = 1(2)15$. MC_n is called B_n by Legendre and T_n by Schlömilch and Láska.

On coefficients in Stirling's series for $\log_e \Gamma(x)$, see Art. 4·13.

14·35. $\displaystyle\int_{n-1}^{n} \frac{dx}{\Gamma(x)}$

16 dec.	$n = 0(1)3$	Bourguet 1883

14·36. $\displaystyle\int_0^\infty \sin(x^n)\,dx$ and $\displaystyle\int_0^\infty \cos(x^n)\,dx$

Glaisher 1871*d* gives these functions to 7 decimals without differences, the first for $\pm n = 2(1)10$ and the second for $n = 2(1)10$. For these values of n the integrals are respectively $\Gamma\left(1+\frac{1}{n}\right)\sin\frac{\pi}{2n}$ and $\Gamma\left(1+\frac{1}{n}\right)\cos\frac{\pi}{2n}$.

14·37. $\dfrac{x^x e^{-x}\sqrt{x+1}}{\Gamma(x+1)}$

By Stirling's formula, this expression $\to \dfrac{1}{\sqrt{2\pi}}$ as $x \to \infty$.

7 dec.	$x = -1(\cdot05)0(\cdot1)50\cdot1$	No Δ	K. Pearson 1922 (T. 4)

14·38. $\text{Log}_{10} \sin \pi x$

This is useful since $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$, or $x!(-x)! = \pi x/\sin \pi x$. Davis 1933 (189) gives $\log_{10} \sin \pi x$ to 10 decimals without differences for $x = 0(\cdot01)1$. See also Arts. 8·31 and 14·58.

14·4. $\psi(x) = \dfrac{d}{dx}(\log_e x!) = \dfrac{\Gamma'(1+x)}{\Gamma(1+x)}$

This is the psi function or "digamma function", with the properties

$$\psi(0) = -\gamma \qquad \psi(n) = -\gamma + 1 + \tfrac{1}{2} + \ldots + \frac{1}{n}$$

For zeros, see Arts. 14·32 and 14·325.

35 dec.	$x = 0(1)11$		Miller (MS.)
23 dec.	$x = -\frac{1}{2}, \frac{1}{2}, 10$		Gauss 1813
18 dec.	$x = 0(\cdot 01)1$	No Δ	**Gauss 1813** (Nicolai)

Reproduced in Davis 1933 (T. 12) and Brunel 1886.

13 dec.	$x = 0(\cdot 5)100$		No Δ	**B.A. 1916** (Watson)
12 dec.	$x = 0(\cdot 01)1$ and $10(\cdot 1)60$		δ_m^{2n}	**B.A. 1, 1931**
10 dec.	$x = 0(\cdot 0001)\cdot 1$	(T. 7)	Δ^2	**Davis 1933**
10 dec.	$x = 0(\cdot 001)1$	(T. 8)	Δ^3	**Davis 1933**
12 dec.	$x = 1(\cdot 02)19$	(T. 9)	δ^{2n}	**Davis 1933**
16 dec.	$x = -\cdot 5(\cdot 5)+99$	(T. 10)	No Δ	**Davis 1933**
10 dec.	$x = 100(1)449$	(T. 10)	No Δ	**Davis 1933**
15 dec.	$x = -11(\cdot 01)0$	(T. 11)	No Δ	**Davis 1933**

On errors detected by Comrie in Table 8, see *Scripta Math.*, **3**, 363–364, 1935 and Davis 1935 (xii).

8 dec.	$x = 0(\cdot 02)20$	δ^{2n}	Pairman 1919
8 dec.	$x = \cdot 5(\cdot 5)10$	No Δ	Jeffreys 1939 (375)
7; 4 dec.	$x = 0(\cdot 01)1$; $-1(\cdot 1)+10$		Cassinis 1930
7 dec.	$x = 0(\cdot 01)\cdot 5$	v^3	Knar 1865 (168)
4 fig.; 4 d.	$x = 0(\cdot 01)1$; $0(1)10$	No Δ	Jahnke-Emde 1909
4 dec.	$x = 0(\cdot 01)1$	IΔ	Jahnke-Emde 1933, 1938
4 dec.	$x = 0(\cdot 01)1$	No Δ	Russell 1914 (241)
4 dec.	$x = 0(\cdot 05)1(1)10$	No Δ	Silberstein 1923

A manuscript table by D. R. Hartree and S. Johnston tabulates the real part of $\psi(z) - \log_e z + \frac{1}{2}z^{-1}$, where $\psi(z) = \Gamma'(z)/\Gamma(z) =$ our $\psi(z-1)$, to 10 decimals for $z^{-2} = -1(\cdot 02)+1$. When $z^2 > 0$, it is understood that $z > 0$. The function tabulated vanishes at $z^{-2} = 0$. Notice that half the range of the table corresponds to pure imaginary z; when $z = ia$ $(a > 0)$, the function tabulated is the real part of $\psi(ia) - \log_e a$, that is,

$$\frac{d}{da}\{\arg \Gamma(ia)\} - \log_e a$$

The imaginary part of $\psi(ia)$ is $-\frac{d}{da}\log|\Gamma(ia)|$, that is, since $|\Gamma(ia)| = \sqrt{\frac{\pi}{a \sinh \pi a}}$, simply $\frac{1}{2}\pi \coth \pi a + \frac{1}{2}a^{-1}$. See also Art. 14·6.

14·45. $\mathrm{Log}_{10}\, \psi(x)$

10 dec.	$x = 0(\cdot 0001)\cdot 1$	(T. 7)	No Δ	Davis 1933
10 dec.	$x = 0(\cdot 001)1$	(T. 8)	No Δ	Davis 1933
10 dec.	$x = -11(\cdot 01)0$	(T. 11)	No Δ	Davis 1933

On errors detected by Comrie in Table 8, see *Scripta Math.*, **3**, 364, 1935.

14·48. $\phi(x) = \sum_{r=1}^{\infty} \frac{x}{r(r+x)} = \psi(x) + \gamma$

$\phi(0) = 0$, $\phi(n) = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$, where n is a positive integer. For this case see Art. 4·612.

16; 10 dec.	$x = 0(\cdot 1)60\cdot 3$; $50(\cdot 01)51$	No Δ; Δ^2	B.A. 1929 (Lodge)

$$14\cdot 5.\quad \psi^{(n-1)}(x) = \left(\frac{d}{dx}\right)^n (\log_e x!) = \left(\frac{d}{dx}\right)^{n-1} \frac{\Gamma'(1+x)}{\Gamma(1+x)}$$

The notation is:

digamma function	$= \psi(x)$,	case $n = 1$ (see Art. 14·4 for this)
trigamma function	$= \psi'(x)$,	case $n = 2$
tetragamma function	$= \psi''(x)$,	case $n = 3$
pentagamma function	$= \psi'''(x)$,	case $n = 4$
hexagamma function	$= \psi^{\mathrm{iv}}(x)$,	case $n = 5$

12 dec.	$n = 2(1)4$, $x = 0(\cdot 01)1$ and $10(\cdot 1)60$	δ_m^{2n}	**B.A. 1, 1931**
	Only 10 decimals for $n = 4$, $x = 0(\cdot 01)1$.		
10–12 d.	$n = 2(1)5$, $x = -11(\cdot 1)-1(\cdot 01)0$	No Δ	**Davis 1935**
10–19 d.	$n = 2(1)5$, $x = 0(\cdot 01)3(\cdot 02)19(\cdot 1)99$	δ^{2n}	**Davis 1935**
8 dec.	$n = 2$, $x = 0(\cdot 02)20$	δ^{2n}	Pairman 1919
5 dec.	$n = 2$, $x = 0(\cdot 01)\cdot 5$ See Art. 14·4	$wD/2$, $w^2D^2/6$	Knar 1865 (168)
4 dec.	$n = 2$, $x = 0(\cdot 01)1$	IΔ	Jahnke–Emde 1933, 1938
3 dec.	$n = 2$, $x = 0(\cdot 05)1$ From Emde 1906	No Δ	Jahnke–Emde 1909; Silberstein 1923

$$14\cdot 51.\quad 1/\psi'(x)$$

3 dec.	$x = 0(\cdot 05)1$ From Emde 1906	No Δ	Jahnke–Emde 1909

$$14\cdot 53.\quad \frac{1}{(n-1)!}\psi^{(n-1)}(x) = (-)^n \sum_{r=1}^{\infty} \frac{1}{(r+x)^n}$$

32 dec.	$n = 2(1)41$, $x = 2(1)9$	Miller (MS.)
	For $x = 0, 1$, see S_n, Arts. 4·6 ff.	
5; 3; 1 d.	$n = 2;\ 3;\ 4$, $x = 0(\cdot 01)\cdot 5$ See Art. 14·4	Knar 1865 (168)

$$14\cdot 55.\quad \mathrm{Log}_{10}\, \psi^{(n-1)}(x)$$

10 dec.	$n = 2(1)5$, $x = -11(\cdot 1)-1(\cdot 01)0$	No Δ	Davis 1935

14·58. $\pi \cot \pi x$ and Derivatives

These functions are useful in connection with the psi function and its derivatives, since $\psi(-x) - \psi(x-1) = \pi \cot \pi x$.

Davis 1933 (288) gives $\pi \cot \pi x$ to 15 decimals without differences for $x = 0(\cdot 01)1$.

Davis 1935 (23–24) gives $(-)^n\left(\frac{d}{dx}\right)^n(\pi \cot \pi x)$, $n = 1(1)4$, without differences for $x = 0(\cdot 01)1$. The first derivative is given to 12 decimals, the second and third to 10 decimals, and the fourth to 9–14 decimals (17 or more figures).

See also Art. 14·38.

14·6. Gamma Function of Complex Argument

Davis 1933 (T. 6) tabulates the real and imaginary parts of $1/\Gamma(z)$ to 12 decimals, where $z = re^{i\theta}$, $r = -1(\cdot 1)+1$, $\theta = 0, 30^\circ, 45^\circ, 60^\circ, 90^\circ, 120^\circ, 135^\circ, 150^\circ$.

Diagrams and a large amount of numerical information about the gamma function of complex argument are given in Ginzel 1931. The statement in Jahnke & Emde 1933 (95), 1938 (21) that Ginzel gives in this work a 6-figure table of $(x+iy)!$ is, however, incorrect. Presumably the reference is to Tables 1 and 2 (pp. 345–347), which in fact give, to 5 decimals, co-ordinates of points on the curves

$$\arg \Gamma(x+iy) = \tfrac{1}{2}k\pi \quad \text{and} \quad \log_e |\Gamma(x+iy)| = \tfrac{1}{2}k\pi$$

for various integral values of k. We may add that entries on page 346 for $x = \frac{5}{2}(\frac{1}{2})\frac{17}{2}$ should be lowered by one line.

We have not seen Meissner 1939, but detailed and accordant descriptions of the tables have been published in *Math. Reviews*, **1**, 31–32, January 1940 and in *Zs. f. angew. Math. u. Mech.*, **20**, 295, October 1940 (see also *M.T.A.C.*, **1**, 177, 1944). The following account appears to be warranted.

The tables relate to the 192 points in the z-plane which are defined by

$$z = \frac{2m + i2n\sqrt{3}}{24} \quad \text{and} \quad z = \frac{(2m+1) + i(2n+1)\sqrt{3}}{24}$$

with $m = 6(1)17$, $n = 0(1)7$. These points form a lattice within the rectangle $\frac{1}{2} < x < \frac{3}{2}$, $0 < y < \frac{5\sqrt{3}}{8}$. If $\Gamma(z) = e^{u+iv} = 10^{\lambda}e^{iv}$, the tables give, for each of the 192 values of z, 7-decimal values of u, v and $\lambda = u \log_{10} e$, and values of v in sexagesimal measure to the nearest second. For 24 of the values of z 15-decimal values of u and v are also given.

Stieltjes 1886 tabulates modulus to 3 decimals and argument to $0'{\cdot}1$ of $1/\Gamma(ia)$ for $a = 0(\cdot 2)4$. Beckerley 1941 tabulates $\arg \Gamma(ia+1)$ to 4 decimals of a radian for $a = 0(\cdot 1)2$, and suggests that no special table is needed for the modulus of $\Gamma(ia+1)$, which is simply $\sqrt{\pi a/\sinh \pi a}$ when a is real. See also Art. 14·4.

Approximate values may be read off from the diagram in Jahnke & Emde 1933 (90), 1938 (12).

Airey (MS.) gives $1/\Gamma(1+i)$ as $1{\cdot}83074\,43965\,9053 + 0{\cdot}56960\,76410\,3667i$.

The New York W.P.A. workers reported as under consideration in August 1941 a "table of the gamma function for complex arguments, $x+iy$, for both x and y ranging from 0 to 5 at intervals of 0·05".

14·65. Generalized Gamma Function

A generalized gamma function $\Gamma_k(x)$ with useful analytical properties is tabulated in Bendersky 1933. When x is integral, $\Gamma_k(x+1) = 1^{1^k}2^{2^k}\ldots x^{x^k}$. See Bendersky for the definition when x is not integral. The functions tabulated are $\Gamma_1(x)$ and $\Gamma_2(x)$ to 6 decimals, $\log \Gamma_1(x)$ to 8 decimals, and $\log \Gamma_2(x)$ to 10 decimals, all for $x = 0(\cdot 1)2$ and without differences. See also Art. 5·835.

14·7. Beta Function, etc.

Pearson 1934 (T. 1) gives at the head of each column values of

$$\mathrm{B}(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$$

to 8 figures for p and q equal to $\cdot 5(\cdot 5)11(1)50$, $p > q$.

Soper, etc. 1917 gives $\int_0^{\pi/2} \sin^{n-1} x\, dx = \frac{1}{2}B(\frac{1}{2}, \frac{1}{2}n)$ to 10 decimals without differences for $n = 1(1)105$. The table is reprinted in Pearson 1931 (179).

Higuchi 1930 gives to 7 decimals for $n = 1(1)7, 50, \infty$ the values of

$$\int_0^1 (1-x^{2n})^{\frac{1}{2n}} dx \qquad \int_0^1 (1-x^{2n})^{\frac{1}{n}} dx \qquad \frac{\int_0^1 (1-x^{2n})^{\frac{3}{2n}} dx}{\int_0^1 (1-x^{2n})^{\frac{1}{2n}} dx}$$

14·8. Incomplete Gamma Function

This has been tabulated in various forms, but the only general and extensive tables that we have encountered are contained in K. Pearson 1922.

$\int_x^\infty e^{-t}t^n dt$ is perhaps a better form than $\int_0^x e^{-t}t^n dt$, since if $x > 0$ it is unnecessary to stipulate that $n > -1$. The integral is an elementary function if n is zero or a positive integer, depends on the exponential integral if n is a negative integer, and depends on moments of the error function ($\int e^{-u^2}u^{2n+1}du$, $u^2 = t$) if $2n$ is an integer > -1. A slight amount of information already given in Sections 10 and 13 is repeated in Art. 14·82; for error integrals, see Section 15. See also Poisson sums in Art. 23·862 and areas of Pearson Type III curves (Salvosa 1930 and Burgess 1927) in Art. 23·87.

14·81. $\int_0^x e^{-t}t^p dt \Big/ \int_0^\infty e^{-t}t^p dt$

We use p instead of n in this article to conform in essentials to the notation of Pearson 1922. This uses $u = \dfrac{x}{\sqrt{p+1}}$ as an argument, and usually denotes the above ratio of incomplete to complete integral by $I(u, p)$; sometimes $I(x, p)$ is used.

In Tables 1 and 2 it gives $I(u, p)$ to 7 decimals for $p = -1(\cdot 05)0(\cdot 1)5(\cdot 2)50$, $u = 0(\cdot 1)\ldots$ until $I(u, p) = 1$ to 7-decimal accuracy. δ^{2n} is given in both u and p.

In Table 5 it gives $I(u, p)$ to 5 decimals without differences for $p = -1(\cdot 01)-\cdot 75$, $u = 0(\cdot 1)6$.

In Table 3 it gives $\log_{10} \dfrac{I(u, p)}{u^{p+1}}$ to 8 decimals for $p = -1(\cdot 05)0(\cdot 1)10$, $u = 0(\cdot 1)1\cdot 5$, with δ^{2n} in both u and p.

On page xiv it gives logarithmically to 7 decimals without differences

$$\frac{1}{x^{p+1}\Gamma(p+1)} \int_0^x e^{-t}t^p dt$$

for $p = -1(\cdot 01)-\cdot 9$, $x = 0(\cdot 01)\cdot 3$.

Constants of the curves $y = y_0 x^p e^{-x}$ are given in Table 4.

14·811. $x^{-n}e^{\pm x}\int_0^x e^{\mp t}t^n dt$

Coulson & Duncanson 1942 gives small tables to 2–12 decimals for $x = \cdot 1(\cdot 1)\cdot 5$, $n = 0(1)5 + 10x$ in connection with the tabulation of the exponential integral.

14·82. $\int_x^\infty e^{-t}t^n dt$

Miller & Rosebrugh 1903 gives this to 9 decimals without differences for $n = -2(1)+2$, $x = 0(\cdot 001)1(\cdot 01)2$, except that for $n = -1, -2$ an auxiliary function is given instead in the range $x = 0(\cdot 001)\cdot 1$. See Arts. 10·35–10·36 and 13·2–13·26 for more details.

14·83. $\int_1^\infty e^{-xt}t^n dt = \frac{1}{x^{n+1}}\int_x^\infty e^{-t}t^n dt$

This would be denoted in the following way by the various authors:

$\phi_n(x)$	Born, Misra
$A_n(x)$	Rosen 1931*b*, Kotani, etc.
$A_n(1, x)$	Bartlett, Rosen 1931*a*
$H_{-n}(x)$	Gold, Roberts
$E_{-n}(x)$	Mian & Chapman

No authors give differences.

11 fig.	$\begin{cases} n = 0(1)15 \\ x = \cdot 25(\cdot 25)8(\cdot 5)13(1)16(2)20(1)22, 24 \end{cases}$	Kotani, etc. 1938

Errata in Kotani, etc. 1940.

7–10 d.	$n = -4\cdot 5(\cdot 5)+7$, $x/\pi = \cdot 5(\cdot 25)5$	Born & Misra 1940
7–8 d.	$\begin{cases} n = -4\cdot 5(\cdot 5)-1, & x/\pi = 1(1)4 \\ n = -\cdot 5(\cdot 5)+6, & x/\pi = \cdot 5(\cdot 5)4, \text{ also } \cdot 75, 2\cdot 75 \end{cases}$	Misra 1940
6 fig.	$\begin{cases} n = 0(1)10, & x = \cdot 5(\cdot 5)5(1)10(2)14 \\ n = 11(1)20, & x = 1(1)10(2)14 \end{cases}$	Rosen 1931*b*

This is an enlarged and corrected version of a similar 6-figure table in Rosen 1931*a*.

4–5 fig.	$\begin{cases} n = 0(1)6, & x = 1\cdot 5(\cdot 5)5(1)10 \\ n = 7(1)11, & x = 3(1)10 \end{cases}$	Bartlett 1931
5 dec.	$\begin{cases} n = -1(1)-3 \\ x = \cdot 01(\cdot 01)\cdot 1(\cdot 05)1(\cdot 1)2(\cdot 2)3(\cdot 5)5, 6 \end{cases}$	Gold 1909
5 dec.	$n = -2(1)-8$, $x = \cdot 01, \cdot 05, \cdot 1, \cdot 25, \cdot 5(\cdot 5)3(1)6$	Mian & Chapman 1942
3 dec.	$n = -1, -3$, $x = \cdot 01, \cdot 05, \cdot 1, \cdot 2, \cdot 5(\cdot 5)2, 5$	Roberts 1930

14·84. $\int_{-1}^\infty e^{-xt}t^n dt = \frac{1}{x^{n+1}}\int_{-x}^\infty e^{-t}t^n dt$

6 fig.	$\begin{cases} n = 0(1)8, & x = 1\cdot 5(\cdot 5)2\cdot 5(\cdot 25)4\cdot 5, \\ & \quad 5(1)8(\cdot 5)9, 10(2)14 \\ n = 9(1)16, & x = 3(\text{var.})5(1)8(\cdot 5)9, \\ & \quad 10(2)14 \end{cases}$	No Δ	Rosen 1931*a* (T. 2, Ikehara)

14·85. $\int_{-1}^{+1} e^{-xt}t^n dt = \frac{1}{x^{n+1}}\int_{-x}^{x} e^{-t}t^n dt$

11 fig.	$\begin{cases} n = 0(1)8 \\ x = \cdot 25(\cdot 25)8(\cdot 5)13(1)16(2)20(1)22, 24 \end{cases}$	No Δ	Kotani, etc. 1938

Errata in Kotani, etc. 1940.

14·88. $\int_0^x e^{-t^n} dt$

4 dec. $n = 4$, $x = 0(\cdot 1)1$ Schumann 1923 (235)

14·89. $n\int_0^1 10^{-t} t^{n-1} dt$

This is tabulated naturally to 11 decimals and logarithmically to 10 decimals for $n = 1(1)29$ in Bessel 1812*a*, in connection with the logarithmic integral. There are smaller tables relating to similar integrals in which 10 is replaced by 2, $\frac{3}{2}$ and $\frac{7}{6}$.

14·9. Incomplete Beta Function

The integral $B_x(p, q) = \int_0^x x^{p-1}(1-x)^{q-1}dx$ occurs in various guises with considerable frequency, particularly in statistics. We give below some account of the more important and standard varieties of table. Some statistical tables have been rather widely copied in textbooks, and we limit our references for the most part to works which we have seen. Some errors in statistical tables are given in *M.T.A.C.*, **1**, 85–86, 1943.

14·91. Direct Tables relating to $B_x(p, q)$

Pearson 1934 (Table 1) tabulates $I_x(p, q) = B_x(p, q)/B(p, q)$ to 7 decimals without differences for p and q equal to $\cdot 5(\cdot 5)11(1)50$, $p \geqslant q$, $x = 0(\cdot 01)1$. The values of the complete integral in the denominator are given (see Art. 14·7). This important table occupies 430 pages. Constants of the curves $y = y_0 x^{p-1}(1-x)^{q-1}$ are given in Table 2.

Seven-decimal values of $\frac{1}{2} + \frac{1}{2}I_x(p, q)$ are given for $p = \frac{1}{2}$, $q = \cdot 5(\cdot 5)15$, $x = 0(\cdot 01)1$, without differences, in Pearson & Stoessiger 1931, and are reproduced in Pearson 1931 (169). An auxiliary table of $I_x(p, q)/2\sqrt{x}$ is provided to assist interpolation. Compare Art. 14·95.

For tables to facilitate the computation of $B_x(p, q)$ for values of p and q exceeding 50, see Wishart 1927 and Pearson 1931 (240).

14·92. Tables of x

Inverse tables of the incomplete beta function are given in **C. M. Thompson 1941*a*.** This tabulates to 5 figures the values of x for which $I_x(p, q) = P$, where

$$P = \cdot 50, \cdot 25, \cdot 10, \cdot 05, \cdot 025, \cdot 01, \cdot 005$$
$$n_1 = 2q = 1(1)10, 12, 15, 20, 24, 30, 40, 60, 120, \infty$$
$$n_2 = 2p = 1(1)30, 40, 60, 120, \infty$$

This table may easily be converted into tables of Snedecor's F or Fisher's z, which are considered in the next two articles; also into a table of Student's t, see Art. 14·97. (We conform to the commonest notation in using n_1 and n_2 for numbers of "degrees of freedom". Thompson and others use ν_1 and ν_2.)

14·93. Tables of Snedecor's F (Variance Ratio)

The transformation $F = \dfrac{p(1-x)}{qx}$, where $p = \frac{1}{2}n_2$, $q = \frac{1}{2}n_1$, changes the element $x^{p-1}(1-x)^{q-1}dx$ into one proportional to

$$\phi(F)dF = F^{\frac{1}{2}n_1-1}dF/(n_2+n_1F)^{\frac{1}{2}(n_1+n_2)}$$

and the expression for P becomes

$$\int_F^\infty \phi(F)dF \Big/ \int_0^\infty \phi(F)dF$$

Recently the table of x in Thompson 1941*a* has been converted into a table of F in **Merrington & Thompson 1943**. The values of F are given to 5 figures for the same values of P; n_1 and n_2 as are mentioned in Art. 14·92 above.

Several tables have been given by Snedecor, who introduced $F = e^{2z}$ as a variant of Fisher's z, which preceded F historically. Fisher's tables of z (see Art. 14·94 below) were converted into tables of F. All Snedecor's tables are for 5 per cent and 1 per cent points ($P = \cdot 05$ and $\cdot 01$) and are to about 3 figures. The first table appeared in Snedecor 1934, and was reproduced, with additions by Snedecor, in Davenport & Ekas 1936 (180). A considerably enlarged table appeared in Snedecor 1937, and is reproduced in Goulden 1939 and Kenney 1940 (**2**, 194). This enlarged table is for

$$n_1 = 1(1)12(2)16(4)24, 30, 40, 50, 75, 100, 200, 500, \infty$$
$$n_2 = 1(1)30(2)50(5)70, 80, 100, 125, 150, 200, 400, 1000, \infty$$

Fisher & Yates 1938 (Table V) gives tables of F, here called the variance ratio, to about 3 figures for

$P = \cdot 20, \cdot 05, \cdot 01, \cdot 001$ $\quad n_1 = 1(1)6, 8, 12, 24, \infty$ $\quad n_2 = 1(1)30, 40, 60, 120, \infty$

The table of 20 per cent points is due to H. W. Norton.

14·94. Tables of Fisher's z

In terms of $z = \frac{1}{2}\log_e F$ the probability element becomes proportional to

$$\psi(z)dz = e^{n_1 z}dz/(n_2+n_1e^{2z})^{\frac{1}{2}(n_1+n_2)}$$

and the expression for P is

$$\int_z^\infty \psi(z)dz \Big/ \int_{-\infty}^\infty \psi(z)dz$$

Fisher 1941 (Table VI) gives z to 4 decimals for

$P = \cdot 05, \cdot 01, \cdot 001$ $\quad n_1 = 1(1)6, 8, 12, 24, \infty$ $\quad n_2 = 1(1)30, 60, \infty$

The table for $P = \cdot 001$ is due to W. E. Deming and appeared for the first time in the 1936 edition. The tables are reproduced in Yule & Kendall 1937 (538). The tables of 5 per cent and 1 per cent points are also given in Kendall 1943 (442) and (with additional arguments $n_1 = 7, 9$ and $n_2 = 40, 120$) in Rider 1939 (204).

Fisher & Yates 1938 (Table V) gives 4-decimal values of z for

$P = \cdot 20, \cdot 05, \cdot 01, \cdot 001$ $\quad n_1 = 1(1)6, 8, 12, 24, \infty$ $\quad n_2 = 1(1)30, 40, 60, 120, \infty$

The table of 20 per cent points is due to H. W. Norton, and that of 0·1 per cent points is taken from C. G. Colcord & L. S. Deming 1935.

14·95. Integral of Student's *z* Distribution

Values of $\int_{-\infty}^{z} \frac{dz}{(1+z^2)^{\frac{1}{2}n}} \Big/ \int_{-\infty}^{\infty} \frac{dz}{(1+z^2)^{\frac{1}{2}n}}$ are given to 5 decimals without differences for $n = 2(1)10$ and $z = 0(\cdot 1)3$ in Pearson 1930 (36). This table is a slight extension of the original one in Gosset ("Student") 1908, which omits values for $n = 2, 3$ and gives only 4 decimals (except 5 decimals for $n = 10$). The transformation $z^2 = x/(1-x)$ reduces the expression tabulated to $\frac{1}{2}+\frac{1}{2}I_x(p, q)$, where $p = \frac{1}{2}$, $q = \frac{1}{2}(n-1)$, and $I_x(p, q)$ has the same meaning as in Art. 14·91. A more extensive tabulation of the same functions is contained in Gosset 1917, where 4-decimal values without differences are given for $n = 2(1)30$ and $x = 0(\cdot 1)3\cdot 5(\cdot 5)10(5)50(\text{var.})3000$; the highest value of the argument x for which the function differs from 1·0000 varies strongly with n, being 3000, 45, 15, 6 and ·8 when n is 2, 3, 4, 5 and 30 respectively.

For statistical purposes Student's z distribution is now less used than his t distribution (see the next two articles), suggested by R. A. Fisher. The difference between the two is mathematically trivial. The n of this article is now usually called $n+1$, as below.

14·96. Integral of Student's *t* Distribution

The ordinate in this distribution is proportional to $f_n(t)$, where

$$f_n(t) = \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{1}{2}(n+1)}}$$

We shall use the notation $P_n = \int_t^{\infty} f_n(t)\,dt \Big/ \int_0^{\infty} f_n(t)\,dt$. Thus P_n is the ratio of the area in the two tails to the whole area. It may be noticed that

$$P_1 = \frac{2}{\pi}\cot^{-1} t \qquad P_2 = 1 - \frac{t}{\sqrt{t^2+2}}$$

As n approaches infinity $f_n(t)$ approaches $e^{-\frac{1}{2}t^2}$, and P_∞ is expressible in terms of the normal error integral. The substitution $t^2 = n(1-x)/x$ gives $P_n = I_x(p, q)$, where $p = \frac{1}{2}n$, $q = \frac{1}{2}$, and $I_x(p, q)$ has the same meaning as in Art. 14·91.

Gosset ("Student") 1925 tabulated $1 - \frac{1}{2}P_n = \int_{-\infty}^{t} f_n(t)\,dt \Big/ \int_{-\infty}^{\infty} f_n(t)\,dt$, and the table is reprinted, with a few corrections due to Gosset, in Peters & Van Voorhis 1940 (488), except that these authors subtract Gosset's values from unity so as to tabulate $\frac{1}{2}P_n$. The main part of this table gives 4 decimals for $n = 1(1)20$ and $t = 0(\cdot 1)6$. A small supplementary table for high values of t gives 6 decimals. Gosset's main table is also reproduced, with his corrections, in Yule & Kendall 1937 (536) and Kendall 1943 (440); only 3 decimals are given, and the supplementary table is omitted.

An auxiliary table to facilitate the calculation of the t integral when $n > 20$ is given in Gosset 1925 and reproduced without change in Peters & Van Voorhis 1940 (493).

14·97. Tables of Student's t

It is now more usual to give an inverse table, in which t is tabulated with arguments P (or $\frac{1}{2}P$) and n.

Merrington 1942 gives a table derived by taking $t^2 = n(1-x)/x$, $n_1 = 1$, $n_2 = n$ in the table of x in Thompson 1941*a* (see Art. 14·92). Merrington's table gives t to 5 figures for

$$P = \cdot 50, \cdot 25, \cdot 10, \cdot 05, \cdot 025, \cdot 01, \cdot 005 \qquad n = 1(1)30, 40, 60, 120, \infty$$

Fisher & Yates 1938 (Table III) gives t to 3 decimals for

$$P = \cdot 9(\cdot 1)\cdot 1, \cdot 05, \cdot 02, \cdot 01, \cdot 001 \qquad n = 1(1)30, 40, 60, 120, \infty$$

Most other tables are based wholly or partly on tables in the various editions of Fisher 1941. Some have rather a large number of arguments n greater than 30, but these high values of n are of little importance. Fisher 1941 (Table IV) gives t to 3 decimals for

$$P = \cdot 9(\cdot 1)\cdot 1, \cdot 05, \cdot 02, \cdot 01 \qquad n = 1(1)30, \infty$$

The table is reprinted in Rider 1939, except that the P-argument is halved.

Goulden 1939 gives t to 2 decimals (3 decimals when $P = \cdot 50$) for

$$P = \cdot 50, \cdot 10, \cdot 05, \cdot 02, \cdot 01 \qquad n = 1(1)30(5)50(10)100(25)150, 200(100)500, 1000, \infty$$

Snedecor 1934 (88) gives 3-decimal values for $P = \cdot 05$ and $\cdot 01$ and for the same n as Goulden. Snedecor's values are reproduced in Davenport & Ekas 1936 (180). A small table is given in Kenney 1940 (**2**, 138).

14·98. Trigonometrical Forms of the Incomplete Beta Function

Legendre 1816 (178) gives $\int_0^\theta \sin^n \theta \, d\theta$ for $\theta = 0(1°)90°$, to 10 decimals with Δ^3 when $n = 2$, and to 9 decimals with Δ when $n = 4$ and 6. See also Arts. 7·66 and 7·84.

Tables to facilitate the evaluation of $\int_0^\theta \cos^n \theta \, d\theta$ are given in Wishart 1925*a*, *b*, and are reproduced in Pearson 1931 (236, 239).

SECTION 15

THE ERROR INTEGRAL, HIGHER INTEGRALS, DERIVATIVES. HERMITE POLYNOMIALS AND FUNCTIONS. MOMENTS

15·0. Introduction

This section deals with tables connected with the exponential e^{-x^2} or $e^{-\frac{1}{2}x^2}$, depending on the unit adopted for the independent variable. A third unit for the independent variable is ρ, a constant such that

$$\frac{2}{\sqrt{\pi}}\int_0^\rho e^{-t^2}dt = \sqrt{\frac{2}{\pi}}\int_0^{\rho\sqrt{2}} e^{-\frac{1}{2}t^2}dt = \tfrac{1}{2}$$

We find the value $\rho = 0{\cdot}47693\ 62762\ 04469\ 87338\ 14183\ 53643\ldots$ See also Art. 5·81. In general, only those functions connected fairly simply with the exponential concerned are considered in this section; more complicated functions, of statistical or other special interest only, have often been omitted.

In the case of tables connected with the Error Integral, it is a very common practice to increase the number of decimals given as the integral approaches its limiting value. These alterations are indicated in a condensed form special to this section; for example

8; 9, 10, 11 dec. 0(·01)3; 1·25, 1·51, 2 Kramp 1799 (195)

indicates a total range of the independent variable from 0 to 3, with interval 0·01 throughout; the table starts on page 195 with 8 decimals, and increases to 9, 10 and 11 decimals occur at arguments 1·25, 1·51 and 2·00 respectively.

Many statistical textbooks contain tables related to the Error Integral; our references concern mainly those which were fairly easily accessible to us. We have not attempted a complete examination.

15·01. Notation

There is considerable diversity of notation amongst users of these functions. Accounts of some of these notations will be found in Glaisher 1908 (127) and in H. Jeffreys, *Operational Methods in Mathematical Physics*, University Press, Cambridge, 1927 (94); others have been used subsequently. We adopt the notation of Burgess and of New York W.P.A., namely

$$H'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2} \qquad H(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

More uniformity seems to occur amongst statistical users; with them (except that the argument x is not given here as a suffix) we write

$$z(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \qquad \alpha(x) = \int_{-x}^{x} z(t)\,dt = \sqrt{\frac{2}{\pi}}\int_0^x e^{-\frac{1}{2}t^2}dt$$

and, as an extension of this notation, we use

$$z_\rho(x) = \frac{1}{\sqrt{\pi}}e^{-\rho^2x^2} \qquad \alpha_\rho(x) = \frac{2}{\sqrt{\pi}}\int_0^{\rho x} e^{-t^2}dt$$

The connection with the Incomplete Gamma Function, Section 14, should be noted. This is exhibited by the substitution $x^2 = u/\sqrt{2}$ or $\frac{1}{2}x^2 = u/\sqrt{2}$.

As an illustration of the diversity of notation we may note that Erf x or erf x has been given the following meanings: $\frac{1}{2}\sqrt{\pi}(1-H)$, Glaisher, Pendlebury; $\frac{1}{2}\sqrt{\pi}H$, Glaisher, Pendlebury, Whittaker and Watson, etc.; $1-H$, Jeans; H, Jeffreys, etc.; α, Aitken.

15·02. Errata and Bibliographies

Important lists of errata are given in Burgess 1898, Conrad & Krause 1937*a* and *b*, 1938*a* and *b*. Many tables are described or listed in B.A., 7, 1939 and in Glaisher 1871*a*, 1908 and 1911*b*.

15·03. Arrangement

The subdivision of the section is as follows:

15·0 Introduction
15·1 Tables of the Ordinate
15·2 Tables of the Integral
15·3 Ratio of Integral to Ordinate
15·4 Inverse Tables, having the Integral as Argument
15·5 Higher Integrals and Derivatives, etc.
15·6 Hermite Polynomials and Functions
15·7 Moments. "Chi-squared" Tables
15·8 The Error Integral with Complex Argument
15·9 Miscellaneous

15·1. Tables of the Ordinate

15·11. $H'(x) = \dfrac{2}{\sqrt{\pi}} e^{-x^2}$

21–26 dec.	1, 1·1, 1·25, 1·4(·2)3, 1·5, 2·5			Burgess 1898 (273)
15 dec.	0(·0001)1(·001)6		No Δ	**New York W.P.A. 1941*e*** (2)
15; 18, 21, etc. dec. 9 dec.	1·25(·01)1·5(·02)2·12, 2·17, 2·15(·05)3(·1)5·1(·2)5·5, 6; 3·5, 4·4, etc. 0(·001)1·249		No Δ	**Burgess 1898** (295, 283)

Major errors at ·017, ·155(·001)·160, ·188, 3·00 (see New York W.P.A. 1941*e*).

4 dec.	0(·01)4	Called $\Phi(x)_1$	$\Delta, (\frac{1}{2}D)^5$	Bruns 1906 (A. 6) Czuber 1924 (462)
4 dec.	0(·01)3	Called $\Phi_1(x)$	$(\frac{1}{2}D)^5$	Jahnke–Emde 1909 (37), 1933 (99), 1938 (26)
8 fig.	4(·01)10		No Δ	**New York W.P.A. 1941*e*** (296)

15·121. $\frac{1}{2}H'(x)$

5–4 fig.	0(·1)5	No Δ	Jones 1893 (160)
4 dec.	0(·01)2·99	No Δ	Jordan 1927 (329)

15·1221. $\frac{1}{2}\sqrt{\pi}H'(x) = e^{-x^2}$

24–27 dec.	1, 1·1, 1·25, 1·4(·2)3, 1·5, 2·5		No Δ	Burgess 1898 (273)
15; 18, 21, etc. dec.	3(·1)5(·5)6; 3·4, 4·3, etc.		No Δ	Burgess 1898 (321)
9 dec.	·05(·1)4·55	$\frac{1}{10}e^{-x^2}$ to 10 dec.	Δ^7	Oppolzer 1880 (37)
5 fig.	0(·1)5		No Δ	Potin 1925 (443); Mellor 1931 (626); Fowle 1933 (54)
5 dec.	0(·1)3·4		No Δ	Stumpff 1939 (132)
5 dec.	0(·1)3	$e^{-.01}$ in error	No Δ	Jeans 1925 (App. B)
4 fig.	0(·1)5		No Δ	Airy 1861 (16)

See also Emde 1941.

15·1222. e^{x^2}

5 fig.	0(·1)5	No Δ	Potin 1925 (443); Mellor 1931 (626); Fowle 1933 (54)
5; 4 fig.	0(·1)5; 3·4	No Δ	Stumpff 1939 (132)

15·1241. $e^{-k^2x^2}$

3 dec.	$k = \cdot1(\cdot1)\cdot6$	$x = \mu = 0(\cdot5)4(1)8$	No Δ	Smart 1938 (214)
5 dec.	$k = \frac{1}{2}\sqrt{\pi}$	$x = 0(\cdot1)3\cdot9$	No Δ	Stumpff 1939 (132)

See also Art. 15·16 for $k^2 = \frac{1}{2}$.

15·1242. $e^{k^2x^2}$

5; 4 fig.	$k = \frac{1}{2}\sqrt{\pi}$	$x = 0(\cdot1)5;\ 3\cdot9$	No Δ	Stumpff 1939 (132)

15·131. $\text{Log}_{10} H'(x)$

16 dec.	1(·001)1·5(·002)3	No Δ	Burgess 1898 (296)

15·132. $\text{Log}_{10} e^{\pm x^2}$

5 dec.	0(·1)5	No Δ	Fowle 1933 (54)

15·14. $z(x) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

15 dec.	0(·0001)1(·001)8·285		No Δ	**New York W.P.A. 1942b** (2)
7 dec.	0(·01)4·5		Δ^2	**Sheppard 1903** (182)
10 dec.	4·5(·01)6		No Δ	**Pearson 1930** (2)
7 dec.	0(·01)4		D^6	Jørgensen 1916 (178)
7 dec.	0(·01)4	Called τ_1	No Δ	Lee 1925 (351); Pearson 1931 (74)
5 dec.	0(·01)4·99		D^n, $n = 2(1)4$	Rietz 1924 (209)
5 dec.	0(·01)4·99		D^n, $n = 2(1)8$	Glover 1930 (394)

5 dec.	0(·01)3·99	No Δ	Kenney 1940 (1, 225; 2, 191) Davis & Nelson 1935 (397)
5 dec.	0(·01)3(·1)5	No Δ	Davenport & Ekas 1936 (165)
5 dec.	0(·1)4·8	Δ^2	Yule & Kendall 1937 (531)
5 dec.	0(·1)4	D^6	Fry 1928 (456)
4; 6, 7 d.	0(·01)4·4(·05)4·7(·1)5; 3, 3·5	No Δ	Peters & Van Voorhis 1940 (485)
4 dec.	0(·01)3·99	No Δ	Rider 1939 (194)
4 dec.	0(·01)3(·1)4·9 or 5	IΔ; no Δ	Charlier 1906 (44); 1920 (123)
4 dec.	0(·01)3(·1)3·9	PPMD	Fisher & Yates 1938, 1943 (T. II)
7 fig.	6(·01)10	No Δ	**New York W.P.A. 1942*b*** (340)

15·15. $\frac{1}{\sqrt{2\pi}} - z(x) = m_1(x)$

7 dec.	0(·1)5	No Δ	Pearson & Lee 1908 (66) Pearson 1930 (22)

15·16. $\sqrt{2\pi}\, z(x) = e^{-\frac{1}{2}x^2}$

10 dec.	−7(·1)+6·6	D^6	B.A. 1, 1931 (60)
5 dec.	0(·01)3(·1)5	No Δ	Davenport & Ekas 1936 (164)
5 dec.	0(·01)3(·1)5	No Δ	Garrett 1937 (127, 469)
4 dec.	0(·1)4·5	D^4	Thiele 1903 (18)

15·18. Miscellaneous Ordinates

Clark 1887 (95) gives $x^3e^{-x^2}$ and $x^4e^{-x^2}$ for $x = 0(\cdot 1)3$ to 6 decimals.

Bruns 1906 (185) gives $x^{n-2}e^{-x^2}/(n-1)!$ for $x = 2(1)7$, $n = 1(1)10$ to 1 decimal.

Elderton 1902 (162) and Pearson 1930 (29) give $\log_{10}\{2x\,z(x)\}$ and $\log_{10}\{\sqrt{2\pi}\,z(x)\}$ for $x^2 = 1(1)100$ to 8 decimals.

15·2. Tables of the Integral

15·211. $H(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$

19–22 d.	1, 1·25, 1·4, 1·5, 1·6, 1·8, 2, 2·5, 3		Burgess 1898 (273)
15 dec.	0(·0001)1(·001)6	D only	**New York W.P.A. 1941*e*** (2)
15 dec.	·1(·001)1·5(·002)3	Δ^4	**Burgess 1898** (296, 321, 283)
15 dec.	3(·1)5(·5)6	No Δ	
9 dec.	0(·001)1·249	Δ^2	

Major errors at ·015, ·055, ·291, ·367, 1·308, 1·564, 1·798, 1·966, 1·998, 3·5, 4·1, 4·7, 6·0 (see New York W.P.A. 1941*e*).

7; 11 d.	0(·01)4(·1)4·8; 3·46	No Δ	Bertrand 1888 (329, $\Theta(x)$)
		Δ; no Δ	Czuber 1891, 1924 (455)
		No Δ	Van Deuren 1934 (540, $\Phi(x)$)
7 dec.	0(·01)3	Δ^2	De Morgan 1838 (App. xxxiv) Faà de Bruno 1869 (T. 2)
7 dec.	0(·01)2·97, 3	Δ^2	Galloway 1839 (212)

7 dec.	0(·01)2	Δ^2	Encke 1832 (305) De Morgan 1845 (483) Kelvin 1890 (434)
6; 7 d.	0(·1)3·9; 3	No Δ	Jones 1893 (160)
6; 7 d.	0(·001)2·5(·01)3·79; 2·5	No Δ	Markoff 1912 (312)
6 dec.	0(·1)3·5	No Δ	Lupton 1898 (122)
5 dec.	0(·001)1·5(·002)2·5(·01)3·09	No Δ	Peirce 1910 (116)
5 dec.	0(·001)2(·01)3	IΔ	Plummer 1940 (270)
5; 6, 7 d.	0(·001)1·6(·01)3·19; 2·7, 3·15	No Δ	Deltheil 1930 (153)
5 dec.	0(·001)1·5(·01)3·09	IΔ	Dwight 1941 (210)
5; 6, etc. d.	0(·01)2·6(·05)3·6(·1)5(·5)6; 2·85, etc.	No Δ	Brunt 1917 (214), 1931 (234)
5 dec.	0(·01)3·09	No Δ	Witkowski 1904 (92) Schorr 1916 (A. 40)
5 dec.	0(·01)3(·1)3·3	No Δ	Fowle 1933 (56)
5 dec.	0(·01)3	No Δ	Dale 1903 (84) Hayashi 1930*b* (165)
5 dec.	0(·01)2·99	No Δ	Allen 1941 (158) Castelnuovo 1919 (T. 2)
5 dec.	0(·01)2·4(·1)3·2	Δ	Bauschinger 1901 (141) B'schinger–Stracke 1934 (188)
5 dec.	0(·01)2	Δ	Oppolzer 1880 (603) Chauvenet 1863 (2, 593)
5 dec.	0(·02)3	No Δ	Levy & Roth 1936 (197)
5 dec.	0(·01)·1(·1)3	No Δ	Edgeworth 1911 (391)
$4\frac{1}{2}$; $5\frac{1}{2}$ d.	0(·01)3; 2	Δ	MT.C. 1931 (211)
4; 5 dec.	0(·001)1·5(·01)3·09; 1·5	No Δ	Silberstein 1923 (117)
4 dec.	0(·001)1·51(·01)2·89	No Δ	Kämpfe 1893 (147) Bruns 1906 (A. 2) Jahnke–Emde 1909 (33) Czuber 1924 (458)
4; 5, 6 d.	0(·01)2·99; 2, 2·8	IΔ	Jahnke–Emde 1933 (98), 1938 (24)
4; 6, 10 d.	0(·01)2·5(·1)3, 4; 2·5, 4	No Δ	Gibbs 1929 (86)
4 dec.	0(·01)2·69	No Δ	Mellor 1931 (621)
4 dec.	0(·01)2·5(·1)2·9	IΔ	Merriman 1910 (220)
4 dec.	0(·01)2(·1)2·9	IΔ	Hoüel 1901 (61)
4 dec.	0(·025)2·875	No Δ	Pearson & Hartley 1942 (302)

Given in the column $n = 2$, with argument $W = 2x$.

See also lists of tables of the Incomplete Gamma Function, since $I(u, -\frac{1}{2}) = H(x)$, with $u = \sqrt{2}x^2$; Arts. 14·8 ff.

$$\mathbf{15{\cdot}212.}\quad 1 - H(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$$

8 fig.	4(·01)10	D only	**New York W.P.A. 1941*e*** (296)

$$\mathbf{15{\cdot}215.}\quad \tfrac{1}{2}H(x) = \frac{1}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

6 dec.	0(·1)3	No Δ	Airy 1861 (22) Edwards 1922 (858)

15·221. $\frac{1}{2}\sqrt{\pi}H(x) = \int_0^x e^{-t^2}dt$

10 dec.	0(·01)4·52	No Δ	Oppolzer 1880 (587)

15·222. $\frac{1}{2}\sqrt{\pi}(1 - H(x)) = \int_x^\infty e^{-t^2}dt$

15; 21 dec.	3(·1)5(·5)6; 5·5	No Δ	Burgess 1898 (321)
11 dec.	0(·001)3(·01)4·8	Δ^n	Markoff 1888 (1)
11; 13, 14 d.	3(·01)4·5; 3·51, 4·01	No Δ	Glaisher 1871*a* (436)
8; 9, 10, 11 dec.	0(·01)3; 1·25, 1·51, 2	Δ^n	Kramp 1799 (195); De Morgan 1845 (477)
4 dec.	0(·05)2·6	No Δ	De Morgan 1842*a* (657)

15·223. $\sqrt{\pi}(1 - H(x)) = 2\int_x^\infty e^{-t^2}dt$

10 dec.	$x = 0(\cdot 01)\cdot 5$	Δ^n	Legendre 1826 (520)
10 dec.	$e^{-x^2} = 0(\cdot 01)\cdot 8$		

15·226. $\text{Log}_{10}\,\frac{1}{2}\sqrt{\pi}(1 - H(x))$

7 dec.	0(·01)3	Δ^2	Kramp 1799 (203); De Morgan 1845 (477)

15·231. $\frac{1}{2}\alpha(x) = \int_0^x z(t)\,dt = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{1}{2}t^2}dt$

6 dec.	0(·01)4·99	D^n, $n = 1, 3, 4, 5$	Rietz 1924 (209)
5 dec.	0(·001)2·5(·01)4(·1)5	$I\Delta$, PPd	Davenport & Ekas 1936 (166)
5 dec.	0(·01)4·99	D^n, $n = 1, 3(1)9$	Glover 1930 (394)
5 dec.	0(·01)3·99	D only	Kenney 1940 (**1**, 225; **2**, 191)
5 dec.	0(·01)3·99	No Δ	Davis & Nelson 1935 (398)
5 dec.	±0(·1)2, 2·2, 2·5, 3, 4	No Δ	R. W. Burgess 1927 (296)
4; 6, 7 dec.	0(·01)4·4(·05)4·7(·1)5; 3, 3·5	No Δ	Peters & Van Voorhis 1940 (485)
4; 5, 7, etc. dec.	0(·01)3·2(·1)4(·5)5; 3, 3·2, etc.	No Δ	Garrett 1937 (110, 467)
4; 5, 6 dec.	0(·01)2(·02)3(·2)4(·5)5; 3, 3·6	No Δ	Uspensky 1937 (407)
4; 5, 6 dec.	0(·01)2(·02)3(·2)4, 4·5; 3, 4·5	No Δ	Bowley 1937 (271)
4 dec.	0(·02)1·5(·05)2(·1)3(·2)4	No Δ	Aitken 1939 (144)

15·232. $\frac{1}{2}(1 + \alpha(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{1}{2}t^2}dt$

7 dec.	0(·01)4·5	Δ^3	**Sheppard 1903** (182)
10 dec.	4·5(·01)6	No Δ	**Pearson 1930** (2)
7 dec.	0(·01)4	D^7	Jørgensen 1916 (178)
7 dec.	0(·1)5·4	No Δ	Gosset 1925 (114)

5 dec.	0(·1)4·4	Δ^2	Yule & Kendall 1937 (532)
5 dec.	0(·1)4	D^7	Fry 1928 (456)
5 dec.	0(·1)·4(·05)·7(·01)·8(·05)1·1, 1·5(·5)3·5	No Δ	Jones 1924 (284)
4 dec.	−2·9(·1)−2(·01)+2(·1)2·9	No Δ	Charlier 1920 (121)
4 dec.	0(·05)3·95	D^n, n = 1, 4(1)7	A. Fisher 1922 (280)

Kelley 1938 (3) gives 13-decimal ratios of each coefficient to the next in the expansion in powers of x, for terms up to that in x^{61}.

Pearson 1931 (162), from Tippett 1925 (384), gives $\{\frac{1}{2}(1+\alpha(x))\}^n$ to 7 decimals for n = 3, 5, 10 at interval 0·2 and for n = 20, 30, 50, 100(100)1000 at interval 0·1, for all x for which the value is not 0 or 1 to 7 decimals.

$$15{\cdot}233.\quad \tfrac{1}{2}(1-\alpha(x)) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{1}{2}t^2}\,dt$$

7 dec.	0(·01)4	Called τ_0	No Δ	Lee 1925 (351); Pearson 1931 (74)
4 dec.	0(·01)3·99		D only	Rider 1939 (194)
7; 10 dec. / 5 fig.	0(·5)5; 4·5 / 6(1)10		No Δ	Rider 1939 (199)

$$15{\cdot}241.\quad \alpha(x) = \int_{-x}^{x} z(t)\,dt = \sqrt{\frac{2}{\pi}}\int_0^x e^{-\frac{1}{2}t^2}\,dt$$

15 dec.	0(·0001)1(·001)8·112	$\frac{1}{2}D$ only	**New York W.P.A. 1942*b*** (2)
5 dec.	As in Art. 15·232	No Δ	Jones 1924 (284)
4; 5, 6, etc. d.	0(·01)3(·1)6; 4, 4·5, etc.	No Δ	Charlier 1906 (43)

See also lists of tables of the Incomplete Gamma Function, since $I(u, -\frac{1}{2}) = \alpha(x)$, with $u = x^2/\sqrt{2}$; Arts. 14·8 ff.

$$15{\cdot}242.\quad 1-\alpha(x) = \sqrt{\frac{2}{\pi}}\int_x^\infty e^{-\frac{1}{2}t^2}\,dt$$

5 dec.	0(·1)4·5	Δ^2	Yule & Kendall 1937 (533)
4 dec.	0(·01)4	No Δ	Fry 1928 (453)
4 dec.	0(·1)4·1	No Δ	Kelley 1938 (116, n = 1)
7 fig.	6(·01)10	$\frac{1}{2}D$ only	**New York W.P.A. 1942*b*** (240)
7 dec.	x^2 = 1(1)30	No Δ	Elderton 1902 (163); Pearson 1930 (30)
5 dec.	x^2 = 0(·01)1(·1)10	Δ	Yule & Kendall 1937 (534)

$$15{\cdot}243.\quad \sqrt{\tfrac{1}{2}\pi}\,\alpha(x) = \int_0^x e^{-\frac{1}{2}t^2}\,dt$$

5 dec.	0(·1)4·5	D^5	Thiele 1903 (18)

$$15{\cdot}244.\quad \sqrt{\tfrac{1}{2}\pi}(1-\alpha(x)) = \int_x^\infty e^{-\frac{1}{2}t^2}\,dt$$

10 dec.	−7(·1)+6·6	D^7	B.A. **1**, 1931 (63)

15·251. $-\mathrm{Log}_e \frac{1}{2}(1-\alpha(x))$

24 dec.	0(1)10	No Δ	B.A. **7**, 1939 (23)
16 dec.	0(·1)10	v^n	B.A. **7**, 1939 (24)

15·252. $\pm\mathrm{Log}_{10} \frac{1}{2}(1-\alpha(x))$

12 dec.	0(·1)10	v^n	B.A. **7**, 1939 (28)
8 dec.	0(·01)10	δ^2	B.A. **7**, 1939 (30)
5 dec.	5(1)50(10)100(50)50δ	No Δ	Pearson 1930 (11, Julia Bell)

15·261. $\frac{1}{2}\alpha_\rho(x) = \frac{1}{2}H(\rho x) = \frac{1}{2}\alpha(\sqrt{2}\rho x) = \frac{1}{\sqrt{\pi}}\int_0^{\rho x} e^{-t^2}dt$

8; 9(1)13 d.	0(·01)9(·1)10; 3·81, 7·43, 8·08, 8·68, 9	Δ	**Conrad, Krause &, 1938*a*** (400)
4; 5, etc. d.	0(·05)5·95; 5·1, etc.	No Δ	Garrett 1937 (111, 468)
4; 5(1)13 d.	0(·01)5(·1)10; 4·57, 5·5, etc.	No Δ	Conrad & Krause 1938*b* (492)

15·262. $\alpha_\rho(x) = H(\rho x) = \alpha(\sqrt{2}\rho x) = \frac{2}{\sqrt{\pi}}\int_0^{\rho x} e^{-t^2}dt$

7 dec.	0(·01)7	No Δ	Conrad, Krause &, 1937*b* (56)
5 dec.	0(·01)3·5(·1)5	Δ	Chauvenet 1863 (**2**, 594)
5 dec.	0(·01)3·4(·1)5	Δ	Encke 1832 (309)

This table is reproduced in De Morgan 1838 (App. xxxviii), 1845 (484), Faà de Bruno 1869 (T. 3) and Czuber 1891.

5 dec.	0(·01)3(·1)5·9	No Δ	Fowle 1933 (57)
4 dec.	0(·01)3·5(·1)6·1	No Δ	Johnson 1892 (T. 2)
4 dec.	0(·01)3·5(·1)5·9	IΔ	Merriman 1910 (221)
		No Δ	Mellor 1931 (622)

Numerous other references are given in Conrad, Krause &, 1937*b*; this also gives valuable lists of errata. A selection of references follows: Brunt 1931 (19), Cooke 1936 (247), Gregory & Renfrow 1929 (97), Guilford 1936 (535), Holzinger 1928 (237), Howe 1890 (24), Jones 1893 (160), Leland 1921 (231), Lupton 1898 (122), Morton 1928 (xlvii), Odell 1935 (444). In these tables errors, due to copying, tend to occur at $x = \cdot 9$, $\cdot 97$ and $1\cdot 8$.

Czuber 1924 (1914 edition, p. 314) gives 5-decimal values of $\alpha_\rho(x)$ and $H(x/\sqrt{\pi})$ for $x = 0(\cdot 2)4$.

15·281. $\int_0^x e^{t^2}dt$

6 dec.	0(·01)2	No Δ	Terrill & Sweeny 1944*b* (220), 1944*a* (496)
7–9 fig.	2(·01)4		
6 dec.	0(·01)2	No Δ	Dawson 1898 (521)
4; 3, 2 dec.	0(·01)1·99; 1, 1·8	IΔ	J. E. 1933 (106), 1938 (32)

Schumann 1923 gives a 4- to 5-figure table of the function $i^{-1}E_2(ix) = \frac{2}{\sqrt{\pi}}\int_0^x e^{t^2}dt$, for $x = 0(\cdot 1)2\cdot 6(\cdot 2)7\cdot 4$.

See also Art. 15·312; and Art. 22·56, since $\int_0^x e^{t^2}dt = \frac{1}{2x}\{e^{x^2} - M(-\frac{1}{2}, \frac{1}{2}, x^2)\}$.

15·3. Ratio of Integral to Ordinate

15·311. $\dfrac{1-H(x)}{H'(x)} = e^{x^2}\displaystyle\int_x^\infty e^{-t^2}\,dt$

12 dec.	$x^2 = 3(1)16$	No Δ	Radau 1885 (B. 5)

15·312†. $F(x) = e^{-x^2}\displaystyle\int_0^x e^{t^2}\,dt$

6 dec.	$x = 0(\cdot1)1$, $\frac{1}{2}x^2 = \cdot6(\cdot2)6, 16, 25, 100$	D^n	Terazawa 1917 (172)

Terazawa gives the first four derivatives to 5, 4, 4 and 3 decimals respectively. The table contains a number of errors.

6; 8, 9 dec.	$x = 0(\cdot01)4(\cdot05)7\cdot5(\cdot1)10(\cdot2)12$; 2, 5	No Δ	Lash Miller & Gordon 1931 (2878)
4; 5 dec.	$x = 0(\cdot2)12$; 5·2	No Δ	Mitchell & Zemansky 1934 (322)

A 4-figure table of $1-2xF(x) = e^{-x^2}M(-\frac{1}{2}, \frac{1}{2}, x^2)$, see Art. 22·56, is also given.

3–4 dec.	$x^2 = 3(1)10(2)20, 21(3)30$	No Δ	Nottingham 1936 (85)

15·32†. $\dfrac{2x(1-H(x))}{H'(x)}$

15 dec.	$3(\cdot1)5(\cdot5)6$	No Δ	Burgess 1898 (321, *L*)

15·33. $\dfrac{\frac{1}{2}(1-\alpha(x))}{z(x)} = e^{\frac{1}{2}x^2}\displaystyle\int_x^\infty e^{-\frac{1}{2}t^2}\,dt$

24 dec.	$0(\cdot1)10$	v^n	B.A. 7, 1939 (16)
12 dec.	$0(\cdot01)10$	v^n	B.A. 7, 1939 (2)
5 dec.	$0(\cdot01)4(\cdot05)5(\cdot1)10$	Δ	Mills 1926 (396); Pearson 1931 (11)

15·34. $\dfrac{\frac{1}{2}x(1-\alpha(x))}{z(x)} - 1$

23 dec.	$0(\cdot1)10$	$\dfrac{v^{n-1}}{n}$	B.A. 7, 1939 (16, *F′*)
10 dec.	$0(\cdot01)10$	$\dfrac{v^{n-1}}{n}$	B.A. 7, 1939 (2, *F′*)

15·35. $\mathrm{Log}_{10}\dfrac{1-H(x)}{H'(x)}$

7 dec.	$x = -\cdot12(\cdot001)+1\cdot010$; $x = 10^t$, $t = 0(\cdot001)1$	Δ	Radau 1885 (B. 11)
7 dec.	$x = 0(\cdot01)3$	Δ^2	Kramp 1799 (207)
7 dec.	$x = 0(\cdot01)1$; $x = 10^t$, $t = 0(\cdot01)1$	Δ^2	Bessel 1818 (36)
10 dec.	$x^2 = 3(1)16$	No Δ	Radau 1885 (B. 5)

† In Section 15, items added in proof are on page 231.

15·36. $\text{Log}_{10} \dfrac{1-\alpha(x)}{\sqrt{2\pi e}\, z(x)}$

4 dec.	1(·01)3	Called $\text{Log}_{10} R$	No Δ	Chauvenet 1863 (2, 599)

This table is used in the application of Peirce's Criterion for the rejection of doubtful observations. This involves solution of the equation

$$x^2 - 1 = \frac{m-\mu-n}{n}\left\{1 - \left(\frac{T}{R}\right)^{2n/(m-n)}\right\}$$

in which $T = n^n(m-n)^{m-n}/m^m$. Chauvenet gives tables of x^2, when this exceeds unity, to 4 decimals for

$$\mu = 1, 2 \qquad m = 3(1)60 \qquad n = 1(1)9$$

These tables were given first by B. A. Gould, see *Report of the Superintendent of the U.S. Coast Survey for 1854*, Appendix, p. 131*–138*; also *Astron. Journ.*, 4, 81–87.

15·37. $\dfrac{z(x)}{\frac{1}{2}(1-\alpha(x))}$

15 dec.	0(·1)10	v^{n-1}/n	B.A. 7, 1939 (24, L')

15·39. $x + \dfrac{z(x)}{\frac{1}{2}(1+\alpha(x))}$

3 dec.	−4(·2)+3	Smart 1938 (289)

15·4. Inverse Tables, having the Integral as Argument

15·411. General Tables of the Abscissa or Deviate x

10 dec.	$\frac{1}{2}(1+\alpha) = \cdot 5(\cdot 001)1$	No Δ	**Kondo & Elderton 1931** (368); **Pearson 1931** (2)
8 dec.	$\frac{1}{2}(1+\alpha) = \cdot 5(\cdot 0001)1$	No Δ	**Kelley 1938** (14)
6 dec.	$\frac{1}{2}\alpha = 0(\cdot 001)\cdot 499$	No Δ	Kelley 1923 (373, Kelley, Wood & Kelley)
5 dec.	$\frac{1}{2}(1-\alpha) = 0(\cdot 001)\cdot 5$	No Δ	Everitt 1910 (442); Pearson 1930 (42)
4 dec.	$\frac{1}{2}(1+\alpha) = 0(\cdot 001)\cdot 98(\cdot 0001)\cdot 9999$	PPMD	Fisher & Yates 1938, 1943 (T. IX)

Called "Probits"; the quantity tabulated is $x+5$.

4 dec.	$\frac{1}{2}(1+\alpha) = \cdot 5(\cdot 001)1$	No Δ	Sheppard 1907 (405); Pearson 1930 (1)
4 dec.	$\frac{1}{2}(1-\alpha) = 0(\cdot 001)\cdot 5$	No Δ	Peters & Van Voorhis 1940 (481)
4 dec.	$\frac{1}{2}(1-\alpha) = 0(\cdot 005)\cdot 5$	No Δ	Rider 1939 (198)
4 dec.	$\frac{1}{2}(1+\alpha) = 0(\cdot 01)1$	No Δ	Charlier 1920 (122)
3 dec.	$\frac{1}{2}\alpha = 0(\cdot 01)\cdot 5$	No Δ	Holzinger 1928 (212); Garrett 1937 (370, 473)

4 dec.	$\frac{1}{2}(1-\alpha) = \frac{m}{n}$; $2m < n < 30$		Fisher & Yates 1938, 1943 (T. X)
7 dec.	$\alpha = 0(\cdot 01)\cdot 8$	Δ^3	Sheppard 1903 (189); Pearson 1930 (9)
6 dec.	$1-\alpha = 0(\cdot 01)1$	No Δ	Fisher 1941 (T. I); Fisher & Yates 1938, 1943 (T. I)
4 dec.	$\alpha = 0(\cdot 01)1$	No Δ	Charlier 1912 (34)

Error; at $\alpha = 0{\cdot}36$, for 0·4667 read 0·4677.

5 f. or d.	$1-H = 0(\cdot 01)1$	No Δ	Fowle 1933 (60)

$2x$ is tabulated, also $\log_{10} 2x$ to 5 decimals. This table is also in the 1914 and 1920 editions, and is headed "Diffusion Integral".

4 dec.	$\frac{1}{2}(1+H) = \cdot 5(\cdot 01)\cdot 83(\cdot 0025)\cdot 97(\cdot 001)1$	Δ	Fechner 1889 (108–111)
	$\frac{1}{2}(1+H) = \frac{1}{2}(1/64)1$	No Δ	
7; 6 dec.	$\frac{1}{2}\alpha_p = 0(\cdot 001)\cdot 45(\cdot 0001)\cdot 4999$; $\cdot 45$	No Δ	**Conrad, Krause &,** ***1938a*** (416)
2 dec.	$50\alpha_p = 0(\cdot 1)45(\cdot 01)49{\cdot}99$	No Δ	Conrad & Krause 1938*b* (496)
2 dec.	$1-\alpha_p = 1/2n$; $n = 0(1)99$		Mellor 1931 (623)
2 dec.	$1-\alpha_p = 1/2n$; $n = 3(1)25(\text{var.})500$		Merriman 1910 (228)

These two tables are adapted for use when applying Chauvenet's Criterion for the rejection of doubtful observations.

15·412. Short Tables of *x*

9 dec.	$H = 0(\cdot 1)1$	No Δ	Burgess 1898 (281)

Burgess also tabulates $\log_{10} x$ to 10 decimals.

9 dec.	$\alpha_p = 0(\cdot 1)1$	No Δ	Burgess 1898 (281)

15·413. Tail-Area Tables of *x*

Differences are not given with these tables. A special notation is used in this article, where, for example, $49{\cdot}9^5 3$ stands for 49·999993.

8–5 dec.	$\frac{1}{2}(1-\alpha) = 10^{-4}, 10^{-p}, 5\times 10^{-p}, p = 5(1)9$	Kelley 1938 (114)
6; 5, 4 dec.	$\frac{1}{2}\alpha = \cdot 45(\cdot 0001)\cdot 4999, \cdot 49^p r$, $p = 3(1)8, r = 0(1)9$; $\cdot 49^4 1, \cdot 49^8 1$	**Conrad & Krause** ***1937a*** (279)
5 dec.	$\frac{1}{2}(1-\alpha) = 5\times 10^{-p}, p = 3(1)10$	Rider 1939 (199)
5 dec.	$1-\alpha = 10^{-p}, p = 3(1)9$	Fisher 1941 (T. II); Fisher & Yates 1938, 1943 (T. I)
4–2 dec.	$\alpha = \cdot 9^p r, p = 2(1)8, r = 1(1)9$	Charlier 1912 (34)
3 dec.	$1-\alpha = 10^{-p}, p = 1(1)6$	Fry 1928 (455)
3 dec.	$1-H = 10^{-p}, p = 5(1)15$ and 20	Plummer 1940 (274)
3 dec.	$1-H = 10^{-p}, p = 5(1)11$	MT.C. 1931 (211)
6; 5, 4 dec.	$\frac{1}{2}\alpha_p = \cdot 49^p r, p = 3(1)9, r = 0(1)9$; $\cdot 49^4 1, \cdot 49^8 1$	**Conrad, Krause &,** ***1938a*** (416)
2 dec.	$50\alpha_p = 49{\cdot}9^p r, p = 2(1)8, r = 0(1)9$	Conrad & Krause 1938*b* (498)

15·414. Tables of x^2

R. A. Fisher's "chi-squared" table, in the row $n = 1$, gives values of x^2 to 3 decimals (with extension to give a minimum of 3 figures) for $\alpha = \cdot 99, \cdot 98, \cdot 95, \cdot 9, \cdot 8, \cdot 7, \cdot 5, \cdot 3, \cdot 2, \cdot 1, \cdot 05, \cdot 02, \cdot 01$. This table is given in Fisher 1941 (T. III) and has been widely reproduced (see Art. 15·74). C. M. Thompson 1941*b* (188) in the row $\nu = 1$ gives x^2 to 6 decimals for $\alpha = P = \cdot 995, \cdot 99, \cdot 975, \cdot 95, \cdot 9, \cdot 75, \cdot 5, \cdot 25, \cdot 1, \cdot 05, \cdot 025, \cdot 01, \cdot 005$.

15·42. The Ordinate z

10 dec.	$\frac{1}{2}(1+\alpha) = \cdot 5(\cdot 001)1$	No Δ	**Kondo & Elderton 1931** (368); **Pearson 1931** (2)
8 dec.	$\frac{1}{2}(1+\alpha) = \cdot 5(\cdot 0001)1$	No Δ	**Kelley 1938** (14)
6 dec.	$\frac{1}{2}\alpha = 0(\cdot 001)\cdot 499$		Kelley 1923 (373, Kelley, Wood & Kelley)
5 dec.	$\frac{1}{2}(1+\alpha) = 0(\cdot 001)1$	No Δ	Pearson & Soper 1921 (428); Pearson 1931 (1)
5 dec.	$\frac{1}{2}(1-\alpha) = 0(\cdot 001)\cdot 5$	No Δ	Everitt 1910 (442); Pearson 1930 (42)
Denoted by τ_1.			
4 dec.	$\frac{1}{2}(1-\alpha) = 0(\cdot 001)\cdot 5$	No Δ	Peters & Van Voorhis 1940 (481)
3 dec.	$\frac{1}{2}\alpha = 0(\cdot 01)\cdot 5$	No Δ	Holzinger 1928 (212); Garrett 1937 (370, 473)
7 dec.	$\alpha = 0(\cdot 01)\cdot 8$	Δ^3	Sheppard 1903 (189); Pearson 1930 (9)
9–14 d.	$\left\{\begin{array}{l}\frac{1}{2}(1-\alpha) = 10^{-4},\ 10^{-p},\ 5\times 10^{-p},\\ p = 5(1)9\end{array}\right\}$		Kelley 1938 (114)

15·43. Auxiliary Functions

Kondo & Elderton 1931 (368) and Pearson 1931 (2) both give 10-decimal tables, for $\frac{1}{2}(1+\alpha) = \cdot 5(\cdot 001)1$, of the four functions $\frac{1}{2}(1\pm\alpha)/z$ and $z/\frac{1}{2}(1\pm\alpha)$. Kelley 1923 (373) gives the two latter functions for $\frac{1}{2}\alpha = 0(\cdot 001)\cdot 499$, mainly to 5 decimals.

Pearson 1913 (27, Julia Bell) and 1930 (35, T. 24) give 4-decimal values of $\chi_\alpha = \sqrt{1-\alpha^2}/2z$ for $\frac{1}{2}(1+\alpha) = \cdot 5(\cdot 01)\cdot 98, \cdot 985, \cdot 99(\cdot 001)\cdot 999$.

15·5. Higher Integrals and Derivatives, etc.

It is a peculiarity of functions involving the exponential e^{-x^2} or $e^{-\frac{1}{2}x^2}$ that repeated integrals of one function and higher derivatives of another are often expressible in terms of one another; thus, using the notation of B.A. **1**, 1931 (xxiv), namely

$$Hh_1(x) = e^{-\frac{1}{2}x^2} \qquad Hh_0(x) = \int_x^\infty e^{-\frac{1}{2}t^2}dt$$

$$Hh_n(x) = \int_x^\infty Hh_{n-1}(t)dt = \frac{1}{n!}\int_x^\infty (t-x)^n e^{-\frac{1}{2}t^2}dt$$

we find that, with

$$F(x) = \frac{\frac{1}{2}(1 - \alpha(x))}{z(x)} = \frac{Hh_0(x)}{Hh_{-1}(x)} = e^{\frac{1}{2}x^2} Hh_0(x)$$

we have

$$\frac{d^n F(x)}{dx^n} = (-1)^n n! \frac{Hh_n(x)}{Hh_{-1}(x)}$$

Many functions tabulated seem to be expressible more easily as integrals; we list them in the simpler form, even though they may be given as derivatives.

In the following articles X is an argument such that $f(X+h)$ vanishes to tabular accuracy, where $f(x)$ is the function tabulated and h is the tabular interval.

15·51. Higher Integrals

We consider first the more extensive tables, involving $e^{-\frac{1}{2}x^2}$.

15·511. $Hh_n(x) = \int_x^\infty Hh_{n-1}(t)\,dt = \frac{1}{n!}\int_x^\infty (t-x)^n e^{-\frac{1}{2}t^2}\,dt$

	n	x			
10 d.	0(1)11	$-7(\cdot 1)X$	All $X < 6{\cdot}6$	D^m	B.A. 1, 1931 (63, Airey)
	12(1)17	$-5(\cdot 1)X$			
	18(1)21	$-2{\cdot}5(\cdot 1)X$			
10 d.	0(1)11	$-10(1)-7(\cdot 1)X$		D^m, $m < n$	B.A. 1928 (325, I_n, Airey)
	12(1)15	$-5(\cdot 1)X$			
	16(1)21	$-2{\cdot}5(\cdot 1)X$			

15·512. $(-1)^n e^{\frac{1}{2}x^2} Hh_n(x)$

These are given as reduced derivatives $h^n F^{(n)}(x)/n!$ of $F(x) = e^{\frac{1}{2}x^2} Hh_0(x)$, i.e. they are multiplied by $h^n (h = \cdot 01 \text{ or } \cdot 1)$ in the tables listed below.

	n	x		
$(24-n)$ dec.	0(1)11	0(·1)10	$\frac{h^m}{m!}F^{(m)}$	B.A. 7, 1939 (16)
	12(1)16	0(·1)X		
$(12-2n)$ dec.	0(1)3	0(·01)10	$\frac{h^m}{m!}F^{(m)}$	B.A. 7, 1939 (2)
	4 and 5	0(·01)X		

15·513. Miscellaneous Combinations of $Hh_n(x)$

15 dec.	$L' = e^{-\frac{1}{2}x^2}/Hh_0(x)$	$x = 0(\cdot 1)10$	$\frac{h^m}{m!}L^{(m)}$	B.A. 7, 1939 (24)
14 dec.	$\frac{1}{2}L'' = \frac{e^{-\frac{1}{2}x^2} Hh_1(x)}{2Hh_0^2(x)}$			
9; 8, 7, etc. dec.	$\frac{Hh_0(x)Hh_2(x)}{Hh_1^2(x)}$	$x = -7(\cdot 1)+5$; 0, 1, etc.	δ^2	B.A. 1, 1931 (72)

$$\psi_1 = 2\frac{Hh_0(x)Hh_2(x)}{Hh_1^2(x)} - 1 \qquad \psi_2 = \frac{Hh_0(x)}{Hh_1(x)} \qquad \psi_3 = \frac{\sqrt{2\pi}}{Hh_0(x)}$$

3 dec.	$x = 0(\cdot 01)\cdot 1(\cdot 1)3$	No Δ	Pearson 1930 (25)
3 dec.	$-x = 0(\cdot 01)\cdot 1(\cdot 1)3$	Δ	Lee 1914 (214); Pearson 1931 (146)

Part of this table appears in Pearson & Lee 1908 (68).

15·516. Integrals involving e^{-x^2}

In the absence of a better notation, and in spite of our previous avoidance of the symbols erf, erfc, we here use Hartree's notation, namely

$$\text{ierfc}\, x = \int_x^\infty \int_t^\infty \frac{2}{\sqrt{\pi}} e^{-u^2} du\, dt = \int_x^\infty (1 - H(t))dt = \frac{2}{\sqrt{\pi}} \int_x^\infty (t-x)e^{-t^2} dt$$

$$\text{iierfc}\, x = \int_x^\infty \text{ierfc}\, t\, dt = \frac{1}{\sqrt{\pi}} \int_x^\infty (t-x)^2 e^{-t^2} dt$$

15·5161. 2 ierfc x and 4 iierfc x

4 dec.	2 ierfc x	$x = 0(\cdot01)1(\cdot02)2\cdot18$	Δ	Hartree 1936 (99)
4 dec.	4 iierfc x	$x = 0(\cdot01)1(\cdot02)2$	Δ	

Two small critical tables carry on the tabulation until the functions vanish to 4 decimals.

15·5163. $f(x) = e^{x^2}$ ierfc $(-x)$, etc.

8–9 f.	$f(x)$	$x = -3(\cdot1)+2$		Miller (MS.)
4 dec.	$\log_{10} f(x)$	$x = -1\cdot2(\cdot1)+2$	No Δ	Eddington 1914 (129); Smart 1938 (39)
2 dec.	$e^{-x^2} f(x \cos\theta)$	$x = \cdot1(\cdot1)1\cdot8,\ \theta = 0(5^\circ)180^\circ$	No Δ	Smart 1938 (104)

15·5164. $g(x) = 2$ iierfc $(-x)$/ierfc $(-x)$

3 dec.	$x = -1(\cdot1)+1\cdot6$	No Δ	Eddington 1914 (141)
3 dec.	$x = -1(\cdot1)+1\cdot7$	No Δ	Smart 1938 (41)

15·5165. $F(x) = \sqrt{\pi}(x + \text{ierfc}\, x)$

3 dec.	$x = 0(\cdot05)2$	No Δ	Smart 1938 (49)

15·5166. $\psi(x) = \frac{1}{2}\sqrt{\pi}(1 + 2x^2 - 4\ \text{iierfc}\, x)$

7 fig.	$x = 0(\cdot05)1\cdot5(\cdot1)10(\cdot2)20\cdot4$	No Δ	Rosenberg 1942 (529)

Rosenberg also gives $x^5/\psi(x)$ to 7 figures; from about $x = 3$, $\psi(x) = \frac{1}{2}\sqrt{\pi}(1+2x^2)$ to 5 decimals.

6 dec.	$x = 0(\cdot1)3$	No Δ	Clark 1887 (95)

Clark also gives 5-decimal tables of $x^3 e^{-x^2}/\psi(x)$ and $x^4 e^{-x^2}/\psi(x)$.

5 dec.	$x = 0(\cdot1)3$	No Δ	Jeans 1925 (App. B)

15·5167. $\phi(x) = \frac{1}{4}\sqrt{\pi}\left\{2x + \frac{1 - 4\ \text{iierfc}\, x}{x}\right\}$

3 dec.	$x = 0(\cdot1)2$	No Δ	Smart 1938 (51)

15·5168. Miscellaneous Functions of Integrals

Crout 1936 (42, 43) and Giddings & Crout 1936 (166–168) give tables. All listed below are to 5 decimals, except for h_L/h_0 which is given to 6 decimals. The arguments are $x = \rho = 0, \cdot03, \cdot06, \cdot1(\cdot05)\cdot5(\cdot1)1(\cdot2)2, \infty$, also (in Crout 1936 only) $x = \cdot55$. We write $\phi_n = n!\, i^n\text{erfc}\,(-x)$, not quite in agreement with the notation used by Crout, etc. for $n \neq 3$.

The functions tabulated include

$$\frac{T_L}{T_0} = \frac{\pi}{4}\left(\frac{\phi_1}{\phi_2}\right)^2 \qquad \frac{\nu}{\nu_0} = \frac{2}{\pi}\frac{\phi_2}{\phi_1^2} \qquad \frac{M}{M_0^+} = \frac{2x}{\phi_1} \qquad \frac{p_s}{p_0} = \frac{x^2+\frac{1}{2}}{\phi_2} \qquad \bar{\sigma} = \frac{6}{\pi}\left(\frac{\phi_2}{\phi_1}\right)^2 - \frac{1}{2}$$

$$\frac{h_L}{h_0} = \frac{T_0}{T_L} = \frac{4}{\pi}\left(\frac{\phi_2}{\phi_1}\right)^2 \qquad \sigma(\text{p. }42) = \sigma_m = \sigma(\gamma = 0) = \frac{8}{\pi}\left(\frac{\phi_2}{\phi_1}\right)^2 - \frac{\phi_3}{\phi_1} \qquad \sigma_m\frac{T_L}{T_0} = 2 - \frac{\pi}{4}\frac{\phi_1\phi_3}{\phi_2^2}$$

Further, more complicated functions depending on these are also tabulated. Graphs are given.

$$\textbf{15·517.}\quad \frac{1}{2\pi\sqrt{1-r^2}}\int_h^\infty\int_k^\infty \exp\left\{-\frac{1}{2(1-r^2)}(x^2+y^2-2rxy)\right\}dx\,dy$$

This is the volume cut off from the normal bivariate or correlation surface for $x > h$, $y > k$, and represents the probability that the variables x and y, having mutual correlation coefficient r, simultaneously exceed these values h and k respectively.

In all of the following tables the ranges of h and k are each $0(\cdot 1)2\cdot 6$. No differences are given.

8+ dec.	$-r = \cdot 65(\cdot 05)\cdot 75$	Moul, etc. 1930 (12)
7 dec.	$-r = 0(\cdot 05)\cdot 6$	
7 dec.	$-r = \cdot 8(\cdot 05)1$	Lee 1917 (288)
6 dec.	$r = 0(\cdot 05)1$	Lee 1927 (373)

All the preceding items are reproduced in Pearson 1931 (78).

4 dec.	$r = \cdot 8(\cdot 05)1$	Everitt 1912 (390); Pearson 1930 (52)

$$\textbf{15·518.}\quad \int_{-\infty}^{\infty}\left[1-\left(\frac{1+\alpha(x)}{2}\right)^n-\left(\frac{1-\alpha(x)}{2}\right)^n\right]dx,\ \text{etc.}$$

These integrals give the average range of a set of n samples from a normal population with unit standard deviation.

5 dec.	$n = 2(1)1000$	Tippett 1925 (386); Pearson 1931 (165)

The probability distribution of the range, W, is given in Pearson & Hartley 1942 (302) to 4 decimals, for $W = 0(\cdot 05)7\cdot 25$ and $n = 2(1)20$.

15·52. Derivatives

No provision for interpolation is usually made, since the derivatives themselves may be used for this purpose.

$$\textbf{15·521.}\quad \left(\frac{d}{dx}\right)^{n+1} H(x)$$

20 dec.	$n = 1(1)14$	$x = 0(\cdot 1)8$	New York W.P.A. (MS.)

$$\textbf{15·522.}\quad \frac{\Phi_n(x)}{2^{n-1}} = 2^{-(n-1)}\left(\frac{d}{dx}\right)^n H(x)$$

4 dec.	$n = 1(1)6$	$x = 0(\cdot 01)4$	Δ Bruns 1906 (A. 6); Czuber 1924 (462)
4 dec.	$n = 1(1)6$	$x = 0(\cdot 01)3$	Jahnke-Emde 1909 (37), 1933 (99), 1938 (26)

15·523. $z^{(n)}(x) = \left(\dfrac{d}{dx}\right)^n z(x)$

	n	x		
20 dec.	1(1)14	0(·1)8·4		New York W.P.A. (MS.)
7 dec.	1(1)6	0(·01)4		Jørgensen 1916 (178)
5 dec.	2(1)8	0(·01)4·99		Glover 1930 (394)
5 dec.	2(1)4	0(·01)4·99		Rietz 1924 (209)
5 dec.	1(1)6	0(·1)4		Fry 1928 (456)
4; 5 dec.	3	0(·01)3(·1)6; 5·2	1Δ	Charlier 1906 (45, 46)
4 dec.	4	0(·01)3(·1)6		
4 dec.	3, 4	0(·01)3(·1)4·9	No Δ	Charlier 1920 (124)
4 dec.	3(1)6	0(·05)3·95		A. Fisher 1922 (280)

Values for $n = -1$ and 0 are always given as well, with possible minor variations in range, etc.

15·524. $(-1)^{n-1}\sqrt{2\pi}\, z^{(n-1)}(x) = Hh_{-n}(x)$

10 dec.	$n = 1(1)3$	$x = -7(·1)+6·6$	B.A. **1**, 1931 (60)
9, 8, 7, 6 dec.	$n = 4, 5, 6, 7$		
4, 3, 2, 1, 0 d.	$n = 1, 2, 3, 4, 5$	$x = 0(·1)4·5$	Thiele 1903 (18)

15·525. $\dfrac{1}{3!}\left\{z^{(2)}(x) + \dfrac{1}{\sqrt{2\pi}}\right\}$

4 dec.	0(·01)2(·02)3(·2)4·2, ∞	Bowley 1937 (303)
4 dec.	0(·1)3(·2)4·2, ∞	Darmois 1928 (309)

15·54. Tetrachoric Functions, $\tau_n = (-1)^{n-1} z^{(n-1)}/\sqrt{n!}$

Both x and $\tau_0 = \frac{1}{2}(1-\alpha)$ have been used as independent variable; the notation $z^{(n)}$, however, indicates x-derivatives in both cases.

7 dec.	$n = 0(1)19$	$x = 0(·1)4$	Lee 1925 (351); Pearson 1931 (74)
5 dec.	$n = 1(1)6$	$\tau_0 = 0(·001)·5$	Everitt 1910 (442); Pearson 1930 (42)

15·56. Miscellaneous Derivatives

B.A. **7**, 1939 (2) tabulates $(h^n/n!)D^nF$, where $F = \frac{1}{2}(1-\alpha)/z$, with $h = 0·01$, for $x = 0(·01)10$ and $n = 1(1)N$ to 12 decimals; $N = 5$, 4 or 3.

B.A. **7**, 1939 (16) tabulates $(h^n/n!)D^nF$, with $h = 0·1$, for $x = 0(·1)10$ and $n = 1(1)N$ to 24 decimals; $N = 16$, 15, 14, 13, 12 or 11.

B.A. **7**, 1939 (24) tabulates $(h^n/n!)D^nL$, where $L = -\log_e \frac{1}{2}(1-\alpha)$ with $h = 0·1$, for $x = 0(·1)10$ and $n = 1(1)N$ to 16 decimals; $N = 10$, 9, 8 or 7.

B.A. **7**, 1939 (28) tabulates $(h^n/n!)D^nl$, where $l = \log_{10} \frac{1}{2}(1-\alpha)$ with $h = 0·1$, for $x = 0(·1)10$ and $n = 1(1)N$ to 12 decimals; $N = 7$, 6 or 5.

In each case N is such that the corresponding functions include all that are significant to the number of decimals tabulated. The derivatives are given as an aid to interpolation, but some have independent interest; see Arts. 15·34, 15·37, 15·512 and 15·513. See also Art. 15·312, Terazawa 1917.

15·6. Hermite Polynomials and Functions

We use the notation suggested by Davis, *M.T.A.C.*, **1**, 50, 1943, namely

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2}\left(\frac{d}{dx}\right)^n e^{-\frac{1}{2}x^2} \qquad H_n(x) = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}$$

Klose 1934 has announced a programme of "Hermite and Laguerre polynomials and orthogonal functions". Peters & Van Voorhis 1940 (369, 378) gives the coefficients in $h_n(x)$ for $n = 1(1)8(2)16$ and Sasuly 1934 (289) for $n = 0(1)10$.

15·61. Hermite Polynomials $h_n(x) = (-1)^n e^{\frac{1}{2}x^2} D^n e^{-\frac{1}{2}x^2}$

Exact	$n = 1(1)6$	$x = 0(\cdot 01)4$	Jørgensen 1916 (196)
Exact	$n = 0(1)10$	$x = 0(\cdot 1)10$	Miller (MS.)

See also Thiele 1889.

15·615. $\dfrac{h_n(x)}{\sqrt{(n+1)!}}$

Peters & Van Voorhis 1940 (501, H. P. Peters) gives 4-decimal tables of these functions for $n+1 = 2(1)25$, x being such that $\frac{1}{2}(1-\alpha(x)) = \cdot 05(\cdot 01)\cdot 4$.

15·64. Hermite Functions $e^{-\frac{1}{2}x^2} H_n(x) = (-1)^n e^{\frac{1}{2}x^2} D^n e^{-x^2}$

5 fig.	$n = 0(1)11$	$x = 0(\cdot 04)1(\cdot 1)4(\cdot 2)7(\cdot 5)8$	Russell 1933 (292)

15·66. Zeros of Hermite Polynomials

10 dec.	Zeros of $h_n(x)$	$n = 1(1)8$	Miller (MS.)
6 dec.	Zeros of $h_n(x)$	$n = 1(1)27$	E. R. Smith 1936

See also Thiele 1889.

15·68. Expressions involving Hermite Polynomials

Peters & Van Voorhis 1940 (505) gives tables of

$$r'(r, k) = r - \frac{r^3}{2.3}h_2(k) + \frac{r^5}{2.4.5}h_4(k) - \frac{r^7}{2.4.6.7}h_6(k) + \dots$$

These tables give 4-decimal values for $r = \cdot 01(\cdot 01)\cdot 9$ and 3-decimal estimates for $r = \cdot 91(\cdot 01)1$, in each case for k such that $\frac{1}{2}(1-\alpha(k)) = \cdot 09(\cdot 01)\cdot 25$.

15·7. Moments. "Chi-squared" Tables

These can also be expressed in terms of the Incomplete Gamma Function; see Section 14.

$$\mu_n(x) = \frac{1}{\sqrt{2\pi}}\int_0^x t^n e^{-\frac{1}{2}t^2} dt = \int_0^x t^n z\, dt = \sqrt{\frac{2^{n-2}}{\pi}}\,\Gamma_{\frac{1}{2}x^2}\left(\frac{n+1}{2}\right)$$

$$m_n(x) = \frac{\mu_n(x)}{(n-1)!!} \qquad (n-1)!! = (n-1)(n-3)\dots(2 \text{ or } 1)$$

See Pearson & Lee 1908 (60, 61), Pearson 1930 (xxiv), etc. Note that

$$m_0(x) = \tfrac{1}{2}\alpha(x) \qquad m_1(x) = \frac{1}{\sqrt{2\pi}} - z(x)$$

Also

$$P_{n'}(x^2) = \sqrt{\frac{2}{\pi}}\int_x^\infty e^{-\frac{1}{2}t^2}dt + \sqrt{\frac{2}{\pi}}e^{-\frac{1}{2}x^2}\left(\frac{x}{1}+\frac{x^3}{3!!}+\ldots+\frac{x^{n'-3}}{(n'-3)!!}\right)$$

$$= \sqrt{\frac{2}{\pi}}\int_x^\infty \frac{t^{n'-2}}{(n'-3)!!}e^{-\frac{1}{2}t^2}dt = 1 - 2m_{n'-2}(x) \qquad n' \text{ even}$$

$$P_{n'}(x^2) = e^{-\frac{1}{2}x^2}\left(1+\frac{x^2}{2!!}+\frac{x^4}{4!!}+\ldots+\frac{x^{n'-3}}{(n'-3)!!}\right)$$

$$= \int_x^\infty \frac{t^{n'-2}}{(n'-3)!!}e^{-\frac{1}{2}t^2}dt = 1 - \sqrt{2\pi}\, m_{n'-2}(x) \qquad n' \text{ odd}$$

See Elderton 1902 (156), Pearson 1930 (xxxi), etc. We may also write

$$P_{n'}(x^2) = \int_x^\infty \frac{(\frac{1}{2}t^2)^{\frac{1}{2}n-1}}{(\frac{1}{2}n-1)!}e^{-\frac{1}{2}t^2}d(\tfrac{1}{2}t^2) = 1 - \frac{\Gamma_{\frac{1}{2}x^2}(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)} \qquad n = n' - 1$$

See Fisher & Yates 1938 (1), 1943 (1).

These are the functions involved in Pearson's "chi-squared" test, χ^2 being generally used in place of x^2; they are usually considered as functions of x^2 rather than of x, and so they are very closely connected with the Incomplete Gamma Function. For n' odd, the close connection with the Poisson distribution should be noted; see Art. 23·86 for tables.

Moments of the tail area about the bounding ordinate may be expressed in terms of repeated integrals; see Art. 15·51.

15·71. $m_n(x)$

7 dec.	$n = 1(1)10$	$x = 0(\cdot 1)5$	No Δ	Pearson & Lee 1908 (66) Pearson 1930 (22)
7 dec.	$n = 11$ and 12	$x = 0(\cdot 1)5$	δ^2	Pearson 1931 (147, E. M. Elderton and Wishart)

15·72. $1 - \sqrt{2\pi}\dfrac{\mu_{n-1}(\sqrt{n}x)}{2^{\frac{1}{2}n-1}(\frac{1}{2}n-1)!}$

These functions reduce to $1-\sqrt{2\pi}m_{n-1}(\sqrt{n}x)$ for n even and $1-2m_{n-1}(\sqrt{n}x)$ for n odd.

4 dec.	$n = 1(1)10, 12, 15, 19, 24, 30$	$x = 0(\cdot 1)4$	Kelley 1938 (116)

15·731. $P_{n'}(x^2)$

10 dec.	$n' = 3(1)30$	$x^2 = 1(1)30(10)70$	No Δ	Davis & Nelson 1935 (399)
6 dec.	$n' = 3(1)30$	$x^2 = 1(1)30(10)70$	No Δ	W. P. Elderton 1902 (159) Pearson 1930 (26) Peters & Van Voorhis 1940 (498)
5 dec.	$n' = 7(1)15$	$x^2 = 4(1)15$	No Δ	Jones 1924 (285)

15·732. $-\text{Log}_{10} P_4(x^2)$

3 dec. $x^2 = 1(1)50(10)100(50)2100(100)$ $3000(500)25{,}000$ No Δ Pearson 1930 (31)

15·74. Inverse Tables

The main inverse table is due to R. A. Fisher and has been reproduced many times. Fisher 1941 (T. III) gives values of x^2 to 3 decimals (with extensions to a minimum of 3 figures) for $P_{n-1}(x^2) = ·99, ·98, ·95, ·9, ·8, ·7, ·5, ·3, ·2, ·1, ·05, ·02, ·01$ with $n = 1(1)30$. Reproductions are given in Davenport & Ekas 1936 (179); Fisher & Yates 1938, 1943 (T. IV), with column $P = ·001$ added; Fry 1928 (468); Garrett 1937 (379, 474); Goulden 1939 (268), $P = ·99, ·95, ·5, ·3, ·2, ·1, ·05, ·01$; Kenney 1940 (2, 198), to 2 decimals and with $P < ·5$ only; Rider 1939 (202).

Values of x^2 to 6 figures for $P_{n-1}(x^2) = ·995, ·99, ·975, ·95, ·9, ·75, ·5, ·25, ·1, ·05, ·025, ·01, ·005$ are given in C. M. Thompson 1941*b* (188) for $n = \nu = 1(1)30(10)100$.

7 dec. x $n = 0(1)10$ Pearson & Lee 1908 (62, 63)
3 dec. x $n = 11(1)14$ Pearson 1930 (24, xxv)

$$\frac{m_n(x)}{m_n(\infty)} = \frac{\mu_n(x)}{\mu_n(\infty)} = \tfrac{1}{2}$$

15·8. The Error Integral with Complex Argument

If $z = x+iy$, we have

$$\int_z^\infty e^{-t^2}dt = e^{y^2}\int_x^\infty e^{-t^2}(\cos 2yt - i\sin 2yt)\,dt$$

$$= \tfrac{1}{2}e^{-x^2}\int_0^\infty e^{-xt-\frac{1}{4}t^2}(\cos yt - i\sin yt)\,dt$$

If $z = \frac{1}{2}\sqrt{\pi}(1+i)u = \sqrt{\frac{1}{2}\pi i}\,u$, then

$$\int_0^z e^{-t^2}dt = \sqrt{\tfrac{1}{2}\pi i}\int_0^u e^{-\frac{1}{2}\pi i t^2}dt$$

$$= \sqrt{\tfrac{1}{2}\pi i}\{C(u) - iS(u)\}$$

where $C(u)$ and $S(u)$ are the Fresnel Integrals; see Sub-section 20·6.

No differences are given with any of the tables listed below.

15·85. $e^{z^2}\int_z^\infty e^{-t^2}dt$

With $z = (1-iu)/\eta$, Born 1933 (486) tabulates real and imaginary parts of

$$\frac{1}{\eta\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{1+iv}{1+v^2}\exp\left\{-\frac{(u-v)^2}{\eta^2}\right\}dv = \frac{2}{\eta}e^{z^2}\int_z^\infty e^{-t^2}dt$$

giving 3 figures for $u = 0(·2)4$, $\eta = 2, 1, \frac{1}{2}$ and $\frac{1}{10}$.

15·86. $\frac{2}{\pi}\mathcal{R}e^{z^2}\int_z^\infty e^{-t^2}dt$

Although there are many variations of notation, the function tabulated is nearly always, with $z = x+iy$,

$$\frac{k_\nu}{k_0} = \frac{x}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}dt}{x^2+(t-y)^2} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-xt-\frac{1}{4}t^2}\cos yt\,dt = \frac{2}{\sqrt{\pi}}\mathcal{R}e^{z^2}\int_z^\infty e^{-t^2}dt$$

A formula appropriate for small x (see Arts. 15·1221 and 15·312) is

$$\frac{k_\nu}{k_0} = e^{-y^2} - \frac{2x}{\sqrt{\pi}}\left(1-2ye^{-y^2}\int_0^y e^{t^2}dt\right) + O(x^2)$$

4 dec.	$x = 0(\cdot5)1\cdot5$	$y = 0(\cdot2)2$	Zemansky 1930 (232)
4 dec.	$x = 0$	$y = 0(\cdot2)3$	Mitchell & Zemansky 1934 (329)
3 or 4 dec.	$x = \cdot5,\ 1\cdot5$	$y = 0(\cdot2)2(2)10$	
3 or 4 dec.	$x = 1$	$y = 0(\cdot2)4(2)10$	
3 fig.	$x = 2$ and 10	$y/x = 0(\cdot2)4$	
3 fig.	$x = 0(\cdot01)\cdot2$	$y = 0(\frac{1}{4})5$	Hjerting 1938 (512)
2 f. or 4 d.	$x = 0(\cdot1)\cdot5$	$y = 0(\frac{1}{4})2(\frac{1}{2})5$	

Hjerting also gives $k_\nu/x\,k_0$ for $y = 5(\frac{1}{2})20$; if $x \leqslant \cdot2$ this function is independent of x to the 3 figures given.

15·9. Miscellaneous

Lash Miller & Gordon 1931 (2877) gives 11-decimal tables for $x = \cdot4(\cdot1)2\cdot5$ for each of the functions

$$G(x) = \int_0^x y^{-1/2}e^{-\pi^2/y}dy = 2\sqrt{x}e^{-\pi^2/x} - 2\pi^{3/2}\{1-H(\pi/\sqrt{x})\}$$

and

$$\int_0^x G(y)dy = \tfrac{4}{3}\sqrt{x}(x+\pi^2)e^{-\pi^2/x} - 2\pi^{3/2}(x+\tfrac{2}{3}\pi^2)\{1-H(\pi/\sqrt{x})\}$$

Items added in Proof

15·312, 15·32. Kapzov & Gwosdower 1927 (133) gives

5 dec.	$xe^{-x^2}\int_0^x e^{t^2}dt$	$\cdot1,\ \cdot5,\ \cdot8,\ 1\cdot0,\ 1\cdot2(\cdot05)2\cdot2$	With Δ
5 dec.	$xe^{x^2}\int_0^x e^{-t^2}dt$	$\cdot7(\cdot01)1\cdot2$	With Δ

15·312. Sakamoto 1928 (218) gives

4 dec.	$e^{-x^2}\int_0^x e^{t^2}dt,\ \frac{e^{-x^2}}{x}\int_0^x e^{t^2}dt$	$0(\cdot01)3\cdot6,\ 4,\ 4\cdot5,\ 5,\ 6,\ 8,\ 10$	No Δ

SECTION 16

LEGENDRE FUNCTIONS

16·0. Introduction

A variety of notations is in use. We denote by $P_n(x)$ as usual the Legendre polynomial of the nth order, such that $P_n(1) = 1$ and the coefficient of x^n is $\dfrac{(2n-1)!!}{n!}$. We shall denote by $P^n(x)$ the polynomial $\dfrac{n!\,P_n(x)}{(2n-1)!!}$, used by Laplace and Gauss, in which the coefficient of x^n is unity.

When we turn to associated functions, we find numerous forms in use. We shall adopt the notation of Schmidt 1935 for the functions considered in that work.

The usual form of associated function in non-numerical work in the case $|x| < 1$ is Ferrers's function $(1-x^2)^{m/2}\dfrac{d^m P_n(x)}{dx^m}$, which we denote by $P_{nm}(x)$. This has the grave disadvantage in numerical work that for different values of n and m the order of magnitude of the function varies enormously, since the quadratic mean of the tesseral* harmonic $P_{nm}(\cos\theta)\cos(m\phi+\epsilon)$ over the surface of a sphere is

$$\sqrt{\frac{1}{2(2n+1)}\,\frac{(n+m)!}{(n-m)!}} \qquad (m \neq 0)$$

Five other forms of associated function are defined in the scheme in Art. 16·01, in which the first column gives our notation, the second the ratio of the function to P_{nm}, the third the ratio of the zonal harmonic (case $m = 0$) to P_n, while the fourth to sixth columns give $\int S^2 d\omega$ over all solid angle ω; in the fourth column S is the associated function $X_{nm}(\cos\theta)$, $(m \neq 0)$, in the fifth S is the zonal harmonic $X_{n0}(\cos\theta)$, and in the sixth S is the tesseral harmonic obtained by multiplying the associated function by $\cos m\phi$ or $\sin m\phi$ $(m \neq 0)$. The values in the sixth column are thus halves of those in the fourth. The value of ϵ_m is 1 when $m = 0$, and 2 when m is a positive integer. (We are here using X as a general symbol, and not in the special sense in which it is used in Art. 16·6.)

R_n^m is a function introduced by Schmidt and abandoned by him in favour of P_n^m. It will be seen that in the case of R_n^m the harmonics (zonal and tesseral) are fully normalized. F_n^m is used in Doodson 1936, his $F_n^m(\theta)$ being our $F_n^m(\cos\theta)$. R_{nm} (called R_n^m by the Egersdörfers) is used in Egersdörfer 1936. It will be seen that $F_n^m = \sqrt{\pi}R_{nm}$, and that the Doodson and Egersdörfer functions (rather than harmonics) are fully normalized.

Complete normalization in any of these fashions gives a zonal harmonic ($m = 0$) not equal to P_n, but differing from P_n by a factor depending on n. Schmidt has therefore introduced the function P_n^m which is normalized in m only. The quadratic mean of both zonal and tesseral harmonics is $1/\sqrt{(2n+1)}$, which does not vary to an inconvenient extent with n, and we have $P_n^0 = P_n$. A resolution recommending the use of Schmidt's partly normalized P_n^m was adopted by the International Association of Terrestrial Magnetism and Electricity at its Washington assembly in 1939.

* We use the term *tesseral* in this section as including *sectorial* (case $m = n$).

The remaining function P^{nm} is the one used by Laplace and Gauss, the coefficient of x^{n-m} in $P^{nm}(x)$ being unity. The expressions for $\int S^2 d\omega$ in this case are $\left\{\frac{(n-m)!}{(2n-1)!!}\right\}^2$ times those for P_{nm}, and are of little interest.

It seems best to give some comparison between the functional notations of various authors. We shall always use (n, m) as above, although Tallqvist, Prévost and the Egersdörfers use (n, j), Doodson uses (r, n), and so on.

P_{nm} of this *Index*, Schmidt 1935, Tallqvist, Prévost, Prey
 $= P_n^m$ of Doodson, most English writers, the Egersdörfers, Jahnke & Emde, Mursi
 $= T_n^m$ of Ferrers

P_n^m of this *Index* is that of Schmidt 1935

R_n^m of this *Index*, Schmidt 1935, Schuster
 $= R_m^n$ of Schmidt (last century)

F_n^m of this *Index*, Doodson
 $= \overline{P}_n^m$ of Jahnke & Emde 1933, 1938

R_{nm} of this *Index*
 $= R_n^m$ of the Egersdörfers

P^{nm} of this *Index*, Schmidt 1935, Laplace, Gauss
 $= H_n^m$ of Adams

We also use $P_n^{(m)}$ as usual for $d^m P_n/dx^m$. The notation for functions of the second kind is explained in Art. 16·7.

We have not attempted to index all explicit algebraic developments of the various functions, but have mentioned some of the more extensive in the appropriate articles. For convenience we use the term "Fourier series" throughout for series of sines and cosines of multiples of an angle θ, although the series terminate. Except in Arts. 16·8 ff., and where otherwise stated, the range in x is from 0 to 1.

For tables, schedules, etc. to assist in the expansion of an arbitrary function in spherical harmonics, see Schuster 1902, Seeliger 1890, Bauschinger 1915, Carse & Shearer 1915 and **Prey 1922.**

On important manuscript tables see H. T. Davis, *M.T.A.C.*, **1**, 119, 1943, and A. N. Lowan, *M.T.A.C.*, **1**, 164, 1944.

The general arrangement of this section is as follows:

16·01. Table of Notation

For explanation, see Art. 16·0.

X_{nm}	$\dfrac{X_{nm}}{P_{nm}}$	$\dfrac{X_{n0}}{P_n}$	Integral over unit sphere of square of $X_{nm}(m\neq 0)$	X_{n0}	$X_{nm}\cos m\phi(m\neq 0)$
P_{nm}	1	1	$\dfrac{4\pi}{2n+1}\dfrac{(n+m)!}{(n-m)!}$	$\dfrac{4\pi}{2n+1}$	$\dfrac{2\pi}{2n+1}\dfrac{(n+m)!}{(n-m)!}$
P_n^m	$\sqrt{\epsilon_m\dfrac{(n-m)!}{(n+m)!}}$	1	$\dfrac{8\pi}{2n+1}$	$\dfrac{4\pi}{2n+1}$	$\dfrac{4\pi}{2n+1}$
R_n^m	$\sqrt{\dfrac{(2n+1)\epsilon_m(n-m)!}{(n+m)!}}$	$\sqrt{2n+1}$	8π	4π	4π
F_n^m	$\sqrt{\dfrac{(2n+1)(n-m)!}{2(n+m)!}}$	$\sqrt{\dfrac{2n+1}{2}}$	2π	2π	π
R_{nm}	$\sqrt{\dfrac{(2n+1)(n-m)!}{2\pi(n+m)!}}$	$\sqrt{\dfrac{2n+1}{2\pi}}$	2	2	1
p^{nm}	$\dfrac{(n-m)!}{(2n-1)!!}$	$\dfrac{n!}{(2n-1)!!}$			

ϵ_m is 1 when $m = 0$, and 2 when m is a positive integer.

16·02. Conversion Factors

The following tables giving ratios of functions differing only by a factor independent of x (or θ) will facilitate conversions. In each case the tables are for all n up to the limit given and for $m = 0(1)n$.

For tables of $n!$ and $n!!$ see Section 3.

p^{nm}/P_n^m	7–8 fig.	To $n = 6$	Schmidt 1935 (20)
P_n^m/p^{nm}	7–8 fig.		
$\log(P_n^m/p^{nm})$	7 dec.		
$2\log(P_n^m/p^{nm})$	7 dec.		
P_{nm}/p^{nm}	Exact	To $n = 6$	Schmidt 1935 (21)
P_{nm}/P_n^m	7–8 fig.		
$\log(P_{nm}/P_n^m)$	7 dec.		
$2\log(P_n^m/P_{nm})$	7 dec.		
P_{nm}/p^{nm}	Exact	To $n = 7$	Schmidt 1925
P_n^m/p^{nm}	7–8 fig.		
R_n^m/p^{nm}	7 fig.	To $n = 7$	Schmidt 1895
$\log(p^{nm}/R_n^m)$	5 dec.	To $n = 5$	Schmidt 1896
$\log(P_{nm}/R_n^m)$	5 dec.		
P_{nm}/F_n^m	7–8 fig.	To $n = 12$	Doodson 1936
$2\log(F_n^m/P_{nm})$	7 dec.	To $n = 17$	Prey 1922 (T. 4)

16·1. Legendre Polynomials $P_n(x)$, etc.

16·11. $P_n(x)$

Algebraic expressions for $P_n(x)$ up to $n = 16$ are given in Tallqvist 1937, both in the form with a power of 2 as common denominator, and also with each coefficient expressed as an exact decimal. The first forms are given to $n = 18$ in Prévost 1933. Algebraic expressions for $P_n(x)$ up to $n = 20$ are given in Egersdörfer 1936 (18).

For tables dealing primarily with the case $x > 1$, see Art. 16·8.

Exact	·01	$n = 2(1)8$	No Δ	**Tallqvist 1904**
Exact	·01	$n = 2(1)8$	No Δ	Hayashi 1930*a*
Exact	·01	$n = 2(1)7$	No Δ	**B.A. 1879**
7–11 dec.	·01	$n = 2(1)7$	No Δ	Schorr 1916 (from B.A.)
10 dec.	·001	$n = 2(1)8$	Δ^2	Davis (MS.)
7 dec.	·01	$n = 2(1)12$	δ_m^{2n}	B.A. Pt.-Vol. A (in press)
6 dec.	·001	$n = 2(1)16$	No Δ	**Tallqvist 1937**
6; 5 dec.	·1	$n = 2(1)4;\ 5(1)7$	No Δ	Schmidt 1935
5 dec.	·01	$n = 2(1)10$	Δ	Prévost 1933
5 dec.	·01	$n = 2(1)10$	No Δ	Hayashi 1930*b*
5 dec.	·01	$n = 2(1)7$	No Δ	Dale 1903
5; 4 dec.	·01	$n = 2(1)7;\ 8(1)10$	No Δ	Dwight 1941
4 dec.	·01	$n = 2(1)8$	No Δ	Tallqvist 1905*a*
4 dec.	·01	$n = 2(1)7$ (From B.A. 1879)	No Δ	Byerly 1893, Jahnke–Emde 1909, 1933, 1938

16·115. Log $P_n(x)$

6 dec.	·1	$n = 1(1)7$	No Δ	Schmidt 1935

16·12. $P^n(x)$

This is the G_n^0 of Adams. He gives algebraic expressions up to $n = 10$ (page 265).

10 dec.	·01	$n = 2(1)10$	No Δ	Adams 1900 (268)
10 dec.	·05	$n = 11(1)20$	No Δ	Airey (MS.)

16·18. Zeros of $P_n(x)$, etc.

The zeros of $P_n(x)$ are required in Gauss's method of numerical quadrature (see Whittaker & Robinson 1926 (159) and many of the other references below). When the range of integration is from -1 to $+1$, the abscissæ for n ordinates are the n roots x of $P_n(x) = 0$, and the multiplier of the ordinate y_r corresponding to the root x_r is

$$A_r = \frac{1}{F(x_r)} \int_{-1}^{+1} F(x)\,dx$$

where $F(x) = P_n(x)/(x - x_r)$. When the range is from 0 to 1, the abscissæ are $\frac{1}{2}(1+x)$ and the multipliers $\frac{1}{2}A$. Conversion from one range to the other is thus easy. Tables which do not go beyond $n = 7$ are almost always taken, with or without conversion, from Gauss 1816. On errors in several of the following tables see Archibald, *M.T.A.C.*, **1**, 51, 1943.

Roots			Multipliers		
$\frac{1}{2}x$	16 dec.	$n = 2(1)10$	$\frac{1}{2}A$	16 dec.	**Moors 1905**
x	16 dec.	$n = 2(1)7$	$\frac{1}{2}A$ Also 10-dec. logs	16 dec.	Hobson 1931 (80)
$\frac{1}{2}(1+x)$	16 dec.	$n = 2(1)7$	$\frac{1}{2}A$ Also 10-dec. logs	16 dec.	Gauss 1816 Markoff 1896 (70)
$\frac{1}{2}(1+x)$	16 dec.	$n = 8(1)10$	$\frac{1}{2}A$ Also 10-dec. logs	16 dec.	Perott 1891
x	16 dec.	$n = 2(1)7$	$\frac{1}{2}A$ Also 10-dec. logs	16 dec.	Heine 1881
x	15 dec.	$n = 2(1)16$	A	15 dec.	Lowan, Davids & Levenson 1942
$\frac{1}{2}x$ / x	16 dec. / 15 dec.	$n = 2(1)7$	$\frac{1}{2}A$ Also 10-dec. logs	16 dec.	B.A. 1879
$\frac{1}{2}(1+x)$	10; 16 d.	$n = 2(1)6$; 7	$\frac{1}{2}A$	10; 16 d.	Todhunter 1875 (108)
x Also 18-dec. logs	13 dec.	$n = 12$	A	13 dec.	Bayly 1938
$\frac{1}{2}(1+x)$	12 dec.	$n = 7$	$\frac{1}{2}A$	12 dec.	Irwin 1923
x^2, $\log x$	10 dec.	$n = 17$			Prey 1922 (12)
x, $\frac{1}{2}(1+x)$	8; 10 d.	$n = 2(1)7$; $8(1)10$	$\frac{1}{2}A$	8; 10 d.	Radau 1880*b* (301)
$\frac{1}{2}(1+x)$	10 dec.	$n = 3(1)7$	$\frac{1}{2}A$	10 dec.	Scarborough 1930 (135)
x	9 dec.	$n = 2(1)5$	$\frac{1}{2}A$	9 dec.	Runge & König 1924 (283)
$\frac{1}{2}(1+x)$	8 dec.	$n = 2(1)6$	$\frac{1}{2}A$ Also 7-dec. logs	7 dec.	Bertrand 1870 (342) Carr 1886 (440)
x, $\frac{1}{2}(1+x)$	8 dec.	$n = 8, 9$	$\frac{1}{2}A$	8 dec.	Radau 1880*a*
$\frac{1}{2}(1+x)$	8 dec.	$n = 3(1)7$	$\log \frac{1}{2}A$	8 dec.	Bauschinger 1901 Bauschinger–Stracke 1934
$\frac{1}{2}(1+x)$	8 dec.	$n = 2(1)5$	$\frac{1}{2}A$	7 dec.	Whittaker & Robinson 1926 (160)
$\frac{1}{2}x$	7 dec.	$n = 2(1)10$	$\frac{1}{2}A$	7 dec.	Nyström 1930
x	7 dec.	$n = 2(1)8$			Tallqvist 1905*a*
x	7 dec.	$n = 2(1)7$			Silberstein 1923 (125)
x	7 dec.	$n = 2(1)6$			Schmidt 1935 (19)
x	6 dec.	$n = 2(1)40$			**Smith 1938**
x	4 dec.	$n = 2(1)10$			Prévost 1933 (28)

16·19. Zeros of $P'_n(x)$

For these see Art. 16·48 (case $m = 1$).

16·2. Legendre Polynomials $P_n(\cos\theta)$, etc.

For R_n^0, F_n^0 and R_{n0} see Arts. 16·54–16·56, cases $m = 0$.

16·21. $P_n(\cos\theta)$

Tallqvist 1932*a* gives exact expressions for $P_n(\cos\theta)$, $n = 1(1)16$, as Fourier series. Tallqvist 1938 gives Fourier series for $P_n(\cos\theta)$ with exact decimal coefficients for $n = 1(1)8$ and with 12-decimal coefficients for $n = 9(1)32$. Egersdörfer 1936 (27) gives exact expressions for $P_n(\cos\theta)$ as Fourier series for $n = 1(1)20$. See also Prey 1922 (T. 2), which goes up to $n = 17$.

16 dec.	5°	$n = 1(1)20$	No Δ	Airey (MS.)
10 dec.	1°	$n = 1(1)8$	No Δ	**Tallqvist 1905*b***
10 dec.	1°	$n = 1(1)8$	No Δ	Hayashi 1930*a*
7 dec.	5°	$n = 1(1)20$	No Δ	Lodge 1904
6 dec.	10′	$n = 1(1)32$	No Δ	**Tallqvist 1938**
6; 5 dec.	5°	$n = 1(1)4$; $5(1)7$	No Δ	Schmidt 1935

5 dec.	1°	$n = 1(1)24;\ 25(1)32$	No Δ	Tallqvist 1932*a*, *b*
5 dec.	1°	$n = 1(1)10$	No Δ	Hayashi 1930*b*
5; 4 dec.	1°	$n = 1(1)7;\ 8(1)10$	No Δ	Dwight 1941
4 dec.	1°	$n = 1(1)8$	No Δ	Tallqvist 1905*a*
4 dec.	1°	$n = 1(1)7$	No Δ	Fowle 1933 (Van Orstrand)
4 dec.	1°	$n = 1(1)7$	No Δ	Jahnke–Emde 1933, 1938
4 dec.	1°	$n = 1(1)7$	No Δ	Perry 1891

On errors see Farr 1900 and Tallqvist 1905*b*. Perry's table is reproduced with little or no correction in A. Gray 1893, Byerly 1893, Jahnke & Emde 1909 and Silberstein 1923.

See also Nettleton & Llewellyn 1932.

16·215. Log P_n (cos θ)

6 dec.	5°	$n = 1(1)7$	No Δ	Schmidt 1935
4 dec.	5°	$n = 1(1)4$	No Δ	Schmidt 1925

See also Nettleton & Llewellyn 1932.

16·22. P^n (cos θ)

10 dec.	5°	$n = 1(1)8$	No Δ	Schmidt 1925

16·225. Log P^n (cos θ)

This is the log G_n^0 of Adams.

8 dec.	1°	$n = 2(1)10$	No Δ	Adams 1900 (282)

16·28. Zeros of P_n (cos θ)

These may of course be easily derived from the zeros of $P_n(x)$, see Art. 16·18.

0″·000001	$n = 17$	Prey 1922 (12)
0″·0001	$n = 2(1)7$	Schmidt 1898
0″·1	$n = 3(1)7$	Seeliger 1890
1″	$n = 2(1)6$	Schmidt 1935 (19)
1′	$n = 2(1)10$	Prévost 1933 (28)

For $n = 3(1)7$ the first root and the intervals between successive roots are given to 1′ in Lodge 1904 (108). From these by addition roots correct to about 1′ (somewhat greater errors in P_7) may be deduced.

16·29. Applications of P_n (cos θ) to Potential Problems

Tallqvist 1932*c* gives tables of the potential due to:

I A uniform circular wire
II A uniform circular disc
III A circular disc carrying an electric charge in equilibrium

In each case the mass or charge is unity and the radius is unity. If r is the distance of any point P from the centre O of the circle, and θ is the inclination of OP to the axis of the circle, the tables give the potential at P in each case for $\theta = 0(1°)90°$, $r = \cdot01(\cdot01)\cdot1(\cdot1)2(1)10(10)100$. Four decimals are given, except that 5 decimals

are given in case I for $r = 10(10)100$. There are no differences in these double-entry tables.

Tallqvist 1931 contains a similar table of the magnetic potential (solid angle) for a circular current. The solid angle is given to 4 decimals, without differences, for $\theta = 0(5^\circ)85^\circ$, $r = \cdot001(\cdot001)\cdot01(\cdot01)\cdot1(\cdot1)1(\cdot2)2(1)10(10)100(100)500$. Quantities relating to the position and vertical angles of the cone which contains the solid angle are also tabulated with the same arguments.

A number of the quantities tabulated in these two papers may be expressed in two ways, either as series involving P_n or in terms of elliptic integrals (including in some cases those of the third kind).

16·295. Legendre Functions of Fractional Order

Legendre functions of half-integral order may be expressed in terms of complete elliptic integrals. If the modulus of the elliptic integrals is $k = \sin\frac{1}{2}\theta$, so that $k^2 = \frac{1}{2}(1-\cos\theta) = \frac{1}{2}(1-x)$, we have

$$P_{-\frac{1}{2}}(x) = P_{-\frac{1}{2}}(\cos\theta) = \frac{2}{\pi}K$$

$$P_{\frac{1}{2}}(x) = P_{\frac{1}{2}}(\cos\theta) = \frac{2}{\pi}(2E-K)$$

and further Legendre functions of half-integral order may be found by the usual recurrence relation. See also Fouquet and New York W.P.A. in Art. 16·8.

We know of no tables of $P_n(x)$ for general values of n, but a diagram is given in Jahnke & Emde 1933 (174), 1938 (108). Sir Geoffrey Taylor has however pointed out to us that this diagram is quite erroneous in the neighbourhood of $x = -1$.

16·3. Derivatives $P_n^{(m)} = d^m P_n/dx^m$, etc.

16·31. $P_n^{(m)}(x)$

Algebraic expressions for $P_n^{(m)}(x)$ for $n = 0(1)20$, $m = 0(1)n$ are given in Egersdörfer 1936 (18). Prévost 1933 gives them up to $n = 10$.

Exact	·01	$n = 2(1)8,\quad m = 1(1)n$	Tallqvist 1904
4 dec.	·01	$n = 2(1)8,\quad m = 1(1)n$	Tallqvist 1905*a*

For $P_n'(x)$, $x > 1$, see Bateman 1922 and Fouquet 1937 in Art. 16·8.

16·32. $G_n^m(x) = \dfrac{(n-m)!}{(2n-1)!!}P_n^{(m)}(x)$

This function is such that $(1-x^2)^{m/2}G_n^m(x)$ is the $H_n^m(x)$ of Adams, our $P^{nm}(x)$.

10 dec.	·01	$n = 1(1)10,\quad m = 1(1)n$	No Δ	Adams 1900 (270)

Algebraic expressions are given (page 265) for all functions tabulated.

An Airey manuscript evidently planned to give the functions to 10 decimals for $x = 0(\cdot05)1$, $n = 11(1)20$, $m = 1(1)n$. The work is complete for G_n^1 and G_n^2, and much of the remainder is done.

16·35. $P_n^{(m)}(\cos\theta) = \left(\dfrac{d}{d\cos\theta}\right)^m P_n(\cos\theta)$

Exact Fourier series for $P_n^{(m)}(\cos\theta)$ for $n = 0(1)20$, $m = 0(1)n$ are given in Egersdörfer 1936 (27). Prévost 1933 gives them up to $n = 10$.

About 10 fig.	$1°$	$n = 1(1)8,\ m = 1(1)n$	No Δ	Tallqvist 1906

Fourier series are given for all the above.

See also Mises 1928 (21), where manuscript tables by H. Blumer for $n = 0(1)18$, $m = 1, 2$ are mentioned, and Davis, *M.T.A.C.*, **1**, 119, 1943 (Tallqvist MS.).

16·365. Log $G_n^m(x)$

For notation see Art. 16·32.

8 dec.	$1°$	$n = 1(1)10,\ m = 1(1)n$	No Δ	Adams 1900 (284)

16·4. Associated Functions $P_{nm}(x)$, $P_n^m(x)$, etc.

On notation see Arts. 16·0 and 16·01. For P_{n0} and P_n^0, see P_n in Arts. 16·1 ff. See also the description of the projected New York W.P.A. tables in Art. 16·8.

16·41. $P_{nm}(x)$

Algebraic expressions up to $n = 10$ are given in Prévost 1933.

8–13 fig.		·001	$n = 1(1)10,\ m = 1(1)n$	δ_m^2	**Mursi 1941**
Exact About 10 f.	(m even) (m odd)	·01	$n = 1(1)8,\ m = 1(1)n$	No Δ	Tallqvist 1906
5–6 fig.		·01	$n = 1(1)8,\ m = 1(1)n$ Also $n = 9,\ m = 1$	Δ	Prévost 1933
4 dec.		·01	$n = 1(1)8,\ m = 1(1)n$	No Δ	Tallqvist 1905*a*

Doodson 1936 (342) gives $P_{nm}(0)$ to about 8 figures for $n = 0(1)12$, $m = 0(1)n$.

16·42. $P_n^m(x)$

6 dec.	·1	$n = 1(1)3,\ m = 1(1)n$	No Δ	Schmidt 1935
5 dec.	·1	$n = 4(1)6,\ m = 1(1)n$		

16·425. Log $P_n^m(x)$

6 dec.	·1	$n = 1(1)6,\ m = 1(1)n$	No Δ	Schmidt 1935

16·43. $P^{nm}(x)$

Algebraic expressions up to $n = 10$ are given in Adams 1900 (404).

16·48. Zeros of $P_{nm}(x)$, etc.

$P_{nm}(x)$, $P_n^m(x)$, etc. have identical zeros. For $m = 0$, see Art. 16·18.

8 dec. 10 dec.	$n = 3(1)7$ $n = 8(1)10$ } $m = 1$ only	Radau 1880*b* (307)
8 dec.	$n = 8, 9,\ m = 1$ only	Radau 1880*a*
7 dec.	$n = 2(1)8,\ m = 1(1)n$	Tallqvist 1905*a*
7 dec.	$n = 2(1)6,\ m = 1(1)n$	Schmidt 1935 (19)
4 dec.	$n = 2(1)8,\ m = 1(1)n$ Also $n = 9,\ m = 1$	Prévost 1933 (28)

16·49. Zeros of $P'_{nm}(x)$, etc.

These are the values of x for which $P_{nm}(x)$, $P_n^m(x)$, etc. are stationary.

7 dec.	$n = 2(1)6, \quad m = 1(1)n$	Schmidt 1935 (19)

The values for $m = 0$ are not given.

16·5. Associated Functions P_{nm} (cos θ), P_n^m (cos θ), etc.

On notation see Arts. 16·0 and 16·01. For P_{n0} and P_n^0, see P_n in Arts. 16·2 ff. See also *M.T.A.C.*, **1**, 119 (Davis) and 164 (Lowan).

16·51. P_{nm} (cos θ), including $dP_n/d\theta$

Exact Fourier series for P_{nm} (cos θ), $n = 0(1)20$, $m = 0(1)n$ are given in Egersdörfer 1936 (37). Prévost 1933 gives them up to $n = 10$, Tallqvist 1906 up to $n = 8$.

About 10 fig.	1°	$n = 1(1)8, \quad m = 1(1)n$	No Δ	Tallqvist 1906
4 dec.	1°	$n = 1(1)8, \quad m = 1(1)n$	No Δ	Tallqvist 1905*a*

The following is a table of $dP_n/d\theta$ or $-P_{n1}$ (cos θ).

4; 3 dec.	1°	$n = 1(1)6;\ 7$	No Δ	Farr 1899

Reproduced in Jahnke & Emde 1909 (88), 1933 (190), 1938 (124) and in Dwight 1941.

Airey (MS.) gives $dP_n/d\theta$ to 16 decimals for $\theta = 0(5°)90°$, $n = 1(1)20$.

See also $X_n^0 = \dfrac{1}{n}\dfrac{dP_n}{d\theta}$ in Art. 16·6.

16·52. P_n^m (cos θ)

6 dec.	5°	$n = 1(1)3, \quad m = 1(1)n$	No Δ	Schmidt 1935
5 dec.	5°	$n = 4(1)6, \quad m = 1(1)n$	No Δ	Schmidt 1935
4 dec.	5°	$n = 1(1)4, \quad m = 1(1)n$	No Δ	Schmidt 1925

16·525. Log P_n^m (cos θ)

6 dec.	5°	$n = 1(1)6, \quad m = 1(1)n$	No Δ	Schmidt 1935
4 dec.	5°	$n = 1(1)4, \quad m = 1(1)n$	No Δ	Schmidt 1925

16·53. P^{nm}(cos θ)

10 dec.	5°	$n = 1(1)7, \quad m = 1(1)n$	No Δ	Schmidt 1925

16·54. R_n^m (cos θ)

3 dec.	10°	$n = 1(1)4, \quad m = 0(1)n$	No Δ	Schmidt 1899

16·545. Log R_n^m (cos θ)

5 dec.	5°	$n = 1(1)5, \quad m = 0(1)n$	No Δ	Schmidt 1896

16·55. F_n^m (cos θ)

Doodson 1936 gives Fourier series with 7-decimal coefficients for $n = 0(1)12$, $m = 0(1)n$.

16·56. $R_{nm}(\cos\theta)$

Egersdörfer 1936 (59) gives Fourier series with 14-decimal coefficients for $n = 0(1)20$, $m = 0(1)n$.

16·58. Zeros of $P_{nm}(\cos\theta)$, etc.

$P_{nm}(\cos\theta)$, $P_n^m(\cos\theta)$, etc. have identical zeros. For $m = 0$, see Art. 16·28.

$0''\cdot0001$	$n = 2(1)7,\ m = 1(1)n$	Schmidt 1898
$1''$	$n = 2(1)6,\ m = 1(1)n$	Schmidt 1935 (19)
$1'$	$n = 2(1)8,\ m = 1(1)n$	Prévost 1933

16·59. Zeros of $P'_{nm}(\cos\theta)$, etc.

These are the values of θ for which $P_{nm}(\cos\theta)$, $P_n^m(\cos\theta)$, etc. are stationary.

$1''$	$n = 2(1)6,\ m = 1(1)n$	Schmidt 1935 (19)

The values for $m = 0$ are not given.

16·6. $X_n^m = \dfrac{1}{n}\dfrac{dP_n^m}{d\theta}$ and $Y_n^m = \dfrac{m}{n}\dfrac{P_n^m}{\sin\theta}$, etc.

These are connected with $\partial/\partial\theta$ and $\partial/\sin\theta\,\partial\phi$ of $P_n^m(\cos\theta)\cos(m\phi+\epsilon)$. See also $dP_n/d\theta$ in Art. 16·51.

Schmidt 1935 gives X_n^m, Y_n^m, $\log X_n^m$, $\log Y_n^m$ to 5–6 decimals for $\theta = 0(5^\circ)90^\circ$ and also for $x = \cos\theta = 0(\cdot1)1$, all for $n = 1(1)6$, $m = 0(1)n$; also X_7^0 and $\log X_7^0$. Four-decimal values for $\theta = 0(5^\circ)90^\circ$, $n = 1(1)4$, $m = 0(1)n$ are given in Schmidt 1925.

Schmidt 1925 gives $X^{nm} = \dfrac{1}{n}\dfrac{dP^{nm}}{d\theta}$ and $Y^{nm} = \dfrac{m}{n}\dfrac{P^{nm}}{\sin\theta}$ to 8 decimals for $\theta = 0(5^\circ)90^\circ$, $n = 1(1)7$, $m = 0(1)n$; also $X^{8,0}$.

Schmidt 1899 gives $\dfrac{1}{n}\dfrac{dR_n^m}{d\theta}$ and $\dfrac{m}{n}\dfrac{R_n^m}{\sin\theta}$ to 3 decimals for $n = 1(1)4$, $m = 0(1)n$, $\theta = 0(10^\circ)180^\circ$.

16·7. Functions of the Second Kind (Q_n, etc.)

When $|x| < 1$ the second solution, $Q_n(x)$, of Legendre's equation of order n is usually defined so that we have

$$Q_n(x) = P_n(x)\tanh^{-1}x - Z_n(x)$$

where $Z_n(x)$ is a certain polynomial of degree $n-1$. When values of x greater than unity, or complex, are being used, it is usual to take

$$Q_n(x) = P_n(x)\coth^{-1}x - Z_n(x)$$

$Q_0(x)$ is $\tanh^{-1}x$ in the first case and $\coth^{-1}x$ in the second. For tables of these, see Section 11.

Algebraic expressions for the polynomials $P_n(x)$ and $Z_n(x)$, $n = 0(1)10$, are given in Cayley 1887. The coefficients in $P_n(x)$ and $Z_n(x)$, $n = 0(1)10$, are given in decimal form to 5 decimals in Rosen 1931*a* (Tables 5 and 6). Tallqvist 1905*a* tabulates $2Z_n(x)$ to 4 decimals without differences for $n = 0(1)8$, $x = 0(\cdot01)1$.

Vandrey 1940 tabulates $Q_n(x)$ to 5 decimals without differences for $n = 0(1)7$, $x = 0(\cdot01)1$. Tallqvist 1905*a* tabulates $2Q_n(x)$ to 4 decimals without differences for $n = 0(1)8$, $x = 0(\cdot01)1$. See also Webster & Fisher 1919, Bateman 1922 and Fouquet 1937 in Art. 16·8.

Associated functions $Q_n^m(x)$ may be defined as

$$\text{const.} \times (1 - x^2)^{m/2} \frac{d^m Q_n(x)}{dx^m}$$

See the projected New York W.P.A. tables mentioned in Art. 16·8.

Explicit expressions for

$$(-)^m \frac{(n-m)!}{(n+m)!} (x^2 - 1)^{m/2} \frac{d^m Q_n(x)}{dx^m}$$

for $n = 1(1)5$, $m = 0(1)n$ are given in Bryan 1888.

16·8. Functions with Argument Greater than Unity, or Complex

When $x > 1$, the associated functions are usually defined as $(x^2 - 1)^{m/2} \dfrac{d^m P_n}{dx^m}$ and $(x^2 - 1)^{m/2} \dfrac{d^m Q_n}{dx^m}$, or multiples thereof. See Art. 16·7.

Bateman 1922 gives $P_n(x)$, $P_n'(x)$, $Q_n(x)$, $Q_n'(x)$ to 10 decimals for $x = 1{\cdot}1$, $n = 0(1)20$ and for $x = 1{\cdot}2, 2, 3$, $n = 0(1)10$.

Webster & Fisher 1919 gives to 4 decimals without differences $Q_1(x)$ for $x = 1{\cdot}1(\cdot001)1{\cdot}3(\cdot01)3{\cdot}2(\cdot1)6(\cdot5)10$ and $Q_1'(x)$ for $x = 1{\cdot}1(\cdot001)1{\cdot}62(\cdot01)3{\cdot}2(\cdot1)6(\cdot5)10$.

Fouquet 1937 tabulates $P_n(\cosh x)$, $P_n'(\cosh x)$, $Q_n(\cosh x)$, $Q_n'(\cosh x)$ to 4 decimals without differences for $n = -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$, $x = 0(\cdot1)3$. The functions can be expressed in terms of complete elliptic integrals.

The following tables will deal mainly, though not entirely, with arguments greater than unity.

B.A. Mathematical Tables, Part-Volume A (in press), will give $P_n(x)$ to 7–8 figures with δ_m^{2n} for $n = 2(1)12$, $x = 0(\cdot01)6$ and for $n = 2(1)6$, $x = 6(\cdot1)11$.

New York W.P.A. reports in May 1942 among tables for which manuscripts are completed: "Table of the Associated Legendre Functions $P_n^m(x)$ and $Q_n^m(x)$, for arguments x and ix." $n = 1(1)10$, $m = 0(1)4$, $x = 0(\cdot1)10$, 6 figures. "Also corresponding values for half-integral values of n, and values of the functions for arguments in degrees." We do not know which particular associated functions are denoted by the New York P_n^m and Q_n^m. Further details are given by A. N. Lowan, *M.T.A.C.*, **1**, 164, 1944. See also *M.T.A.C.*, **1**, 127, 1943 for details of manuscript tables of $P_{nm}(ix)$ and $Q_{nm}(ix)$ by J. N. Sarmousakis.

16·9. Integrals involving Legendre Functions

We consider here divers integrals involving the various kinds of Legendre functions. See also Art. 16·18. Some rather complicated integrals are tabulated in Proudman 1936. Kotani, etc. 1938 tabulates some complicated functions of 4 and 5 variables in addition to some simpler expressions which we have indexed in detail. Errata are given in Kotani, etc. 1940.

16·91. $\int_0^x P_{nm}(x)\,dx$

This includes $\int_0^x P_n(x)\,dx$ (case $m = 0$). Prévost 1933 gives these integrals to about 6 figures, but not more than 5 decimals, with differences, for $x = 0(\cdot 01)1$, $n = 0(1)8$, $m = 0(1)n$, also $(n, m) = (9, 0)$, $(10, 0)$ and $(9, 1)$.

Algebraic expressions for the integrals when $n = 0(1)10$, $m = 0(1)n$ are also given, as well as the equivalent Fourier series in θ. The complete integrals ($x = 1$) are given for $n = 0(1)10$, $m = 0(1)n$. See also Davis, *M.T.A.C.*, **1**, 119, 1943.

16·92. $\int_{-1}^{+1} x^r P_n(x)\,dx$

6 fig. $n = 0(1)10$, $r = n(2)9$ or 10 Rosen 1931*a* (T. 4)

The integral vanishes when n and r are of different parity.

16·93. $\frac{1}{4}\int_{-1}^{+1} R_n^m(x)\cos p\theta\,dx$ and $\frac{1}{2\epsilon_m}\int_{-1}^{+1} R_n^m(x)\sin p\theta\,dx$

Schuster 1902 tabulates the above integrals, in which $x = \cos\theta$, naturally and logarithmically to 6 decimals for values of n, m and p up to 12. It also gives exactly the values of the integrals multiplied by certain functions of n and m, the results being then all rational.

16·96. $\int_1^\infty e^{-at} t^m Q_n(t)\,dt$

Here $Q_0(t) = \coth^{-1} t$ and $Q_1(t) = t\coth^{-1} t - 1$.

For the case $n = 0$ a table of the integral to 6 figures is given in Rosen 1931*b* (Table 2). The arguments are

$$m = 0(1)10, \quad a = \cdot 5(\cdot 5)5(1)10(2)14$$
$$m = 11(1)20, \quad a = 1(1)10(2)14$$

Tables which relate rather to the work of calculating the integral in the case $n = 0$ are given in Zener & Guillemin 1929 (Table 1) and Rosen 1931*a* (Table 3).

Kotani, etc. 1938 gives the function to 9 figures without differences in the case $n = 0$ for

$$m = 0(1)8, \quad a = \cdot 5,\ 1(\cdot 25)5\cdot 5,\ 6(\cdot 25)6\cdot 5(\cdot 5)11(\text{var.})24$$
$$m = 9(1)15, \quad a = 1,\ 2(\cdot 5)11(\text{var.})24$$

Errata are given in Kotani, etc. 1940.

For the case $n = 1$ a table to 3–5 figures is given in Bartlett 1931 (Table 2). The arguments are

$$m = 0(1)5, \quad a = 1\cdot 5(\cdot 5)5(1)10$$
$$m = 6(1)11, \quad a = 3(1)10$$

Kotani, etc. 1938 gives the function to 9 figures without differences in the case $n = 1$ for

$$m = 0(1)7, \quad a = \cdot 5,\ 1(\cdot 25)5\cdot 5,\ 6(\cdot 25)6\cdot 5(\cdot 5)11(\text{var.})24$$
$$m = 8(1)15, \quad a = 1,\ 2(\cdot 5)11(\text{var.})24$$

Errata are given in Kotani, etc. 1940.

SECTION 17

BESSEL FUNCTIONS OF REAL ARGUMENT

17·0†. Introduction

The notation adopted in this *Index* for Bessel Functions is that used in B.A. **6**, 1937, in Watson 1922, and in Gray, Mathews & MacRobert 1922. The two latter are standard textbooks on Bessel Functions. A discussion on, and table of, notations is given in Jahnke & Emde 1909 (172) and in B.A. **6**, 1937 (xix). An extensive list of formulæ is given in McLachlan 1934 (157). Other lists worthy of special mention are given in B.A. **6**, 1937, Dwight 1934 (158), Adams & Hippisley 1922 (196), Jahnke & Emde 1909 (90), 1933 (194) and 1938 (128).

Certain Bessel Function expansions (in particular the Debye and associated expansions) have coefficients that are somewhat difficult to determine. Most of these (as well as the simpler expansions) may be found in, e.g., Watson 1922; extensions are given in Bickley 1943, which also gives further similar expansions for $J_n'(x)$ and $Y_n'(x)$. Airey 1914*b* gives a useful modification of the asymptotic expansions for functions of order zero and unity, together with a short table illustrating its use.

Bessel Functions of *argument* that is not a linear function of the tabular argument are considered as "Miscellaneous Bessel Functions" in Section 20, but *multiples* of Bessel Functions by simple expressions will be found in the same section as the basic Bessel Function concerned.

Mises 1928 gives a good list of references to tables of Bessel Functions, with considerable detail; he is our only authority in a few cases.

This section is subdivided as follows:

17·1. Bessel Functions of the First Kind: Integral Order

Common early notations for $J_n(x)$ are $I_x^{(n)}$ and $I_n(x)$, which must not be confused with the Modified Bessel Functions of Section 18.

17·11†. $J_0(x)$ and $J_1(x) = -J_0'(x)$: Major Tables

21 dec.	0(·1)6	No Δ	**Aldis 1900** (40)
18 dec.	0(·0001)·0055	No Δ	Hayashi 1933 (34)
18 dec.	0(·001)·11	No Δ	Hayashi 1930*a* (2)
16; 12 dec.	·12(·01)·5; ·5(·01)25·1	No Δ	Hayashi 1930*a* (2)
15 dec.	15·5(·01)25		H. T. Davis (MS.)

Comrie has a copy which he has checked and corrected.

† In Section 17, items added in proof are on pages 269–270.

12 dec.	0(·001)16(·01)25		Δ^2	Comrie (MS.)
12 dec.	0(·01)15·5		No Δ	**Meissel 1889** (4) Gray & Mathews 1895 (247) **G., M. & MacR. 1922** (267)

The two latter tables were reproduced from the former, with $J_0(1·71)$ corrected to begin 0·3922. For further errors (5 in the 1895 edition, 3 in the 1922 edition) see B.A. 1926 (297, Airey). B.A. 6, 1937 (xii) gives an analysis of these errors.

12 dec.	25(·01)25·51		No Δ	Hayashi 1933 (56)
10 dec.	0(·001)16(·01)25		δ^2	**B.A. 6, 1937** (2)
10 dec.	0(·01)10		No Δ	**New York W.P.A. 1943*a*** (2)
10 dec.	0(·01)3·2		Δ^2	Bessel 1826 (46)
7 dec.	0(·02)16		No Δ	Watson 1922 (666)
6 dec.	0(·1)20		v^3	Hansen 1843 (159) Schlömilch 1857 (158) Lommel 1868 (127)
6 dec.	0(·1)15		No Δ	Schorr 1916 (A 46)
6 dec.	20(·1)31(·2)41	J_1 only	No Δ	Steiner 1894 (372)

Errors for $x < 25·5$; $J_1(20·1) = +0·082801$, $J_1(23) = -0·039519$.

6; 4 dec.	0(·01)4; 4(·1)15		No Δ	Fowle 1933 (68)

Also in the 1920 edition, page 66; not in earlier editions.

5 dec.	0(·01)16		No Δ	Hayashi 1930*b* (82)
5 dec.	0(·01)15·09		No Δ	Witkowski 1904 (84, 87)
5 dec.	0(·1)15(1)20		No Δ	Dale 1903 (80)

Error; $J_0(5·9) = +0·12203$.

4 dec.	0(·01)15·5		No Δ; IΔ	Jahnke–Emde 1909 (111); 1933 (228), 1938 (156)
4 dec.	0(·1)15·9		No Δ	McLachlan 1934 (173)
4 dec.	0(·1)15		No Δ	Byerly 1893 (287), 1906 (62)
4 dec.	0(·1)13·4		No Δ	Rayleigh 1877 (265), 1894 (321)
4 dec.	0(·2)10	J_0 only	No Δ	Airy 1841 (7)

Errors at 7·4 and 8·0.

4 dec.	0(·1)·2(·2)8		No Δ	Morse 1936 (333)
4 dec.	0(·2)8	J_0 only	No Δ	Hayashi 1930*b* (106)

From Dinnik 1922.

4 dec.	25·3(var., usually ·3)39·9		No Δ	Clapp 1937 (76)

17·12. $J_0(x)$ and $J_1(x) = -J_0'(x)$: Minor Tables

Most of these tables form parts of tables of $J_n(x)$; see Art. 17·131. No differences are given.

52 dec.	50	Miller (MS.)
35–47 dec. 23–97 dec.	0(·01)·05, ·1(·1)·5 1, 2, 10(10)50 and 100	Hayashi 1930*a* (52)

Some incorrect figures are also given.

18 dec.	1(1)24	Gray & Mathews 1895 (266, Meissel) G., M. & MacR. 1922 (286)

17–20 dec.	1(1)5	J_1 only	Hayashi 1933 (36)
15 dec.	15·5(1)24·5 and 25	J_0	Hayashi 1930*a* (28)
15 dec.	·1(·1)·5, 15·5(·5)25	J_1	Hayashi 1930*a* (28)
14 dec.	9, 10		Airey 1918 (242)
10 dec.	6·5(·5)16		B.A. 1915 (30, Airey)
6 dec.	0(·2)6		B.A. 1915 (29, Airey)
6 dec.	0(1)12		Lommel 1885 (315)
6 dec.	0(1)10, 16 and 20		Meissel 1891 (153)
4 fig.	1(1)29		Jahnke–Emde 1938 (171)
4 fig.	1(1)24		Jahnke–Emde 1909 (149), 1933 (242)
4 dec.	·1, ·4(·1)·6, ·8(·1)1·2, 1·5(·5)3(1)6		Nicholson 1912 (113)

17·121†. $J_0(\pi x)$

6 dec.	1(1)50	Also log $J_0(\pi x)$	No Δ	Nagaoka 1891 (315)
5 dec.	1(1)50		No Δ	Hayashi 1930*b* (99)
3 dec.	$180x$ = 0(10)600		No Δ	Conrady 1919 (577)
3 dec.	$180x$ = 600(10)1040		No Δ	Buxton 1921 (558)

17·131†. $J_n(x)$: General Tables

No special provision for interpolation is made with these tables, except where indicated.

	n	*x*		
52 fig.	0(1)100	50		Miller (MS.)
47–20 dec.	0(1)$N-1$	0(·01)·05, ·1(·1)·5		Hayashi 1930*a* (52)
	$J_N(x) < \frac{1}{2}.10^{-20}$ for each x; $8 \leqslant N \leqslant 15$.			
98–46 dec.	0(1)32	1		Hayashi 1930*a* (52)
97–48 dec.	0(1)40	2		Hayashi 1930*a* (52)
	Some incorrect figures are also given.			
65–30 dec.	0(1)50	10		Hayashi 1930*a* (52)
45–20 dec.	0(1)56	20		Hayashi 1930*a* (52)
36–18 dec.	0(1)69; 82	30; 40		Hayashi 1930*a* (52)
31–15 dec.	0(1)88	50		Hayashi 1930*a* (52)
24–10 dec.	0(1)135	100		Hayashi 1930*a* (52)
18; 20 dec.	0(1)2; 3(1)6	0(·0001)·0055		Hayashi 1933 (34)
18 dec.	0(1)$N-1$	1(1)24		Gray & Mathews 1895 (266); **G., M. & MacR. 1922** (286)
	From MS. table by Meissel. $J_N(x) < \frac{1}{2}.10^{-18}$ for each x; $17 \leqslant N \leqslant 61$. For 4 errors, see B.A. 1926 (297, Airey) or B.A. 6, 1937 (xii). Also $J_{26}(6) = +0{\cdot}0^{14}4\,507$ (Miller).			
12 dec.	2(1)20	0(·1)25·5	Δ^n	B.A. (MS., Comrie, etc.)
10 dec.	0(1)20	0(·1)25		B.A. (MS., Comrie, etc.)
10 dec.	0(1)13	6·5(·5)16		B.A. 1915 (30, Airey)
8 dec.	2(1)20	X_0(·01)X_1(·1)25	δ^2_m	B.A. (MS., Comrie, etc.)

n	2–6	7, 8	9	10	11	12	13	14	15	16	17–20
X_0	0	0·5	1	1·5	2	2·5	3	3·5	4	4·5	—
X_1	10	10	10	10	10	10	6·5	6	5	5	5

For $x < X_0$, $J_n(x)$ consists of at most a unit in the 8th decimal.

	n	x	
8 dec.	967, 968, 981(1)1000	1000	Meissel 1891 (155)
7 dec.	2(1)5	0(·1)5	Watson 1922 (730)
7 dec.	N, $N+1$	$X(\cdot 2)X+2$	v^3 { Hansen 1843 (164); Schlömilch 1857 (162); Lommel 1868 (132) }

N	5	8	11	13	15	18	20	23	25	27
X	0	2	4	6	8	10	12	14	16	18

6 dec.	0(1)13	0(·2)6	B.A. 1915 (29, Airey)
6 dec.	0(1)20	0(1)12	Lommel 1885 (315)

Some values are in error, but only $J_{15}(6)$ and $J_{15}(7)$ by more than 4 final units; these should be $0{\cdot}0^5 6$ and $0{\cdot}0^4 51$.

6 dec.	0(1)20	0(1)12	Watson 1922 (731)

From Lommel, corrected.

6 dec.	$0(1)N-1$	0(1)10, 16 and 20	Meissel 1891 (153)

$J_N(x) < \frac{1}{2}.10^{-6}$ for each x; $8 \leqslant N \leqslant 36$.

6 dec.	100(1)109	109	Airey 1916*a* (525)
5 dec.	0(1)12	0(1)20	Dale 1903 (81)
5 dec.	2	0(·01)10	Tweritin (MS.)
4 fig.	$0(1)N-1$	1(1)24	Jahnke–Emde 1909 (149), 1933 (242), 1938 (171)

$J_N(x) < \frac{1}{2}.10^{-18}$ for each x; $17 \leqslant N \leqslant 61$. At most 18 decimals are given.

4 fig.	$0(1)N-1$	25(1)29	Jahnke–Emde 1938 (179, M. J. di Toro)

$J_N(x) < \frac{1}{2}.10^{-6}$ for each x; $42 \leqslant N \leqslant 45$. At most 6 decimals are given.

4 dec.	{ 0(1)3; 0(1)8	·1, ·4(·1)·6, ·8(·1)1; 1·1, 1·2, 1·5(·5)3(1)6 }	Nicholson 1912 (113)
4 dec.	2(1)4	0(·1)4·9	McLachlan 1934 (175)
4 dec.	2	0(·1)·2(·2)8	Morse 1936 (333)

17·132. $J_n(x)$ for x and n nearly equal

Many values can be extracted from the tables listed in the previous Art. 17·131. No differences are given.

7 dec.	$n = x = 1(1)50$		Watson 1922 (746)
6 dec.	$n = x,\ x-1$	$x = 1(1)50(5)100(10)200(20)$ $400(50)1000(\text{var.})10^6$	B.A. 1916 (93, Airey)
6 dec.	$n = x,\ x-1$	$x = 0(1)10$	B.A. 1923 (290, Airey)
6 dec.	$n = x,\ x\pm 1$	$x = 1(1)20$	Airey 1935*b* (233)

Airey 1916*a* (523) gives $J_n(n)$ for $n = 6$, 48 and 750 to 6, 9 and 14 decimals respectively.

See also Art. 17·556.

17·133. $J_{2n}(n)$

8; 10; 12 dec.	10(1)14; 15(1)19; 20 and 21	Meissel 1892 (283)

17·15†. Derivatives of $J_n(x)$

$(16-p)$ dec.	$J_0^{(p)}(x)$	$p = 1(1)10$	$x = \cdot1(\cdot1)\cdot5$	Hayashi 1930*a* (28)
		$p = 1(1)9$	$x = 16(\cdot5)25$	
20; 17 dec.	$2J_n'(x)$	$n = 1(1)27$	$x = 1, 2;\ 3(1)5$	Hayashi 1933 (36)
3 dec.	$2J_n'(x)$	$n = 1(1)3$	$x = \cdot5, \cdot6, \cdot8, \cdot9, 1$	Nicholson 1912 (122)
4− dec.	$2J_n'(x)$	$n = 1(1)6$	$x = 1\cdot2, 1\cdot5(\cdot5)3(1)5$	
7 dec.	$J_n'(n)$	$n = 1(1)50$		Watson 1922 (746)

See also Art. 17·556.

17·16. Differences of $J_n(x)$

Hayashi 1933 (30) gives tables of $\Delta^p J_n(x)$. Each value of $p = 1(1)4$ occupies a separate page; $n = 0(1)6$, $x = 0(\cdot0001)\cdot0055$, and values are to 20 decimals, except for $n = 0$, $p = 1$, 2 and 3, which are to 19 decimals.

17·2. Bessel Functions: Unrestricted Order

Since functions of the second kind are expressible, when n is not an integer, as linear combinations of functions of the first kind, both are included in this sub-section.

17·21. $J_n(x)$ for $n = m+\frac{1}{2}$, m Integral

These functions are elementary, involving circular functions; see Gray & Mathews 1895 (42), Gray, Mathews & MacRobert 1922 (17), Dale 1903 (83), Jahnke & Emde 1909 (91) or Hayashi 1930*a* (120), where explicit expressions are given for $\pm 2n = 1(2)11$. See also Arts. 17·42 and 17·43. No special provision is made for interpolation. Note that $Y_{m-\frac{1}{2}}(x) = (-1)^m J_{-m+\frac{1}{2}}(x)$.

	n	x	
12 dec.	$\mu+\frac{1}{2}(1)M+\frac{1}{2}$	$1(1)20$	B.A. 1925 (222, Airey)
$\lvert J_{\mu-\frac{1}{2}}(x)\rvert > 1$, $\lvert J_{M+1\frac{1}{2}}(x)\rvert < \frac{1}{2}.10^{-12}$ for each x; tabulated values all lie between these limits. $\mu = -1$ to -23, $M = 11$ to 44. Airey (MS.) has 1 to 5 extra decimals.			
12; 9 d.	$\pm\frac{1}{2}, \pm\frac{3}{2}$	$0(\cdot01)1\cdot99;\ 2(\cdot01)9\cdot89$	Hayashi 1930*a* (30)
12 dec.	$\pm\frac{1}{2}, \pm\frac{3}{2}$	$9\cdot9(\cdot01)10(\cdot1)20(1)100$	
8 dec.	$\pm\frac{1}{2}$	$1\cdot5(\cdot02)20$	Airey (MS.)
6 dec.	$-6\frac{1}{2}(1)+6\frac{1}{2}$	$1(1)50$	Lommel 1886 (644)
	$7\frac{1}{2}(1)34\frac{1}{2}$	$1(1)20$	
6 dec.	$-6\frac{1}{2}(1)+6\frac{1}{2}$	$1(1)50$	Watson 1922 (740)
	$7\frac{1}{2}(1)18\frac{1}{2}$	$1(1)20$	
6 dec.	$\pm\frac{1}{2}$	$0(\cdot02)20$	B.A. 1925 (234, Airey)
5 dec.	$\pm\frac{1}{2}, \pm\frac{3}{2}$	$0(\cdot01)1(\cdot1)10(1)100$	Hayashi 1930*b* (102)
4 fig.	$-6\frac{1}{2}(1)+6\frac{1}{2}$	$1(1)50$	Jahnke–Emde 1909 (108), 1933 (224), 1938 (154)
4 f. or 6 d.	$\frac{1}{2}(1)6\frac{1}{2}$ to $18\frac{1}{2}$	$1(1)24$	Jahnke–Emde 1933 (242), 1938 (171)
4 dec.	$\pm\frac{1}{2}$	$0(\cdot1)15$	Dinnik 1933 (7)
4 dec.	$-2\frac{1}{2}(1)+2\frac{1}{2}$	$0(\cdot2)8$	Dinnik 1913*a* (239)
Errors exceeding two final units in $J_{-3/2}(1\cdot2)$, $J_{-3/2}(3\cdot6)$.			
6 dec.	$\pm x, \pm(x-1)$	$\frac{1}{2}(1)9\frac{1}{2}$	B.A. 1923 (290, Airey)
6 dec.	$x\pm\frac{1}{2}$	$1(1)20$	Airey 1935*b* (233)

17·211. $\dfrac{1}{2.n!}\dfrac{d^{n-1}}{dx^{n-1}}J_{\mp\frac{1}{3}}(x)$

6 dec.	$n = 1(1)6$	$x = 1(1)50$	Lommel 1886 (649, 650)

17·22. Bessel Functions of Order $n = m \pm \frac{1}{3}$, m Integral

See also Art. 20·2, which deals with the Airy Integral.

17·221†. $J_n(x)$ **for** $n = m \pm \frac{1}{3}$, m **Integral**

	n	x		
13 dec.	$\pm\frac{1}{3}, \pm\frac{2}{3}$	10(1)25	D^n	New York W.P.A. (MS.)
10 dec.		0(·01)25	δ^{2n}_m	
7 dec.	$\frac{1}{3}$	0(·02)16	No Δ	Watson 1922 (714)
	$-\frac{1}{3}$	2(2)16		
5 dec.	$\pm\frac{2}{3}$	0(·1)10	No Δ	Karas 1936 (249)
5 dec.	$\pm\frac{1}{3}$	0(·1)10	No Δ	Hayashi 1930*b* (100)
4 dec.	$\pm\frac{1}{3}, \pm\frac{2}{3}$	0(·1)15	No Δ	Dinnik 1933 (10)
4 dec.	$\pm\frac{1}{3}$	0(·05)2(·2)8	No Δ	Watson 1918*a* (204)

See also Grishkov 1925 for $n = \pm\frac{1}{3}, \pm\frac{2}{3}$.

17·222. $n!\,J_n(x)$

4 dec.	$n = \pm\frac{1}{3}, \pm\frac{2}{3}$	$x = 0(·2)8$	Dinnik 1911 (338)

17·231†. $Y_{1/3}(x) = \dfrac{J_{1/3}(x) - 2J_{-1/3}(x)}{\sqrt{3}}$ **and** $Y_{-1/3}(x) = \dfrac{2J_{1/3}(x) - J_{-1/3}(x)}{\sqrt{3}}$

7 dec.	$Y_{1/3}(x)$	0(·02)16	No Δ	Watson 1922 (714)
	$Y_{-1/3}(x)$	2(2)16		
5 dec.		0(·1)10	No Δ	Hayashi 1930*b* (100)

17·232. $Y_{2/3}(x) = -\dfrac{J_{2/3}(x) + 2J_{-2/3}(x)}{\sqrt{3}}$ **and** $Y_{-2/3}(x) = \dfrac{2J_{2/3}(x) + J_{-2/3}(x)}{\sqrt{3}}$

5 dec.	0(·1)10	No Δ	Karas 1936 (249)

17·241. $J_n(x)$ **for** $n = \pm\frac{1}{4}$ **and** $\pm\frac{3}{4}$

13 dec.	10(1)25	D^n	New York W.P.A. (MS.)
10 dec.	0(·01)25	δ^{2n}_m	
10 dec.	10(·1)20·3	δ^2	B.A. (MS., Comrie, etc.)
7 dec.	5(·1)20		
4 dec.	0(·1)15	No Δ	Dinnik 1933 (16)
4 dec.	0(·1)8	No Δ	Karas 1936 (250)
4 dec.	0(·2)8	No Δ	Dinnik 1913*b* (325)

All the values for $x = 4·8$ and $5·4$ are erroneous.

4 dec.	0(·2)8	No Δ	Jahnke–Emde 1933 (223), 1938 (164)

The 1933 table is from Dinnik 1913*b*; the 1938 table has been corrected to agree with Karas 1936.

Dinnik 1916 also gives 4-figure tables. See also Art. 17·45.

17·242. $J_n(n)$ and $J_{n-1}(n)$

6 dec.	$n = 0(\frac{1}{4})10$	No Δ	B.A. 1923 (290, Airey)

17·25. $Y_n(x) = J_n(x) \cot n\pi - J_{-n}(x) \operatorname{cosec} n\pi$ and $Y_{-n}(x) = J_n(x) \operatorname{cosec} n\pi - J_{-n}(x) \cot n\pi$

4 dec.	$n = \pm\frac{1}{4}, \pm\frac{3}{4}$	$x = 0(\cdot 1)8$	No Δ	Karas 1936 (250)
6 dec.	$n = x, x-1$	$x = \frac{1}{4}(\frac{1}{2})9\frac{3}{4}$	No Δ	B.A. 1923 (290, Airey)

Part of a table of $Y_n(n)$ and $Y_{n-1}(n)$ for $n = 0(\frac{1}{4})10$.

17·26. $J_n(x)$ for $n = \pm\frac{1}{8}$ and $\pm\frac{5}{8}$

4 dec.	$0(\cdot 1)15$	No Δ	Dinnik 1933 (21)

17·27. $J_n(x)$ of General Order n

6 dec.	$n-x = -1(\cdot 1)+1$	$x = 1(1)20$	No Δ	Airey 1935*b* (233)

17·28†. Derivatives of $J_n(x)$

See *Items added in Proof*, page 270.

17·29. $\frac{\partial}{\partial n} J_n(x)$ and $\frac{\partial^2}{\partial n^2} J_n(x)$

6 dec.	$\frac{\partial}{\partial n} J_n(x)$	$n = \pm\frac{1}{2}, \pm\frac{3}{2}$	$x = 0(\cdot 1)20$	B.A. 1928 (320, Airey)
5 dec.	$\frac{\partial}{\partial n} J_n(x)$	$n-x = -1(\cdot 1)+1$	$x = 1(1)20$	Airey 1935*c* (238)
4 dec.	$\frac{\partial^2}{\partial n^2} J_n(x)$			

Neither table gives differences.

17·3. Bessel Functions of the Second Kind: Integral Order

Several notations and several different functions have been used for a second solution of Bessel's differential equation; we shall take, with Watson 1922 and B.A. **6**, 1937, Weber's solution $Y_n(x)$ as standard. For useful sets of formulæ see B.A. **6**, 1937 (174, 202).

For purposes of identification we note the following values:

x	$Y_0(x)$	$Y_1(x)$	$Y_2(x)$	$G_0(x)$	$G_1(x)$	$Y^{(0)}(x)$	$Y^{(1)}(x)$
0·1	−1·534	−6·46	−127·6	+2·410	+10·15	−2·294	−10·14
1	+0·088	−0·781	−1·651	−0·138	+1·227	+0·227	−1·176
10	+0·056	+0·249	−0·006	−0·087	−0·391	+0·059	+0·396

$G_n(x)$ and $Y^{(n)}(x)$ are defined in Arts. 17·341 and 17·35 respectively.

When n is not an integer $Y_n(x)$ can be expressed as a linear combination of $J_n(x)$ and $J_{-n}(x)$; tables of all these functions are listed in Sub-section 17·2. The following articles are concerned only with integral n.

$Y_n(x)$ must not be confused with Neumann's solution, $Y^{(n)}(x)$ in Watson's notation, but also frequently denoted by $Y_n(x)$.

17·31†. Weber's $Y_0(x)$ and $Y_1(x) = -Y_0'(x)$

These functions are sometimes denoted by $N_0(x)$ and $N_1(x)$.

19 dec.	0(·1)6	From Aldis's $G_0(x)$, $G_1(x)$		Miller (MS.)
18 dec.	0(1)26			Miller (MS.)
15+ dec.	0(·1)25·5			B.A. (MS., Miller, Gwyther)
14–17 dec.	16(1)25		No Δ	Hayashi 1933 (58)
10 dec.	16(·01)25·51		No Δ	Hayashi 1933 (50)
10 dec.	0(·01)10			New York W.P.A. (in progress)
8 dec.	0(·01)25		δ^2_m	**B.A. 6, 1937** (176)
7 dec.	0(·02)16		No Δ	Watson 1922 (666)
5 dec.	0(·01)16		No Δ	Hayashi 1930*b* (82)
4 dec.	0(·01)15·99		IΔ	Jahnke–Emde 1933 (260), 1938 (190)
Fewer decimals for $x < \cdot 8$ or $\cdot 9$.				
4 dec.	0(·02)16			Glazenap 1932
4 dec.	0(·1)15·9		No Δ	McLachlan 1934 (174)
4 dec.	0(·1)·2(·2)8		No Δ	Morse 1936 (333)
4 dec.	0(·1)10·2		No Δ	Jahnke–Emde 1909 (129)

Errata: $-N_1(0{\cdot}1) = 6{\cdot}4590$, $-N_1(5{\cdot}2) = -0{\cdot}0792$, $N_0(9{\cdot}9) = 0{\cdot}0804$, $-N_1(10{\cdot}2) = -0{\cdot}2502$; there are also others up to 3 units of the final digit.

17·321†. $Y_n(x)$: General Tables

No differences are given in these tables.

	n	x	
18± fig.	0(1)20+	0(1)40	B.A. (MS., Miller, etc.)
13 to 17 dec.	0(1)15	16(1)25	Hayashi 1933 (58)
15 to −2 dec.	16(1)30	16(1)25	Hayashi 1933 (58)
13+ fig.	0(1)21	0(·1)25·5	B.A. (MS., Miller, etc.)

These tables are the basic material from which 10-figure tables (without differences) have been prepared for publication. Tables to 8 figures, at interval 0·01 when $x \leqslant 10$ and $n \leqslant 11$, are also in preparation for use when interpolation with modified second differences is possible, likewise based on this material. In cases of failure $y_n(x) = x^n Y_n(x)$ is to be used; see Art. 17·416.

7+ fig. or 7 dec.	0(1)10	0(·1)5	Watson 1922 (732)
7 dec.	0(1)13	6(1)12	Watson 1922 (732)
4 dec. or 5 fig.	2	0(·1)·2(·2)8	Morse 1936 (333)

17·322. $Y_n(x)$ for x and n nearly equal

7 dec.	$n = x = 1(1)50$		No Δ	Watson 1922 (747)
6 dec.	$n = x,\ x-1$	$x = 0(1)10$	No Δ	B.A. 1923 (290, Airey)

See also Art. 17·556.

17·33†. Heine's $G_0(x) = -\frac{1}{2}\pi Y_0(x)$ and $G_1(x) = -\frac{1}{2}\pi Y_1(x)$

21 dec.	0(·1)6	No Δ	Aldis 1900 (41)
14 dec.	9 and 10		Airey 1918 (242)
10 dec.	6·5(·5)15·5	No Δ	B.A. 1915 (32, Airey)
7 dec.	0(·01)16	No Δ	B.A. 1913 (116, Airey)

7 dec.	0(·1)16	No Δ	B.A. 1911 (75, Airey) Airey 1911e (660)
5 dec.	0(·1)6(·5)16	No Δ	B.A. 1914 (83, Airey)
4 dec.	0(·01)16	No Δ	Airey 1914a (32)
4 dec.	0(·01)1(·1)10·3 or 10·2	No Δ	Smith 1898 (122) Jahnke–Emde 1909 (127)

Jahnke & Emde 1909 (126) also reproduces Aldis's values, cut down to 4 decimals.

4 fig. or dec.	$2G_0(x)$ and $2G_1(x)$ $x = \cdot1,\ \cdot4(\cdot1)\cdot6,\ \cdot8(\cdot1)1\cdot2,$ $1\cdot5(\cdot5)3(1)6$	No Δ	Nicholson 1912 (113)

17·341. $G_n(x) = -\frac{1}{2}\pi Y_n(x)$: General Tables

5 dec. $n = 0(1)13$ $x = 0(\cdot1)6(\cdot5)16$ No Δ B.A. 1914 (83, Airey)

Values of $G_n(x)$ exceeding 10·2 are omitted; the least value of x for which $G_n(x)$ is tabulated is given below:

n	0, 1	2	3	4	5	6	7	8	9	10	11	12	13
x	0·1	0·5	1·0	1·6	2·3	3·0	3·7	4·4	5·2	6	7	8	8·5

3–4 f.	$n = 0(1)4$ $n = 0(1)6$ $n = 7(1)10$	$x = \cdot1,\ \cdot4(\cdot1)\cdot6,\ \cdot8(\cdot1)1$ $x = 1\cdot1,\ 1\cdot2,\ 1\cdot5(\cdot5)3(1)6$ $x = n-4(1)6$	No Δ	Nicholson 1912 (113)

Nicholson tabulates $2G_n(x)$.

17·342. $G_n(x)$ for x and n nearly equal

6 dec.	$n = x,\ x-1$	$x = 1(1)50(5)100(10)200(20)$ $400(50)1000(\text{var.})10^6$	No Δ	B.A. 1916 (93, Airey)
6 dec.	$n = x,\ x-1$ $n = 100(1)107$	$x = 6(1)13$ $x = 104$		Airey 1916a (527)

17·35. Neumann's $Y^{(0)}(x)$ and $Y^{(1)}(x)$

$$Y^{(n)}(x) = \tfrac{1}{2}\pi Y_n(x) + (\log_e 2 - \gamma) J_n(x)$$

14 dec.	9 and 10		Airey 1918 (242)
10 dec.	0(·1)6(·5)15·5	No Δ	B.A. 1915 (33, Airey)
7 dec.	0(·1)16	No Δ	B.A. 1911 (77, Airey) Airey 1911e (662)
6 dec.	0(·02)15·5	No Δ	B.A. 1914 (76, Airey?)
6 dec.	0(·2)6(·5)15·5	No Δ	B.A. 1915 (34, Airey)
4 dec.	0(·01)1(·1)10·2	No Δ	Smith 1896 (99) Jahnke–Emde 1909 (124)

$Y^{(1)}(x)$ contains errors exceeding 2 units in the last decimal when $x > 8\cdot5$.

17·361. $Y^{(n)}(x)$: General Tables

6 dec. $n = 0(1)13$ $x = 0(\cdot2)6(\cdot5)15\cdot5$ No Δ B.A. 1915 (34, Airey)

Values of $Y^{(n)}(x)$ exceeding about 10 are omitted; the least value of x for which $Y^{(n)}(x)$ is tabulated is given below:

n	0, 1	2	3	4	5	6	7	8	9	10	11	12	13
x	0·2	0·6	1·0	1·6	2·2	3·0	3·6	4·4	5·2	6	7	7·5	8·5

17·362. $Y^{(n)}(x)$ for x and n nearly equal

6 dec.	$n = x, x-1$	$x = 1(1)50(5)100(10)200(20)400(50)1000(\text{var.})10^6$	No Δ	B.A. 1916 (93, Airey)

17·381. Derivatives of $Y_n(x)$

$(16-p)$ dec.	$\dfrac{d^p}{dx^p} Y_0(x)$	$p = 1(1)14$	$x = 15(1)25$	Hayashi 1933 (57)
7 dec.	$Y_n'(n)$	$n = 1(1)50$		Watson 1922 (747)

Neither table gives differences. See also Art. 17·556.

17·382. The Derivative $-2G_n'(x)$

4– f.	$n = 1(1)3$	$x = \cdot5, \cdot6, \cdot8(\cdot1)1$	No Δ	Nicholson 1912 (123)
	$n = 1(1)6$	$x = 1{\cdot}2, 1{\cdot}5(\cdot5)3(1)5$		
	$n = 7, 8;\ 7(1)10$	$x = 4;\ 5$		

17·4. Expressions involving either $J_n(x)$ or $Y_n(x)$. Spherical Bessel Functions and Riccati-Bessel Functions

See also Sub-section 17·5 for functions of a more auxiliary nature.

17·411†. $\Lambda_n(x) = \dfrac{n!J_n(x)}{(\frac{1}{2}x)^n}$

In particular $\Lambda_0(x) = J_0(x)$ and $\Lambda_1(x) = \dfrac{2}{x}J_1(x)$.

4–5 dec.	$n = 0(1)8$	$x = 0(\cdot02)9{\cdot}98$	IΔ	Jahnke–Emde 1933 (250), 1938 (180)

From 7-decimal tables, Darmstadt (MS.), calculated by A. Walther, S. Gradstein and K. Hessenberg.

6 dec.	$n = 1$	$x = 0(\cdot1)20$	No Δ	Lommel 1870 (164), 1885 (312)
4 dec.	$n = 1$	$x = 0(\cdot2)12$	No Δ	Airy 1835 (291), 1858 (400); Mascart 1889 (310); Pernter 1906 (440)
4; 5 dec.	$n = 1$	$180x/\pi = 0(15)195;\ 210(15)1125$	No Δ	Schwerd 1835 (T. 3); Mascart 1889 (311); Pernter 1906 (441)

Schwerd and Pernter also give values for $180x/\pi = 1210(180)1930$.

4 dec.	$10\Lambda_1(x)$	$180x/\pi = 0(40)600$	No Δ	Conrady 1919 (579)

A short extension in Buxton 1921 is unreliable.

17·412. $1-\Lambda_1(x) = 1-\dfrac{2}{x}J_1(x)$

4 dec.	$0(\cdot1)\cdot2(\cdot2)4(\cdot5)10(1)15$	No Δ	Morse 1936 (337)

17·413. $\{J_0(x)\}^2$

4 dec.	$0(\cdot2)10$	No Δ	Airy 1841 (7)

17·414. $\left\{\frac{2}{x}J_1(x)\right\}^2$

6 dec.	1(·02)15·2	No Δ	Köhler 1934 (24)
6 dec.	0(·1)20	No Δ	Lommel 1870 (164), 1885 (312)
4 dec.	0(·2)12	No Δ	Mascart 1889 (310); Pernter 1906 (440)
4; 5 dec.	$180x/\pi$ = 0(15)150; 165(15)1125	No Δ	Schwerd 1835 (T. 3); Mascart 1889 (311); Pernter 1906 (441)

Schwerd and Pernter also give values for $180x/\pi$ = 1210(180)1930.

4 dec.	$180x/\pi$ = 0(40)600	No Δ	Conrady 1919 (579)

17·416. $y_n(x) = x^n Y_n(x)$

18 dec.	$n = 1$	x = 0(·1)6	Miller (MS.)

From Aldis's $G_1(x)$.

13+ fig.	n = 0(1)8; n = 9(1)20	x = 0(·1)10·4; x = 0(·1)X	B.A. (MS., Miller, etc.)

X, which exceeds 10, is near the maximum of $y_n(x)$ that precedes the first zero. These tables are the basic material from which 8-figure tables, interpolable with modified second differences, are to be prepared (see Art. 17·321); some of the printer's copy has already (1943) been completed.

17·421. Stokes's Functions or Spherical Bessel Functions $\sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x)$

See Art. 7·63 for a list of tables of $\frac{\sin x}{x}$.

10 dec.	n = −21(1)0	x = 0(·01)10	δ^4	New York W.P.A. (MS.)
8 fig.	n = 1(1)21	x = 0(·01)10	δ^4	New York W.P.A. (MS.)
5 fig.	$\pm n$ = 1(1)10	x = 0(·1)10		New York W.P.A.

Duplicated copies are available.

4 d. or 5 f.	n = −3(1)+2	x = 0(·1)·2(·2)8	No Δ	Morse 1936 (335)

17·422. $\Psi_n(x)$ and $\psi_n(x)$

$$\Psi_n(x) = (-)^n(2n+1)!!\sqrt{\frac{\pi}{2x^{2n+1}}}J_{-n-\frac{1}{2}}(x) = (-)^n(2n+1)!!\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\cos x}{x}$$

$$\psi_n(x) = (2n+1)!!\sqrt{\frac{\pi}{2x^{2n+1}}}J_{n+\frac{1}{2}}(x) = (-)^n(2n+1)!!\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sin x}{x}$$

5 or 6 f. or d.	n = 0(1)3 to 6	$8x$ = 8, 12(2)18, 21(3)27	Rayleigh 1910 (39)
5 f. or d.	n = 0(1)4 or 6	x = 1·25	Paris 1915 (471)

Both Rayleigh and Paris also give short tables of more complicated functions. Neither gives differences.

17·426. $D_n = \left[\left\{\frac{d}{dx}\left(\sqrt{\frac{\pi}{2x}}J_{n+\frac{1}{2}}(x)\right)\right\}^2 + \left\{\frac{d}{dx}\left(\sqrt{\frac{\pi}{2x}}J_{-n-\frac{1}{2}}(x)\right)\right\}^2\right]^{\frac{1}{2}}$

In particular $D_0 = \frac{\sqrt{1+x^2}}{x^2}$ and $D_1 = \frac{\sqrt{4+x^4}}{x^3}$.

4 fig.	n = 0(1)4	x = 0(·1)·2(·2)5	No Δ	Morse 1936 (339)

$$17{\cdot}428.\quad \delta_n = \tan^{-1}\left\{\frac{d}{dx}\left(\sqrt{\frac{\pi}{2x}}\,J_{n+\frac12}(x)\right)\Big/\frac{d}{dx}\left(\sqrt{\frac{\pi}{2x}}\,J_{-n-\frac12}(x)\right)\right\}$$

In particular $\delta_0 = x - \tan^{-1} x$ and $\delta_1 = \tan^{-1}\dfrac{2x}{2-x^2} - x$.

$0°{\cdot}1$ $\quad n = 0(1)4 \quad x = 0(\cdot1)\cdot2(\cdot2)5$ No Δ Morse 1936 (339)

See also Art. 17·527.

17·431. Riccati-Bessel Functions

$$S_n(x) = \sqrt{\tfrac12\pi x}\,J_{n+\frac12}(x) \qquad C_n(x) = (-)^n\sqrt{\tfrac12\pi x}\,J_{-n-\frac12}(x)$$

7 dec. $S_n(x)$, 7 fig. $C_n(x)$ $\quad n = 0(1)N$

- $x = 1(1)10$ No Δ B.A. 1914 (88, Doodson)
- $x = \cdot1(\cdot1)\cdot9$ No Δ B.A. 1922 (264, Doodson)
- $x = 1{\cdot}1(\cdot1)1{\cdot}9$ No Δ B.A. 1916 (98, Doodson)

In each case $S_{N+1}(x) < \frac12 . 10^{-7}$. Some typical values are:

x	·1	·2	·5	1	1·5	2	3	4	6	8	10
N	3	4	6	7	9	10	12	14	17	20	23

4 dec. $F_n = S_n(x)$ $\quad n = 0(1)4 \quad x = 0(\cdot1)5{\cdot}6$ No Δ Yost, Wheeler & Breit 1935

17·432. $S_n'(x)$ and $C_n'(x)$

B.A. 1914 (88), 1922 (264) and 1916 (98) also give $S_n'(x)$ to 7 decimals and $C_n'(x)$ to 7 figures for n and x as listed in Art. 17·431. Yost, Wheeler & Breit 1935 likewise gives $S_n'(x)$ and $xS_n'(x)/S_n(x)$ as well as $S_n(x)$.

17·433. $\mathrm{Log}_{10}|S_n(x)|$, $\log_{10}|C_n(x)|$, $\log_{10}|S_n'(x)|$ and $\log_{10}|C_n'(x)|$

7 dec. $\quad n = 0(1)N'$

- $x = 1(1)10$ No Δ B.A. 1914 (88, Doodson)
- $x = \cdot1(\cdot1)\cdot9$ No Δ B.A. 1922 (264, Doodson)
- $x = 1{\cdot}1(\cdot1)1{\cdot}9$ No Δ B.A. 1916 (98, Doodson)

In these $N' = N$ (Art. 17·431) except in B.A. 1914, where $N' < N$. Typical values are:

x	1	2	4	6	8	10
N'	4	6	9	11	14	17

17·436. $E_n^2(x) = S_n^2(x) + C_n^2(x)$, $E_n'^2(x) = S_n'^2(x) + C_n'^2(x)$, $\log_{10} E_n^2(x)$ and $\log_{10} E_n'^2(x)$

B.A. 1914 (88), 1922 (264) and 1916 (98) give all these functions, some exact, as follows:

7 fig. or dec. $\quad n = 0(1)N'' \quad x = \cdot1(\cdot1)2(1)10$ No Δ

N'' is such that, for $n > N''$, $E_n^2(x) = C_n^2(x)$ and $E_n'^2(x) = C_n'^2(x)$ to tabular accuracy. Typical values are:

x	0·1	0·5	1	2	4	6	8	10
N''	1	2	3	5	8	10	12	15

17·45. $D(x) = 2^{-1/2}(-\frac14)!\,x^{1/4}J_{-1/4}(\frac12 x)$ and $G(x) = -2^{1/2}(\frac14)!\,x^{-1/4}J_{1/4}(\frac12 x)$

3 dec. $\quad 0(\cdot2)8(1)12$

- Prandtl 1901 (25)
- No Δ Jahnke–Emde 1909 (106)

17·471. $C = \frac{1}{8}\{J_0(x) + 2J_0(2x) + 3J_0(3x)\}$

6 dec. 0(·2)7·4 No Δ Lommel 1870 (169)

17·472. $C' = \frac{4}{3}\left\{2\frac{J_1(x)}{x} - \frac{1}{2}\frac{J_1(\frac{1}{2}x)}{\frac{1}{2}x}\right\}$

6 dec. 0(·2)20 No Δ Lommel 1870 (168)

17·473. $\dfrac{xJ_1(x)}{J_0(x)}$

2–5 dec. 0(·01)·1(·05)1·5(var.)14·8 No Δ Nancarrow 1933 (465)

17·48. $\dfrac{J_0(x) \mp \cos x}{x}$ and $\dfrac{J_0^2(x) \mp 2\cos x J_0(x) + 1}{x^2}$

Upper signs; functions denoted by $\frac{2}{x}U_2(x, x)$ and M^2.

Lower signs; functions denoted by $\frac{2}{x}V_0(x, x)$ and M_1^2.

6 dec. 0(·5)12 No Δ Lommel 1885 (323, 328)

17·5. Auxiliary Functions for the Determination of Bessel Functions. Expressions involving both $J_n(x)$ and $Y_n(x)$. Bessel Functions of the Third Kind, or Hankel Functions

For the evaluation of Bessel Functions at intermediate arguments the ordinary methods of interpolation may often be used and the requisite differences or derivatives are sometimes provided, as indicated in the various sections. There are also many special methods (sometimes involving tables of auxiliary functions) for non-tabular arguments, intermediate or beyond the range of tabulation of the main function. The present sub-section lists some of these special methods and indicates where corresponding auxiliary tables can be found. Some of the auxiliary functions are chosen so as to have comparatively slow variation, being thus more easily interpolable, and tables of these will usually be found listed in the articles immediately below; others are functions that have an independent interest of their own, and will be found in other sub-sections as indicated.

17·51. Addition Formulæ

17·511. Write $\mathscr{C}_n(x) = AJ_n(x) + BY_n(x)$ in which A and B are constants independent of n, $\mathscr{C}_n(x)$ being the general solution of Bessel's equation of order n. Then the expansion

$$\mathscr{C}_n(x+h) = \sum_{-\infty}^{\infty} \mathscr{C}_{n-k}(x) J_k(h)$$

may be used for interpolation, by combining tables of $J_n(h)$ (see Art. 17·131), usually with h small, with the appropriate table of $\mathscr{C}_n(x)$ (see Arts. 17·131, 17·321, 17·341 or 17·361). The table in Hayashi 1933 (36) of $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ is designed for the case $n = 1$ (see Art. 17·15).

17·512. Alternatively the Bessel–Lommel expansions may be used, in conjunction with the same tables of $\mathscr{C}_n(x)$, or with the tables listed in Art. 17·21.

$$\mathscr{C}_n(x+h) = \left(\frac{x+h}{x}\right)^n \left\{\mathscr{C}_n(x) - \eta\mathscr{C}_{n+1}(x) + \frac{\eta^2}{2!}\mathscr{C}_{n+2}(x) - \cdots\right\}$$

$$\mathscr{C}_n(x+h) = \left(\frac{x}{x+h}\right)^n \left\{\mathscr{C}_n(x) + \eta\mathscr{C}_{n-1}(x) + \frac{\eta^2}{2!}\mathscr{C}_{n-2}(x) + \cdots\right\}$$

in which $\eta = h(1+h/2x)$.

17·52. Bessel Functions of the Third Kind, or Hankel Functions

$$H_n^{(1)}(x) = J_n(x) + iY_n(x) \qquad H_n^{(2)}(x) = J_n(x) - iY_n(x)$$

Values of $J_n(x)$ and $Y_n(x)$ may conveniently be obtained from tables connected with $|H_n^{(1)}(x)|$ and Arg $H_n^{(1)}(x)$. These functions, being non-oscillatory and relatively slowly varying, are more readily interpolable than $J_n(x)$ or $Y_n(x)$ themselves.

17·5211†. $|H_n^{(1)}(x)| = \sqrt{J_n^2(x) + Y_n^2(x)}$

7 dec.	$n = 0, 1, \frac{1}{3}$	$x = 0(\cdot 02)16$	No Δ	Watson 1922 (666, 714)
4 fig.	$2\,\vert H_1^{(1)}(x)\vert = c_0$	$x = 0(\cdot 1)\cdot 2(\cdot 2)5$	No Δ	Morse 1936 (338)

17·5212. $\rho_n = \pi^2\{J_n^2(x) + Y_n^2(x)\}$

4± f.	$n = 0(1)4$	$x = \cdot 1, \cdot 4(\cdot 1)\cdot 6, \cdot 8(\cdot 1)1$	No Δ	Nicholson 1912 (114)
	$n = 0(1)6$	$x = 1\cdot 1, 1\cdot 2, 1\cdot 5(\cdot 5)3(1)6$		
	$n = 7;\ 8;\ 9, 10$	$x = 4(1)6;\ 5, 6;\ 6$		

Nicholson 1912 (117) also gives 5–6 figure values of $R_n = x\rho_n/2\pi$ for $x = 8$. In each case he tabulates corresponding values of $(\rho_n\rho_{n+1})^{-1}$ or of $(R_nR_{n+1})^{-1}$ to 4 or 5 decimals or figures.

17·5221. $\text{Log}_{10}\sqrt{\frac{1}{2}\pi x\{J_n^2(x) + Y_n^2(x)\}}$

8 dec.	$n = 0(\frac{1}{2})6\frac{1}{2}$	$x = 10(10)100(100)1000$	No Δ	B.A. 1907 (94)

17·5222. $\text{Log}_{10}\sqrt{x\{J_n^2(x) + Y_n^2(x)\}}$

7 dec.	$n = 0(\frac{1}{2})6\frac{1}{2}$	$x = 10(10)100(100)1000$	No Δ	B.A. 1907 (94)

17·5241. $c_n = \sqrt{J_n'^2(x) + Y_n'^2(x)}$

4 fig.	$n = 1(1)4$	$x = 0(\cdot 1)\cdot 2(\cdot 2)5$	No Δ	Morse 1936 (338)

17·5242. $\sigma_n = 4\pi^2\{J_n'^2(x) + Y_n'^2(x)\}$

4± f.	$n = 0(1)3$	$x = \cdot 5, \cdot 6, \cdot 8(\cdot 1)1$	No Δ	Nicholson 1912 (124)
	$n = 0(1)6$	$x = 1\cdot 1, 1\cdot 2, 1\cdot 5(\cdot 5)3(1)5$		
	$n = 7, 8;\ 7(1)10$	$x = 4;\ 5$		

Nicholson also tabulates corresponding values of $(\sigma_n\sigma_{n+1})^{-1}$ to about 4 figures.

17·526†. $\operatorname{Arg} H_n^{(1)}(x) = \tan^{-1}\left\{\dfrac{Y_n(x)}{J_n(x)}\right\}$

$0''.01$	$n = 0, 1, \frac{1}{3}$	$x = 0(.02)16$	No Δ	Watson 1922 (666, 714)

17·527. $\gamma_n = \tan^{-1}\left\{\dfrac{J_n'(x)}{Y_n'(x)}\right\}$

$0^\circ.1$	$n = 0(1)4$	$x = 0(.1).2(.2)5$	No Δ	Morse 1936 (338)

Note that $\operatorname{Arg} H_1^{(1)}(x) = -90^\circ - \gamma_0$. See also Art. 17·428.

17·529. Functions of $\alpha_n = \frac{1}{2}(n+\frac{1}{2})\pi - x + \tan^{-1}\left\{\dfrac{Y_n(x)}{J_n(x)}\right\}$

9 dec.	$-\operatorname{Sin} \alpha_0$	$x = 10(10)100(100)1000$	No Δ	B.A. 1907 (95)
8 dec.	$\operatorname{Log}_{10}(-\sin \alpha_0)$			
8 dec.	$\operatorname{Log}_{10}(-x \sin \alpha_0)$			
7 dec.	$\operatorname{Log}_{10}\left(\dfrac{8x\alpha_n}{4n^2-1}\right)$	$n = 0(\frac{1}{2})6\frac{1}{2}$, $x = 10(10)100(100)1000$	No Δ	B.A. 1909 (35)

17·53. Auxiliary Functions for Large x

17·531†. $A_n(x)$ and $B_n(x)$

B.A. **6**, 1937 (203) tabulates the functions

$$A_0(x) = J_0(x) \sin x - Y_0(x) \cos x \qquad B_0(x) = J_0(x) \cos x + Y_0(x) \sin x$$
$$A_1(x) = -J_1(x) \cos x - Y_1(x) \sin x \qquad B_1(x) = J_1(x) \sin x - Y_1(x) \cos x$$

to 8 decimals with δ_m^2 for $x = 25(.1)50(1)150(10)1150$, $1000(100)6000$. These tables suffice, in conjunction with a table of circular functions (see Art. 7·1), for the determination of $J_0(x)$, $Y_0(x)$, $J_1(x)$ and $Y_1(x)$ to 8 decimals throughout the whole range.

17·536. $U_n(x)$, $V_n(x)$ and $\dfrac{V_n(x)}{U_n(x)}$

These are defined by

$$J_{\pm n}(x) = U_n(x) \cos(x \mp \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi) - V_n(x) \sin(x \mp \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)$$
$$Y_{\pm n}(x) = U_n(x) \sin(x \mp \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi) + V_n(x) \cos(x \mp \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)$$

10 dec.	$n = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$	$x = 25(\text{var.})30,000$		New York W.P.A. (MS.)
4 dec.	$n = \frac{1}{3}$	$x = 0(.05)2(.2)8$	No Δ	Watson 1918*a* (204)

17·537. $Q_n(x) = \sqrt{\frac{1}{2}\pi x}\, V_n(x)$

$$Q_n(x) = \sqrt{\tfrac{1}{2}\pi x\{J_n^2(x) + Y_n^2(x)\}} \sin \alpha_n$$
$$= \sqrt{\tfrac{1}{2}\pi x}\{Y_n(x) \cos(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi) - J_n(x) \sin(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)\}$$

α_n is defined in Art. 17·529. Neither of the following tables gives differences.

9–10 dec.	$Q_n(x)$	$n = 0(1)6$	$x = 10(10)100(100)1000$	B.A. 1909 (34)
9–10 dec.	$Q_0(x)$		$x = 10(10)100(100)1000$	B.A. 1907 (95)
8 dec.	$\operatorname{Log}_{10} \lvert Q_0(x) \rvert$			

17·55. Easily Interpolable Functions for Small x

$Y_n(x)$ has a singularity at $x = 0$, so that special auxiliary functions are needed for interpolation.

17·551. $y_n(x) = x^n Y_n(x)$ contains a logarithmic term involving $x^n \log_e x$ when n is an integer, but this becomes relatively less important as n increases, and interpolation is correspondingly easier. See Art. 17·416 for a list of tables.

17·552. $Y_0(x) = C_0(x) + D_0(x) \log_{10} x \qquad Y_1(x) = \frac{1}{x} C_1(x) + D_1(x) \log_{10} x$

B.A. 6, 1937 (175) gives $C_0(x)$, $D_0(x)$, $C_1(x)$ and $D_1(x)$ with δ^2 in each case to 8 decimals for $x = 0(·01)·5$.

17·556. $n^{1/3}J_n(n)$, $n^{2/3}J_n'(n)$, $n^{1/3}Y_n(n)$ and $n^{2/3}Y_n'(n)$

7 dec.	$n = 1(1)50$	No Δ	Watson 1922 (746)

See also Arts. 17·132, 17·15, 17·322 and 17·381.

17·57. Special Methods

Writing $\mathscr{C}_n(x) = AJ_n(x) + BY_n(x)$, where A and B are constants independent of n, Airey 1918 finds that, to a good degree of approximation,

$$\mathscr{C}_0(\rho+h) = -(\alpha_0 \sin h - \beta_0\eta)\mathscr{C}_1(\rho)$$
$$\mathscr{C}_1(r+h) = (\alpha_1 \sin h + \beta_1\eta)\mathscr{C}_0(r)$$

where ρ and r are zeros of $\mathscr{C}_0(x)$ and $\mathscr{C}_1(x)$ respectively and η is a function of h, while α_0 and β_0 are functions of h/ρ and α_1 and β_1 are functions of h/r. He tabulates (p. 238):

6 dec.	$\alpha_0, \beta_0, \alpha_1, \beta_1$	$\frac{h}{\rho}$ or $\frac{h}{r} = -·1(·01)+·1$	Δ^n
3 dec.	η	$h = ·05(·05)1·6$	No Δ

In conjunction with tables of zeros of $\mathscr{C}_0(x)$ or $\mathscr{C}_1(x)$ and of the corresponding values of $\mathscr{C}_1(x)$ or $\mathscr{C}_0(x)$ (see Arts. 17·7), these suffice to give $\mathscr{C}_0(x)$ or $\mathscr{C}_1(x)$ to 6 decimals when $x > 15·5$.

In his D.Sc. thesis (London University, 1915) Airey also tabulates:

4 dec.	α_0, β_0	$\frac{h}{\rho} = -·3(·01)+·3$	No Δ
2 dec.	η	$h = 0(·1)2·2$	No Δ

These suffice to give $\mathscr{C}_0(x)$ to 4 decimals when $x > 8$.

17·591. $\sqrt{2\pi x}$ and $\sqrt{\frac{2}{\pi x}}$

These are useful in connection with asymptotic expansions of Bessel Functions, and with functions of order half an odd integer. See Arts. 5·153 and 5·156 for lists of such tables, the most important being

6 fig.	$\sqrt{2\pi x}$	$x = 1(1)40$	Wrinch & Wrinch 1923 (848) with 1924 (64)
12 dec.	$\sqrt{2/\pi x}$	$x = 0(·01)2, 10(·1)20$	Hayashi 1930a (29, 48)

17·526†. $\operatorname{Arg} H_n^{(1)}(x) = \tan^{-1}\left\{\frac{Y_n(x)}{J_n(x)}\right\}$

0″·01	$n = 0, 1, \frac{1}{3}$	$x = 0(\cdot 02)16$	No Δ	Watson 1922 (666, 714)

17·527. $\gamma_n = \tan^{-1}\left\{\frac{J_n'(x)}{Y_n'(x)}\right\}$

0°·1	$n = 0(1)4$	$x = 0(\cdot 1)\cdot 2(\cdot 2)5$	No Δ	Morse 1936 (338)

Note that $\operatorname{Arg} H_1^{(1)}(x) = -90° - \gamma_0$. See also Art. 17·428.

17·529. Functions of $\alpha_n = \frac{1}{2}(n+\frac{1}{2})\pi - x + \tan^{-1}\left\{\frac{Y_n(x)}{J_n(x)}\right\}$

9 dec.	$-\operatorname{Sin} \alpha_0$	$x = 10(10)100(100)1000$	No Δ	B.A. 1907 (95)
8 dec.	$\operatorname{Log}_{10}(-\sin \alpha_0)$			
8 dec.	$\operatorname{Log}_{10}(-x \sin \alpha_0)$			
7 dec.	$\operatorname{Log}_{10}\left(\frac{8x\alpha_n}{4n^2-1}\right)$	$n = 0(\frac{1}{2})6\frac{1}{2}$, $x = 10(10)100(100)1000$	No Δ	B.A. 1909 (35)

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B.A. **6**, 1937 (203) tabulates the functions

$$A_0(x) = J_0(x)\sin x - Y_0(x)\cos x \qquad B_0(x) = J_0(x)\cos x + Y_0(x)\sin x$$
$$A_1(x) = -J_1(x)\cos x - Y_1(x)\sin x \qquad B_1(x) = J_1(x)\sin x - Y_1(x)\cos x$$

to 8 decimals with δ_m^2 for $x = 25(\cdot 1)50(1)150(10)1150$, $1000(100)6000$. These tables suffice, in conjunction with a table of circular functions (see Art. 7·1), for the determination of $J_0(x)$, $Y_0(x)$, $J_1(x)$ and $Y_1(x)$ to 8 decimals throughout the whole range.

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$$Y_{\pm n}(x) = U_n(x)\sin(x \mp \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi) + V_n(x)\cos(x \mp \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)$$

10 dec.	$n = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$	$x = 25(\text{var.})30{,}000$		New York W.P.A. (MS.)
4 dec.	$n = \frac{1}{3}$	$x = 0(\cdot 05)2(\cdot 2)8$	No Δ	Watson 1918*a* (204)

17·537. $Q_n(x) = \sqrt{\frac{1}{2}\pi x}\, V_n(x)$

$$Q_n(x) = \sqrt{\tfrac{1}{2}\pi x\{J_n^2(x) + Y_n^2(x)\}}\sin \alpha_n$$
$$= \sqrt{\tfrac{1}{2}\pi x}\{Y_n(x)\cos(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi) - J_n(x)\sin(x - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)\}$$

α_n is defined in Art. 17·529. Neither of the following tables gives differences.

9–10 dec.	$Q_n(x)$	$n = 0(1)6$	$x = 10(10)100(100)1000$	B.A. 1909 (34)
9–10 dec.	$Q_0(x)$		$x = 10(10)100(100)1000$	B.A. 1907 (95)
8 dec.	$\operatorname{Log}_{10}\lvert Q_0(x)\rvert$			

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B.A. **6**, 1937 (175) gives $C_0(x)$, $D_0(x)$, $C_1(x)$ and $D_1(x)$ with δ^2 in each case to 8 decimals for $x = 0(\cdot 01)\cdot 5$.

17·556. $n^{1/3} J_n(n)$, $n^{2/3} J_n'(n)$, $n^{1/3} Y_n(n)$ and $n^{2/3} Y_n'(n)$

7 dec.	$n = 1(1)50$	No Δ	Watson 1922 (746)

See also Arts. 17·132, 17·15, 17·322 and 17·381.

17·57. Special Methods

Writing $\mathscr{C}_n(x) = AJ_n(x) + BY_n(x)$, where A and B are constants independent of n, Airey 1918 finds that, to a good degree of approximation,

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where ρ and r are zeros of $\mathscr{C}_0(x)$ and $\mathscr{C}_1(x)$ respectively and η is a function of h, while α_0 and β_0 are functions of h/ρ and α_1 and β_1 are functions of h/r. He tabulates (p. 238):

6 dec.	$\alpha_0, \beta_0, \alpha_1, \beta_1$	$\frac{h}{\rho}$ or $\frac{h}{r} = -\cdot 1(\cdot 01)+\cdot 1$	Δ^n
3 dec.	η	$h = \cdot 05(\cdot 05)1\cdot 6$	No Δ

In conjunction with tables of zeros of $\mathscr{C}_0(x)$ or $\mathscr{C}_1(x)$ and of the corresponding values of $\mathscr{C}_1(x)$ or $\mathscr{C}_0(x)$ (see Arts. 17·7), these suffice to give $\mathscr{C}_0(x)$ or $\mathscr{C}_1(x)$ to 6 decimals when $x > 15\cdot 5$.

In his D.Sc. thesis (London University, 1915) Airey also tabulates:

4 dec.	α_0, β_0	$\frac{h}{\rho} = -\cdot 3(\cdot 01)+\cdot 3$	No Δ
2 dec.	η	$h = 0(\cdot 1)2\cdot 2$	No Δ

These suffice to give $\mathscr{C}_0(x)$ to 4 decimals when $x > 8$.

17·591. $\sqrt{2\pi x}$ and $\sqrt{\dfrac{2}{\pi x}}$

These are useful in connection with asymptotic expansions of Bessel Functions, and with functions of order half an odd integer. See Arts. 5·153 and 5·156 for lists of such tables, the most important being

6 fig.	$\sqrt{2\pi x}$	$x = 1(1)40$	Wrinch & Wrinch 1923 (848) with 1924 (64)
12 dec.	$\sqrt{2/\pi x}$	$x = 0(\cdot 01)2,\ 10(\cdot 1)20$	Hayashi 1930*a* (29, 48)

17·596. $\text{Log}_e \frac{1}{2}x + \gamma$

These are useful in connection with functions of the second kind of integral order; see Sub-section 17·3.

8 dec. ·2(·2)6 No Δ Jahnke–Emde 1909 (141)

17·7. Zeros and Turning-Values of Bessel Functions

17·71. Expansions for Zeros, etc.

J. McMahon (*Annals of Math.*, **9**, 23–30, 1895) gives asymptotic expansions to 5 terms for zeros of $J_n(x)$ and to 4 terms for zeros of $J_n'(x)$ and of $\frac{d}{dx}\{x^{-\frac{1}{2}}J_n(x)\}$; in these $Y_n(x)$ or $\mathscr{C}_n(x) = AJ_n(x) + BY_n(x)$ may be substituted for $J_n(x)$ with only slight consequent modifications. He also gives 4 terms for zeros of

$$Y_n(x)J_n(kx) - J_n(x)Y_n(kx), \quad \text{of} \quad Y_n'(x)J_n'(kx) - J_n'(x)Y_n'(kx)$$

and of

$$\frac{d}{dx}\{x^{-\frac{1}{2}}Y_n(x)\}\cdot\frac{d}{dx}\{(kx)^{-\frac{1}{2}}J_n(kx)\} - \frac{d}{dx}\{x^{-\frac{1}{2}}J_n(x)\}\cdot\frac{d}{dx}\{(kx)^{-\frac{1}{2}}Y_n(kx)\}$$

These expressions are most useful for n moderate and x large and are reproduced in Gray & Mathews 1895 (241) and in Gray, Mathews & MacRobert 1922 (260). Watson 1922 (506, 507) gives the first three to 5, 3 and 3 terms and Jahnke & Emde 1933 (211), 1938 (143) give the first two to 5 and 4 terms respectively. B.A. **6**, 1937 (202) gives 6 terms when n = 0 and 1 for zeros of $J_n(x)$ only.

The expansions for zeros of $J_n(x)$ and of $J_n'(x)$ are given to 6 terms by Bickley 1943 (40, 41), and the 7th and 8th terms for $J_n(x)$ are given by Bickley & Miller (MS.), with three more terms when $n = 1\frac{1}{2}, 2\frac{1}{2}$.

All the coefficients are polynomials in n; numerical values have been computed by Bickley and Miller for $n = 0(\frac{1}{2})10(1)23, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}$ with the number of figures varying from 15–17 for the early terms to 5–7 for the last. These will be offered for publication shortly.

Airey gives a number of expansions of this type in various papers, mentioned in Arts. 17·8, 18·85 and 18·86.

A series suitable for the computation of a zero, when $\mathscr{C}_n(x)$ and $\mathscr{C}_n'(x)$ are known for a value $x = x_0$, close to the zero, is given by Miller & Jones (MS.) with coefficients as functions of n and x_0, and also as functions of x_0 when $\pm n = 0(\frac{1}{2})3(1)10$ and when $n = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$. Six coefficients are given, all exact.

For the computation of early zeros when n is large, so that n and x are of similar magnitude, expansions are given by Airey 1917 (193, etc.) and reproduced in Jahnke & Emde 1933 (211), 1938 (143). Watson 1922 (516, 521) gives two terms in some of these expansions. Airey's coefficients are in most cases approximations, not always very close to the true values. A similar remark applies to the coefficients given in Emde 1916.

Meissel 1891*a* (437) gives

$$n = x - 1{\cdot}85575\,70814\,89x^{1/3} - 0{\cdot}11479\,44781\,83x^{-1/3} - 0{\cdot}00869\,91858\,12x^{-1}\ldots$$

for the last zero, as n increases, of $J_n(x)$ as a function of n. He also gives

$$n = x - 0{\cdot}80861\,65174\,66x^{1/3} - 0{\cdot}06064\,99879\,10x^{-1/3} - 0{\cdot}03167\,35102\,63x^{-1}\ldots$$

for the last zero of $\dfrac{\partial}{\partial n}J_n(x)$, and

$$0{\cdot}67488\,50964\,30x^{-1/3} + 0{\cdot}07276\,30981\,82x^{-1} - 0{\cdot}01995\,97503\,28x^{-5/3}\ldots$$

for the corresponding absolute maximum of $J_n(x)$ for $n > 0$.

The above remarks refer only to explicit expansions for zeros; various expressions for the calculation of Bessel Functions (see Art. 17·0) may also be used to calculate nearby values and so serve as stepping-stones to the evaluation of zeros.

Asymptotic expansions for $J_1(j_{0,s})$ and $J_0(j_{1,s})$, where $J_0(j_{0,s}) = J_1(j_{1,s}) = 0$, are given to 4 terms in B.A. **6**, 1937 (202); they also apply to $Y_1(y_{0,s})$ and $Y_0(y_{1,s})$. These appeared originally in Davis & Kirkham 1927 (763).

Jahnke & Emde 1933, 1938 give many diagrams from which approximate values of zeros, turning-values, etc. may be estimated.

We shall use the notation of Watson 1922 and of B.A. **6**, 1937 for zeros, namely

$$J_n(j_{n,s}) = 0 \quad \text{and} \quad Y_n(y_{n,s}) = 0$$

The zeros $j_{n,s}$ and $y_{n,s}$ form a single sequence corresponding to values of $\tan^{-1}\dfrac{Y_n(x)}{J_n(x)}$ at interval $\frac{1}{2}\pi$, so that we might write $y_{n,s} = j_{n,s-\frac{1}{2}}$. The combined sequence gives a well-behaved difference-table. A similarly well-behaved combined sequence is given by $|J_n'(j_{n,s})| = |J_{n\pm1}(j_{n,s})|$ and $|Y_n'(y_{n,s})| = |Y_{n\pm1}(y_{n,s})|$.

17·7211. $j_{0,s}$ and $J_1(j_{0,s})$

10 dec.	$s = 1(1)150$	Davis & Kirkham 1927 (765)

Erratum: $J_1(j_{0,4})$, for 98214 read 98314.

10 dec.		$s = 1(1)150$	**B.A. 6, 1937** (171)
10 dec.	$j_{0,s}$	$s = 1(1)40$	Willson & Peirce 1897 (154)
8 dec.	$J_1(j_{0,s})$		G., M. & MacR. 1922 (300)

Erratum: $J_1(j_{0,35})$, for 35913 read 36383. Also in the next two items.

8 dec.		$s = 1(1)40$	Schorr 1916 (A 47)
7 dec.		$s = 1(1)40$	Airey 1918 (241)
6 dec.		$s = 1(1)12$	Peirce 1906 (513)
6 dec.	$j_{0,s}$	$s = 1(1)10$	Fowle 1933 (70)
4 dec.	$J_1(j_{0,s})$		

Fowle 1920 (68) gives $j_{0,s}$ only; earlier editions give neither $j_{0,s}$ nor $J_1(j_{0,s})$.

5 dec.		$s = 1(1)60$	Hayashi 1930*b* (98)
4 dec.	$j_{0,s}$	$s = 1(1)40$	Jahnke–Emde 1909 (122), 1933 (237),
4 fig.	$J_1(j_{0,s})$		1938 (166)

The following items give zeros only; very small tables are omitted.

10 dec.	$s = 1(1)10$	Meissel 1889 (3); Gray & Mathews 1895 (244)
7 dec.	$s = 1(1)40$	Watson 1922 (748)
6 dec.	$s = 1(1)10$	Peirce 1907 (173)
5; 4 dec.	$s = 1(1)10;\ 11(1)25$	Silberstein 1923 (143)
5? dec.	$s = 1(1)40$	Glazenap 1932
5 dec.	$s = 1(1)10$	Witkowski 1904 (84)
4 dec.	$s = 1(1)10$	Byerly 1906 (63)
4 dec.	$s = 1(1)10$	B.A. 1922 (271, Airey)
3 dec.	$s = 1(1)9$	Bourget 1866 (82)

Reproduced in Rayleigh 1877 (274), 1894 (330), Byerly 1893 (286), Thomson 1893 (353) and Dale 1903 (83).

4 dec.	$j_{0,s}/\pi$	$s = 1(1)12$	Stokes 1851 (186)

Reproduced in Rayleigh 1877 (274), 1894 (330), Byerly 1893 (286), Láska 1888–94 (390) and Dale 1903 (83), and for $s = 1(1)10$ in Rayleigh 1888 (432).

Euler 1784 gives approximations to $(\frac{1}{2}j_{0,s})^2$ for $s = 1(1)3$. Smith 1938 gives 7-decimal values for $s = 1(1)6$; $(\frac{1}{2}j_{0,4})^2$ should read 34·7600709. These are zeros of the Bessel–Clifford function $C_0(x) = J_0(2\sqrt{x})$.

17·7212. $\mathrm{Log}_{10} j_{0,s}$ and $|\log_{10} J_1(j_{0,s})|$

10 dec. 9 dec.	$\mathrm{Log}_{10} j_{0,s}$ $\mathrm{Log}_{10} \lvert J_1(j_{0,s}) \rvert$	$s = 1(1)150$	Davis & Kirkham 1927 (765)
10 dec. 7 dec.	$\mathrm{Log}_{10} j_{0,s}$ $^*\mathrm{Log}_{10} \lvert J_1(j_{0,s}) \rvert$	$s = 1(1)40$	Willson & Peirce 1897 (154)

* The value for $s = 35$ is erroneous.

17·722. $y_{0,s}$ and $Y_1(y_{0,s})$

8 dec.	$s = 1(1)50$	**B.A. 6, 1937** (201)

The following tables give zeros only:

7 dec.	$s = 1(1)40$	Watson 1922 (748)
5 dec.	$s = 1(1)40$	Hayashi 1930*b* (99)
5 dec.	$s = 1(1)10$	Airey 1911*b* (223)
4 dec.	$s = 1(1)40$	Glazenap 1932

17·723. Zeros x_s of $J_0(x) + kY_0(x)$ or of $J_0(x) + \kappa Y^{(0)}(x)$

We have $k\kappa(\log_e 2 - \gamma) - \frac{1}{2}\pi\kappa + k = 0.$

$k = \frac{1}{2}\pi/(\log_e 2 - \gamma)$, or $1/\kappa = 0$ Zeros of $Y^{(0)}(x)$

5 dec.	$s = 1(1)40$	Airey 1911*b* (221)
3 dec.	$s = 1(1)4$	Kalähne 1906 (92), 1907; Jahnke–Emde 1909 (130)

$\kappa = \pm 0{\cdot}1,\ \pm 1,\ -2$

5 dec.	$s = 1(1)10$	Airey 1911*b* (224)

$\kappa = 1,\ \frac{1}{2},\ \frac{1}{3}$

Kalähne 1907 (72) gives two zeros, obtained graphically.

17·7311. $j_{1,s}$ and $J_0(j_{1,s})$

16 dec.	$s = 1(1)50$	Meissel 1890 *Gray & Mathews 1895 (280) *G., M. & MacR. 1922 (301)

* Erratum: $j_{1,9}$, for 85340 read 85349. This error also occurs in the next item.

10 dec.	$s = 1(1)150$	Davis & Kirkham 1927 (769)
10 dec.	$s = 1(1)150$	**B.A. 6, 1937** (171)
8 dec.	$s = 1(1)50$	Schorr 1916 (A 47)
7 dec.	$s = 1(1)40$	Airey 1918 (241)
6 dec.	$s = 1(1)15$	Fowle 1920 (68), 1933 (70)

Not in earlier editions.

5+ dec.	$s = 1(1)32$	Silberstein 1923 (147)
5 dec.	$s = 1(1)60$	Hayashi 1930*b* (98)
5 dec.	$s = 1(1)9$, also $j_{1,10}$	Witkowski 1904 (85, 87)
4 dec. 4 fig.	$j_{1,s}$ $J_0(j_{1,s})$ } $s = 1(1)50$	Jahnke–Emde 1909 (123), 1933 (237), 1938 (166)

The following items give zeros only; very small tables are omitted.

7 dec.	$s = 1(1)40$	Watson 1922 (748)
6 dec.	$s = 1(1)6$	Lommel 1870 (167), 1885 (315) Mascart 1893 (668)
4 dec.	$s = 1(1)40$	Glazenap 1932
4 dec.	$s = 1(1)10$	Byerly 1906 (63)
4 dec.	$s = 1(1)10$	B.A. 1922 (271, Airey)
3 dec.	$s = 1(1)9$	Bourget 1866 (83)

Reproduced in Rayleigh 1877 (274), 1894 (330), Byerly 1893 (286), Thomson 1893 (353) and Dale 1903 (83).

6 dec.	$j_{1,s}/\pi$	$s = 1(1)6$		Lommel 1870 (167), 1885 (315) Mascart 1893 (668)
4 dec.	$j_{1,s}/\pi$	$s = 1(1)12$		Stokes 1851 (186)

Reproduced in Rayleigh 1877 (274), 1894 (330), Byerly 1893 (286) and Dale 1903 (83), and for $s = 1(1)10$ in Rayleigh 1888 (432).

3 dec.	$j_{1,s}/2\pi$	$s = 1(1)9$		Pernter 1906 (442)
3 dec.	$j_{1,s}/2\pi$	$s = 1(1)6$	Δ	Mascart 1889 (312)
1 dec.	$180 j_{1,s}/\pi$	$s = 1(1)6$	Δ	Schwerd 1835 (70)

Steiner 1894 (364–367) gives values which seem to be intended as approximations to $j_{1,s}$, $s = 1(1)49$; the second decimal is rarely that of $j_{1,s}$. For $s > 12$ the values are $3{\cdot}14159\,000 \times (s + \frac{1}{4})$ to 5 decimals.

Bates 1938 gives 6-decimal values of $(\frac{1}{2}j_{1,s})^2$ for $s = 1(1)5$, and Ward 1913 (96) gives 5-decimal values for $s = 1(1)4$. These are zeros of the Bessel–Clifford function $C_1(x) = J_1(2\sqrt{x})/\sqrt{x}$.

17·7312. $\text{Log}_{10} j_{1,s}$ and $|\log_{10} J_0(j_{1,s})|$

10 dec. 9 dec.	$\text{Log}_{10} j_{1,s}$ $\text{Log}_{10}\lvert J_0(j_{1,s})\rvert$	$s = 1(1)150$	Davis & Kirkham 1927 (769)

17·732. $y_{1,s}$ and $Y_0(y_{1,s})$

8 dec.	$s = 1(1)50$	**B.A. 6, 1937** (201)

The following tables give zeros only:

7 dec.	$s = 1(1)40$	Watson 1922 (748)

Erratum: in $y_{1,37}$, for 45038 read 45028.

5 dec.	$s = 1(1)40$	Hayashi 1930*b* (99)
5 dec.	$s = 1(1)10$	Airey 1911*b* (223)
4 dec.	$s = 1(1)40$	Glazenap 1932

17·733. Zeros of $J_1(x) + kY_1(x)$ or of $J_1(x) + \kappa Y^{(1)}(x)$

We have $k\kappa(\log_e 2 - \gamma) - \frac{1}{2}\pi\kappa + k = 0.$

$k = \frac{1}{2}\pi/(\log_e 2 - \gamma)$, or $1/\kappa = 0$ Zeros of $Y^{(1)}(x)$

5 dec.	$s = 1(1)40$	Airey 1911*b* (222)
3 dec.	$s = 1(1)3$	Kalähne 1906 (92), 1907; Jahnke–Emde 1909 (130)
	$\kappa = 1$	
5 dec.	$s = 1(1)10$	Airey 1911*b* (224)
	$\kappa = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{10}, \frac{1}{20}$	

Kalähne 1907 (72) gives two zeros, obtained graphically.

17·7411. $j_{n,s}$ for Integral n

	n	s	
10–12 dec.	2(1)20	1(1)53	B.A. (MS.)

The computation is not yet quite complete for small s when n is large.

7 dec.	0(1)5	1(1)40	Watson 1922 (748)
5 dec.	0(1)19	1(1)8 to 1	Smith, Rodgers & Traub 1944
5 dec.	2	1(1)5	Lommel 1870 (167), 1885 (315)

A sixth decimal is given, which is subject to errors up to 16 units. Reproduced in Rayleigh 1888 (433), Gray & Mathews 1895 (180) and G., M. & MacR. 1922 (192).

5 dec.	2	1(1)5	Witkowski 1904 (88)
5 dec.	1000	1	Ikeda 1925 (80); Hayashi 1930*b* (98)
4 dec.	0(1)10(5)20(10)50	1(1)10	B.A. 1922 (271, Airey)
3 dec.	75, 100(100)500	1(1)10	B.A. 1922 (271, Airey)
3; 2 dec.	750; 1000	1(1)10	B.A. 1922 (271, Airey)
6 fig.	10, 100 and 1000	1(1)5	Airey 1916*b* (11)
4 dec.	2	1(1)6	Jahnke–Emde 1909 (163), 1933 (276), 1938 (206)
3 dec.	0(1)10	1(1)10	Thornton 1939 (348)
3 dec.	0(1)5	1(1)9	Bourget 1866 (82–87)

This table contains a number of errors affecting the last two digits; it is reproduced, without correction, in Rayleigh 1877 (274), 1894 (330), Byerly 1893 (286), Thomson 1893 (353), Dale 1903 (83), Jahnke & Emde 1909 (147), 1933 (239), 1938 (168), Gray, Mathews & MacRobert 1922 (302) and Hayashi 1930*b* (52).

2–3 dec.	0 to 10	1(1)5 to 1	Reinstein 1911 (75)

Ward 1913 (96) gives 5-decimal values of $(\frac{1}{2}j_{2,s})^2$ for $s = 1(1)4$, and of $(\frac{1}{2}j_{3,s})^2$ for $s = 1(1)3$.

17·7412. $j_{2,s}/\pi$ and $J_1(j_{2,s})$, etc.

		s	
5 d. correct	$j_{2,s}/\pi$	1(1)5	Lommel 1870 (167), 1885 (315); Mascart 1893 (668)
6 dec.	$\frac{2}{j_{2,s}}J_1(j_{2,s})$, $\frac{4}{j_{2,s}^2}J_1^2(j_{2,s})$	1(1)5	Lommel 1870 (167), 1885 (315); Mascart 1893 (668)

Rayleigh 1888, Gray & Mathews 1895 and G., M. & MacR. 1922 reproduce the latter functions with $j_{1,s}$, see Art. 17·7411.

5 dec.	$j_{2,s}/\pi$, $\frac{4}{j_{2,s}^2}J_1^2(j_{2,s})$	1(1)5	Witkowski 1904 (88)
4 dec.	$\beta_{2,s} = j_{2,s}/\pi$	1(1)3	Morse 1936 (153)
	$J_1(j_{2,s})$	1(1)4	Morse 1936 (153)
3 dec.	$j_{2,s}/2\pi$, Δ	1(1)9	Pernter 1906 (442); Mascart 1889 (312)
5 dec.	$\frac{4}{j_{2,s}^2}J_1^2(j_{2,s})$	1(1)9	Pernter 1906 (442); Mascart 1889 (312)
3 dec.	$j_{2,s}/\pi$	1(1)9	Wrinch 1922 (505)
1 dec.	$180j_{2,s}/\pi$	1(1)5	Schwerd 1835
0 – dec.	$180j_{2,s}/\pi$	6(1)10	Schwerd 1835
5 dec.	$\frac{2}{j_{2,s}}J_1(j_{2,s})$, $\frac{4}{j_{2,s}^2}J_1^2(j_{2,s})$	1(1)10	Schwerd 1835

17·742. $y_{n,s}$ for Integral n

	n	s	
10–12 dec.	2(1)20	1(1)53	B.A. (MS.)

The computation is not yet quite complete for small s when n is large.

7 dec.	0(1)5	1(1)40	Watson 1922 (748)
3 dec.	100	1(1)5	Airey 1916*b* (14)

17·743. Zeros of $J_n(x) + kY_n(x)$ or of $J_n(x) + \kappa Y^{(n)}(x)$

We have $k\kappa(\log_e 2 - \gamma) - \frac{1}{2}\pi\kappa + k = 0.$

$$k = \frac{\frac{1}{2}\pi}{\log_e 2 - \gamma}, \quad \text{or} \quad 1/\kappa = 0 \qquad \text{Zeros of } Y^{(n)}(x)$$

5 dec.	$n = 2$	$s = 1(1)40$	Airey 1911*b* (222)
3 dec.	$n = 100$	$s = 1(1)5$	Airey 1916*b* (14)

17·75. $j_{n,s}$, etc. when n is not an Integer

17·751. $j_{n,s}$ for $n = m + \frac{1}{2}$, n Integral

When $2n$ is an odd integer, $AJ_n(x) + BY_n(x) = 0$ reduces to an equation of the form $\tan x = f(x)$ where $f(x)$ is a rational function. Equations of this type are dealt with in Arts. 5·71, *q.v.* for further details. Here we list only

10–12 dec.	$\pm n = 1\frac{1}{2}(1)10\frac{1}{2}$	$s = 1(1)53$	B.A. (MS.)

This is not yet quite complete.

17·752. $j_{n,s}$ for $n = \pm\frac{1}{3}$ and $\pm\frac{2}{3}$

	n	s	
10 dec.	$\pm\frac{1}{3}, \pm\frac{2}{3}$	1(1)8	New York W.P.A. (MS.)
9 dec.	$-\frac{1}{3}$	1	Ikeda 1925 (81); Hayashi 1930*b* (100)
7 dec.	$\frac{1}{3}$	1(1)40	Watson 1922 (751)
7 dec.	$\frac{2}{3}$	1	Emde 1916
4 dec.	$\frac{1}{3}$	1(1)40	Glazenap 1932
4 dec.	$\pm\frac{1}{3}$	1	Airey 1921 (203); G., M. & MacR. 1922 (317)
3 dec.	$\pm\frac{1}{3}, \pm\frac{2}{3}$	1(1)5	Dinnik 1933 (12)
2 dec.	$\frac{1}{3}, \frac{2}{3}$	1, 2	Dinnik 1911 (338); Airey 1921 (201)
	$-\frac{1}{3}, -\frac{2}{3}$	1(1)3	

17·7531. Zeros $y_{1/3,s}$ of $\sqrt{3}\,Y_{1/3}(x) = J_{1/3}(x) - 2J_{-1/3}(x)$

7 dec.	$s = 1(1)40$	Watson 1922 (751)
4 dec.	$s = 1(1)40$	Glazenap 1932

17·7535. Zeros s_s, d_s of $J_{-1/3}(x) \pm J_{1/3}(x)$

These are connected with the zeros of the Airy Integral, etc.; see Arts. 20·29.

9–11 dec.	$\frac{3}{2}s_s$, $\frac{9}{4}s_s^2$	$s = 4(1)55$	Miller, etc. (MS.)
9–11 dec.	$\frac{3}{2}d_s$, $\frac{9}{4}d_s^2$	$s = 4(1)22$	
7 dec.	s_s, d_s	$s = 1(1)40$	Watson 1922 (751)

Erratum: in s_2, for 5·61 . . . read 5·51 . . .

17·7536. Zeros s'_s, d'_s of $J_{-2/3}(x) \pm J_{2/3}(x)$

These are connected with the zeros of the derivative of the Airy Integral, etc.; see Arts. 20·29.

9–11 dec.	$\frac{3}{2}s'_s$, $\frac{9}{4}s'^2_s$	$s = 5(1)22$	Miller, etc. (MS.)
9–11 dec.	$\frac{3}{2}d'_s$, $\frac{9}{4}d'^2_s$	$s = 4(1)55$	

17·754. $j_{n,s}$ for $n = \pm\frac{1}{6}$ and $\pm\frac{5}{6}$

4 dec.	$n = \pm\frac{1}{6}$	$s = 1$	Airey 1921 (203); G., M. & MacR. 1922 (317)
3 dec.	$n = \pm\frac{1}{6}, \pm\frac{5}{6}$	$s = 1(1)5$	Dinnik 1933 (23)

17·755. $j_{n,s}$ for General n

	n	s	
10 dec.	$\pm\frac{1}{4}, \pm\frac{3}{4}$	1(1)8	New York W.P.A. (MS.)
5 dec.	$-1(\cdot01)+1$	1	B.A. 1927 (252, Airey)
4 dec.	$-\cdot49(\cdot01)+1\cdot49$	1	IΔ Jahnke–Emde 1933 (238), 1938 (167)
	$-\cdot49(\cdot01)+0\cdot99$	2	
4 dec.	$-\frac{1}{2}(\frac{1}{10})+1$	1, 2	Airey 1921 (202); G., M. & MacR. 1922 (317)
3 dec.	$0(\cdot1)1$	1	Emde 1916

3 dec.	$\pm\frac{1}{4}, \pm\frac{3}{4}$	$1(1)5$	Dinnik 1933 (18)
2 dec.	$\frac{1}{4}, \frac{3}{4}$	$1, 2$	Dinnik 1913*b* (326)
	$-\frac{1}{4}, -\frac{3}{4}$	$1(1)3$	Airey 1921 (201)

See also Dinnik 1916, Karas 1936*a* and Ruedy 1938.

17·7555. $\dfrac{j_{n,1}^2}{4(n+1)}$

4 dec.	$n = -1(\frac{1}{10})0$	Airey 1921 (205)

Designed for interpolation.

17·76. Zeros $j'_{n,s}$ of $J'_n(x)$

5 dec.	$n = 0(1)22$	$s = 1(1)9$ to 1	Smith, Rodgers & Traub 1944
3 dec.	$n = 1$	$s = 1(1)6$	Rayleigh 1878 (266), 1896 (298)
	$n = 2, 3$	$s = 1(1)3$	Thomson 1893 (347)

From Rayleigh, *Phil. Mag.*, (4) **44**, 336, 1872.

3 dec.	$n = 1$	$s = 1(1)4$	Chree 1891 (211)
4 dec.	$\dfrac{j'_{n,1}}{\sqrt{2n}}, j'_{n,2}$	$n = -\frac{1}{2}(\frac{1}{10}) + \frac{1}{2}$	Airey 1921 (204)

17·78. Series involving Bessel Function Zeros

See also *M.T.A.C.*, **1** (7), 213–215, 262, 1944.

17·781. $\sum_{s=1}^{\infty} e^{-j_{0,s}^2 t}$ and $4\sum_{s=1}^{\infty} \dfrac{e^{-j_{0,s}^2 t}}{j_{0,s}^2}$

4 dec.	$t = 0(\cdot02)\cdot1(\cdot1)1$	No Δ	Lamb 1884 (143)
			Thomson 1893 (357)

See also Hill 1928 (71) and Buchwald 1921 (12).

17·782. $C(t) = \sum_{s=1}^{\infty} \dfrac{2e^{-j_{0,s}^2 t}}{j_{0,s} J_1(j_{0,s})}$ and $\log_{10} C(t)$

5 dec.	$t = 0(\cdot001)\cdot4$	No Δ	Olson & Schultz 1942 (876)

17·783. $\sum_{s=1}^{\infty} \dfrac{J_0(j_{0,s} r)}{j_{0,s}^p J_1(j_{0,s})} e^{-j_{0,s} z}$

4 dec.	$p = 1$	$z = 0(\cdot05)1\cdot75$	$r = 0(\cdot1)\cdot9$	No Δ	Bertram 1942 (498)
	$p = 2$	$z = 0(\cdot05)1\cdot5$			
	$p = 3, 4$	$z = 0(\cdot05)1$			

Provision is made for extension to greater values of z.

17·784. $\sum_{s=1}^{\infty} \dfrac{e^{-j_{1,s}^2 t}}{j_{1,s}^2}$ and $\sum_{s=1}^{\infty} \dfrac{e^{-j_{1,s}^2 t}}{j_{1,s}^2 J_0(j_{1,s})}$

5 dec.	$t = 0(\cdot005)\cdot01(\cdot01)\cdot1(\cdot02)\cdot2(\cdot05)\cdot5, 1$	Newman & Church 1935

17·785. $\sum_{s=3}^{\infty} (j_{n,s})^{-2p}$

5 fig.	$n = 0, 1$	$p = 2(1)10$	Butterworth 1913 (297)

17·8. Miscellaneous Bessel Function Zeros

See Art. 17·71 for references to various expansions for zeros, and Arts. 18·8 and 19·9 for zeros involving Modified Bessel Functions or Kelvin Functions and for complex zeros. See also *M.T.A.C.*, **1** (7), 215, 218, 221, 223, 1944.

17·811. Zeros $x_{n,s}$ of $Y_n(x)\,J_n(kx) - J_n(x)\,Y_n(kx)$

Kalähne 1907 (68, 81, 73) gives zeros as follows:

		k	n	s
2–4 dec.	$x_{n,s}, (k-1)x_{n,s}$	1·2, 1·5, 2	$0(\frac{1}{2})2\frac{1}{2}$	1(1)6
1–4 dec.	$(k-1)x_{n,1}$	1, 1·2, 1·5, 2(1)11, 19, 39, ∞	$0(\frac{1}{2})2\frac{1}{2}$	
3–4 dec.	$(k-1)x_{n,s}$	3 to 8 var. values	0, 1	1(1)4

These are reprinted in Jahnke & Emde 1909 (162), 1933 (274), 1938 (204). Kalähne 1906 (88) gives the items listed in the first row, except for $k = 2$, $n = 2\frac{1}{2}$.

Reinstein 1911 (88) gives

3 dec.	$x_{n,s}$	1·5, 2·5, 10, 20, ∞	0(1)6	1(1)6 to 1

Muskat, Morgan & Meres 1940 (212) gives

4 dec.	$x_{n,s}$	1·2, 1·5, 2, 3	1	1(1)6
4–5 dec.	$x_{n,s}$	5	1	1(1)7
5–6 dec.	$x_{n,s}$	10	1	1(1)8

Lowan & Hillman 1943 (208) gives

6 dec.	$x_{n,s}$; $(k-1)x_{n,s}$	$1\frac{1}{2}(\frac{1}{2})4$; $1(\frac{1}{2})4$	0	1(1)5

Carsten & McKerrow (MS.) gives

4 dec.	$x_{n,s}$	$2(\frac{1}{2})6$	0	1(1)6

Ruedy 1938 (143) gives

2 dec.	$x_{n,s}$	8; 4	$\frac{1}{3}$; $\frac{1}{4}$	1(1)3; 1(1)2

Ruedy also considers other related equations.

Karas 1936*a* (803, col. 8; 820, col. 13) gives

3 dec.	$(\frac{3}{2}x_{n,s})^2$; $(3x_{n,s})^2$	$2\sqrt{2}$, $3\sqrt{3}$; $3\sqrt{3}$	$\frac{1}{3}$	1, 2
3 dec.	$(8x_{n,s})^2$; $(2x_{n,s})^2$	9; 4	$\frac{1}{4}$	1, 2

Other related equations are also considered.

17·812. Zeros of $Y_1(x)\,J_0(kx) - J_1(x)\,Y_0(kx)$

3–4 dec.	$k = \sqrt{1\cdot1}$, 1·06(·02)1·2(·5)1·5, 2(1)5	First 6 zeros	Carsten & McKerrow (MS.)

17·813. Zeros of $Y_2(x)\,J_0(kx) - J_2(x)\,Y_0(kx)$

3 dec.	k^2 = 1·1, 1·2, 1·5, 2, 2·25, 3, 4, 9, 16, 25	First 4 zeros	Airey 1911*a* (742)

17·821. Zeros of $xJ_0(x) - kJ_1(x) = (2-k)J_1(x) - xJ_2(x)$, or of $(2-k)J_0(x) - kJ_2(x)$

For $k = 2$, the zeros are those of $J_2(x)$, see Art. 17·7411.

3 dec.	$k = \frac{1}{2};\ \frac{2}{3}$	First 4 zeros	Chree 1891 (225; 212)
4 dec.	$k = \frac{1}{2}, \frac{2}{3}$	First 10 zeros	Airey 1913*a* (291)
4 dec.	$k = \frac{2}{3}$	First 8 zeros	Ruedy 1931

17·822. Zeros of $x^2J_0(x) + kJ_2(x)$

Morse 1936 (159) gives 4-decimal values of x/π for the first 4 zeros when $k = 0$, 0·5, 1(1)6(2)10.

17·85. Zeros of $F(x) - F(kx)$, where $F(x) = \dfrac{3xY_0(x) - 2Y_1(x)}{3xJ_0(x) - 2J_1(x)}$

4 dec.	$k = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$	First 10 zeros	Airey 1913*a* (293)

Items added in Proof

17·0. Sections 17 to 20 were some of the earliest to be set up in type, and since then there has been published *A Guide to Tables of Bessel Functions*, by H. Bateman and R. C. Archibald, which forms Part 7 of *M.T.A.C.* (1, 205–308, 1944); it has not been possible at this stage to incorporate in our *Index* all the fresh material therein. Besides lists of tables (including references to many small and special tables that are beyond the scope of the *Index*) the *Guide* contains lists of graphs and diagrams.

This valuable *Guide* is in many ways complementary to the *Index* and should be available to all who work with Bessel Functions.

17·11. $J_0(x)$ and $J_1(x)$

22–25 dec.	0(1)100	v^n	Miller & Gwyther (MS.)
8 dec.	25(·1)50		Comrie (MS.)
5 dec.	0(·01)15·5	No Δ	Silberstein 1923 (143, 147)
4 dec.	0(·02)16		Glazenap 1932

17·121. $J_0(\pi x)$, $J_1(\pi x)$, $\pi J_0(\pi x)$ and $\pi J_1(\pi x)$

9 dec.	0(·2)10	Miller (MS.)

Also $\pi x J_0(\pi x)/\sqrt{x^2-1}$ and $\pi J_1(\pi x)/\sqrt{x^2-1}$ for $x = 1(\cdot 2)10$.

17·131. $J_n(x)$

11–25 dec.	$n = 0(1)N$	$x = 0(1)80$	Miller & Gwyther (MS.)

$J_{N+1}(x) < 10^{-10}$ for each x; 22 decimals at least until n exceeds x.

8 dec.	$n = 0(1)10$	$x = 25(\cdot 1)50$	Comrie (MS.)

17·15. $J_0^{(p)}(x)$

15–8 dec.	$p = 1(1)19$	$x = 0(\cdot 1)10$	New York W.P.A. (MS.)

17·221, 17·231

4 dec.	$J_{1/3}(x)$, $Y_{1/3}(x)$	$0(\cdot 02)14(\cdot 04)16$	Glazenap 1932
4 dec.	$J_{2/3}(x)$, $J_{-2/3}(x)$	$0(\cdot 02)1$	Bell 1944 (585)

17·28. Derivatives $J_n^{(p)}(x)$

Lowan, *M.T.A.C.*, **1**, 163, 1944 describes manuscript tables of derivatives for $\pm n = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$ and $x = 10(\cdot 1)25$; 15- to 8-decimal values are given for $p = 1(1)15$. For $x = 0(\cdot 1)10$, derivatives of $x^{-n}J_n(x)$ for $p = 1(1)18$ are given.

17·31, 17·321. $Y_0(x)$, $Y_1(x)$ and $Y_n(x)$

8 dec.	$n = 0(1)10$	$x = 25(\cdot 1)50$	Comrie (MS.)

17·33. $G_0(x)$

4 dec.	$0(\cdot 01)1(\cdot 1)9\cdot 9$	No Δ	Silberstein 1923 (152)

17·411. $\Lambda_n(x)$

10 dec.	$n = 2(1)20$	$x = 0(\cdot 1)10$	δ_m^2	New York W.P.A. 1943*a* (47)

The interval in x is $\cdot 05$ for $n = 2, 3;\ 4, 5;\ 6(1)8$ when $x < 10;\ 6\cdot 5;\ 1\cdot 5$.

See Lowan, *M.T.A.C.*, **1**, 163, 1944, UMT 16, item (8) for $\pm n = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$.

17·5211. $|H_n^{(1)}(x)|$ and 17·526. Arg $H_n^{(1)}(x)$

7 dec.	$n = 0(1)20$	$x = 0(\cdot 1)25$	Comrie (MS.)
7 dec.	$n = 0(1)10$	$x = 25(\cdot 1)50$	

17·531. Watson 1938 (526) gives a table of

$$\phi(2\lambda) = \frac{1}{4\lambda}[1 + \tfrac{1}{2}\pi\{J_1(\lambda)\cos\lambda + Y_1(\lambda)\sin\lambda\}] = \frac{1}{4\lambda}\{1 - \tfrac{1}{2}\pi A_1(\lambda)\}$$

to 7 decimals, with δ^2, for $\lambda = 0(\cdot 01)\cdot 4$ and for $\lambda = 0(\cdot 1)32$.

SECTION 18

BESSEL FUNCTIONS OF PURE IMAGINARY ARGUMENT, OR MODIFIED BESSEL FUNCTIONS

18·0. Introduction

Most of the remarks of Art. 17·0 apply also to this section, which is subdivided as follows:

18·1 Modified Bessel Functions of the First Kind: Integral Order
18·2 Modified Bessel Functions: Unrestricted Order
18·3 Modified Bessel Functions of the Second Kind: Integral Order
18·4 Expressions involving either $I_n(x)$ or $K_n(x)$
18·5 Auxiliary Functions for the Determination of Modified Bessel Functions. Expressions involving both $I_n(x)$ and $K_n(x)$
18·6 Expressions involving both $J_n(x)$ and $I_n(x)$
18·8 Zeros involving Modified Bessel Functions

18·1. Modified Bessel Functions of the First Kind: Integral Order

18·111†. $I_0(x)$ and $I_1(x) = I'_0(x)$

21 dec.	0(·1)6		No Δ	Aldis 1899 (218, 221)
18 dec.	6(1)11			
15 fig.	6(·1)20·3			B.A. (MS., Thompson, Miller, etc.)
13+ dec.	0(·1)10	See Art. 19·6	D^n	New York W.P.A. (MS.)
12 fig.	0(·2)6		No Δ	B.A. 1889 (29, Lodge); Gray & Mathews 1895 (285); G., M. & MacR. 1922 (309)
10 dec.	0(·01)10		No Δ	**New York W.P.A. 1943*a*** (362)
9 dec.	0(·001)5·1		Δ only	B.A. 1896 (99, I_0, Lodge); B.A. 1893 (229, I_1, Lodge)
9 dec.	0(·01)5·1		No Δ	G., M. & MacR. 1922 (303)

Gray & Mathews 1895 (282) gives the table for $I_1(x)$ only.

8; 7 dec.	0(·001)4; 4(·001)5		δ^2	**B.A. 6, 1937** (214)
6 fig.	5(1)15; 16(1)37		No Δ	Wrinch & Wrinch 1923 (847); 1924 (63)
5 fig.	0(·1)6		No Δ	Dwight 1941*a* (136)
5 fig.	0(·2)5		No Δ	Dale 1903 (82); Isherwood 1904
5 f. or 4 d.	0(·1)·2(·2)8		No Δ	Morse 1936 (334)
4–5 fig. or 4 dec.	0(·01)9·99		IΔ	Jahnke–Emde 1933 (278), 1938 (226)
4 d. or 6 f.	0(·1)6(1)11; 0(·01)5·1(·1)6(1)11	$I_0(x)$; $I_1(x)$	No Δ	Jahnke–Emde 1909 (130)
4 dec.	0(·2)8		No Δ	Hayashi 1930*b* (106, 109)

Called $U_0(x, \frac{1}{2}\pi)$ and $V_1(x, \frac{1}{2}\pi)$. From Dinnik 1922.

4 dec.	0(·2)6		No Δ	Rayleigh 1894 (367)
4 dec.	0(·1)5·9	$I_0(x)$ only	No Δ	Byerly 1906 (63)

† In Section 18, items added in proof are on page 280.

18·112. $\text{Log}_{10} I_0(x)$ and $\log_{10}\{I_1(x)/x\}$

6 dec.	0(·01)10	Δ^2	Anding 1911 (28)

18·131†. $I_n(x)$: General Tables

No differences are given with these tables.

	n	x	
18 fig.	0(1)22	0(·1)6	B.A. (MS., Thompson, etc.)
15 fig.	0(1)21	6(·1)20·3	
17–20 dec.	22(1)28 to 53	5(1)19	Miller (MS.)
12 fig.	0(1)10	0(·2)6	B.A. 1889 (29, Lodge)
10–12 fig.	11		Gray & Mathews 1895 (285)
			G., M. & MacR. 1922 (309)
6 fig.	0(1)6	5(1)15; 16(1)37	Wrinch & Wrinch 1923 (847); 1924 (63)
5 fig.	0(1)11	0(·1)6	Dwight 1941*a* (136)
5 fig.	0(1)10	0(·2)5	Isherwood 1904
5 dec. or fig.	0(1)11	0(·2)5	Dale 1903 (82)
4+ fig.	0(1)11	0(·2)6	Jahnke–Emde 1909 (159), 1933 (282), 1938 (232)
5 f. or 4 d.	0(1)2	0(·1)·2(·2)8	Morse 1936 (334)
55+ fig.	98(1)100	50	Miller (MS.)

18·132. $\text{Log}_{10} I_n(x)$

15 dec.	$n = 20, 21$	$x = 6(·1)20·5$	Thompson (MS.)

18·2. Modified Bessel Functions: Unrestricted Order

Functions of the second kind are expressible as linear combinations of functions of the first kind when n is not an integer; both are included in this sub-section.

18·21. $I_n(x)$ for $n = m+\frac{1}{2}$, m Integral

These can be expressed in terms of elementary functions, for example:

$$I_{1/2}(x) = \frac{\sinh x}{\sqrt{\frac{1}{2}\pi x}} \qquad I_{-1/2}(x) = \frac{\cosh x}{\sqrt{\frac{1}{2}\pi x}}$$

$$I_{3/2}(x) = \frac{x\cosh x - \sinh x}{x\sqrt{\frac{1}{2}\pi x}} \qquad I_{-3/2}(x) = \frac{x\sinh x - \cosh x}{x\sqrt{\frac{1}{2}\pi x}}$$

$$I_{1/2}(x) + I_{-1/2}(x) = \frac{e^x}{\sqrt{\frac{1}{2}\pi x}} \qquad I_{-1/2}(x) - I_{1/2}(x) = \frac{2}{\pi}K_{1/2}(x) = \frac{e^{-x}}{\sqrt{\frac{1}{2}\pi x}}$$

$$I_{3/2}(x) + I_{-3/2}(x) = \frac{e^x}{\sqrt{\frac{1}{2}\pi x}}\left(1 - \frac{1}{x}\right) \qquad I_{3/2}(x) - I_{-3/2}(x) = \frac{2}{\pi}K_{3/2}(x) = \frac{e^{-x}}{\sqrt{\frac{1}{2}\pi x}}\left(1 + \frac{1}{x}\right)$$

There seem to be few published tables of any of these functions, although certain multiples are listed in Arts. 18·43 and 18·44. See also Art. 18·59.

4–6 f.	$I_{\pm 1/2}(x)$	$x = 0(·1)15$	No Δ	Dinnik 1933 (7)
5 fig.	$K_n(x)$, $n = \frac{1}{2}(1)2\frac{1}{2}$	$x = 0(·1)5, 6(2)10$	No Δ	Carsten & McKerrow 1944 (816)

18·221. $I_n(x)$ for $n = \pm\frac{1}{3}$ and $\pm\frac{2}{3}$

13 dec. or fig.	10(1)25 or 13	D^n	New York W.P.A. (MS.)
10 dec.; 10 fig.	0(·01)10; 10(·01)25 or 13	δ_m^{2n}	

The upper limit is 25 for $n > 0$ and 13 for $n < 0$*.

4–6 fig.	0(·1)12	No Δ	Dinnik 1933 (13)
4 dec.	0(·2)4	No Δ	Dinnik 1914 (227)
4–5 fig.	4(·2)8		Jahnke–Emde 1933 (285), 1938 (235)

These tables contain several gross errors. Dinnik's headings are misleading.

See also Dinnik 1916 and Grishkov 1925. These Bessel Functions may also be expressed in terms of Ai x and Bi x and their derivatives, see Arts. 20·2.

18·222. $I_{-1/3}(x) - I_{1/3}(x) = \frac{\sqrt{3}}{\pi} K_{1/3}(x) \doteq 0{\cdot}55132\,890\,K_{1/3}(x)$

7 fig.	0(·2)8	Miller (MS.)

See also Art. 18·426.

18·23. $I_n(x)$ for $n = \pm\frac{1}{4}$ and $\pm\frac{3}{4}$

13 dec. or fig.	10(1)25 or 13	D^n	New York W.P.A. (MS.)
10 dec.; 10 fig.	0(·01)10; 10(·01)25 or 13	δ_m^{2n}	

The upper limit is 25 for $n > 0$ and 13 for $n < 0$*.

4–6 fig.	0(·1)15	No Δ	Dinnik 1933 (18)

Dinnik 1916 also gives 4-decimal tables. Carsten & McKerrow 1944 (817) gives values to 5 or fewer figures of $K_{1/4}(x)$, $K_{3/4}(x)$, $K'_{1/4}(x)$ and $K'_{3/4}(x)$ for $x = 0(\cdot1)5, 6(2)10$.

18·24. $I_n(x)$ for $n = \pm\frac{1}{6}$ and $\pm\frac{5}{6}$

4–6 fig.	0(·1)15	No Δ	Dinnik 1933 (24)

18·3. Modified Bessel Functions of the Second Kind: Integral Order

When n is not an integer, $K_n(x) = \frac{1}{2}\pi \operatorname{cosec} n\pi \{I_{-n}(x) - I_n(x)\}$. Tables of this kind are listed above. See also Arts. 18·44 and 18·59.

18·31. $K_0(x)$ and $K_1(x) = -K_0'(x)$

21 dec.	0(·1)6	No Δ	Aldis 1899 (219)
18 dec.	6(1)12		G., M. & MacR. 1922 (313)
9–16 dec.	5(·1)11·9		
15 dec.	0(·1)6(1)20		Bickley (MS.)
12+ fig.	5·5(·1)20·5		B.A. (MS., Bickley, etc.)
8–9 fig.	0(·01)5	δ_m^2	**B.A. 6, 1937** (266)

* For a description of tables of derivatives, etc. (up to eighteen, with 15 to 8 decimals) see Lowan, *M.T.A.C.*, **1**, 93, 1943 and **1**, 163, 1944.

7 fig.	0(·0001)·03(·001)1	δ^2	New York W.P.A. (MS.)
7 dec.	0(·1)5	No Δ	Watson 1922 (737)
5 fig.	0(·2)5	No Δ	Isherwood 1904
4 fig.	0(·1)12	No Δ	Jahnke–Emde 1909 (135)
4 dec.	0(·1)·9	No Δ	Rayleigh 1892 (179)

The functions tabulated are $-K_0(x)$ and $-xK_1(x)$. This is the earliest table of these functions we have seen.

18·32. $\frac{2}{\pi}K_0(x) = iH_0^{(1)}(ix)$ and $\frac{2}{\pi}K_1(x) = -H_1^{(1)}(ix)$

10 dec.	0(·01)10		New York W.P.A. (in progress)

Part of tables of $Y_0(ix) = -\frac{2}{\pi}K_0(x) + iI_0(x)$ and of $Y_1(ix) = -I_1(x) + \frac{2}{\pi}iK_1(x)$.

4–5 fig.	0(·01)15·99	IΔ	Jahnke–Emde 1933 (286), 1938 (236)

A value is omitted and some are given to 3 figures only at the beginning of the table.

4 fig.	0(·2)6	No Δ	Jahnke–Emde 1909 (134)

18·33. $K_n(x)$: General Tables

	n	x		
15 fig.	0(1)31	1, 3(1)13		Bickley (MS.)
15 fig.	0(1)41	2, 14(1)20		Bickley (MS.)
12 fig.	30(1)41	6(1)13		Bickley (MS.)
15 fig.	0(1)20	0(·1)6		B.A. (MS., Bickley, etc.)
12–13 fig.	0(1)20	5·5(·1)20·5		B.A. (MS., Bickley, etc.)
7+ f. or 7 d.	0(1)10	0(·1)5	No Δ	Watson 1922 (737)
5 fig.	0(1)10	0(·2)5	No Δ	Isherwood 1904; G., M. & MacR. 1922 (316)

18·4. Expressions involving either $I_n(x)$ or $K_n(x)$

See also Arts. 18·5 for functions of a more auxiliary nature.

18·4111. $i_n(x) = x^{-n}I_n(x)$

15 fig.	$n = 1(1)21$ or 22	$x = 0(·1)6$	B.A. (MS., Thompson, etc.)

See also Lowan, *M.T.A.C.*, 1, 163, 1944, for a description of MS. tables with $\pm n = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$. Up to 18 derivatives are included, with 8 to 15 decimals for $x = 0(·1)10$.

18·4112. $(\frac{1}{2}x)^{-1/3}I_{1/3}(x)$

6 dec.	0(·02)1·2	Bursian 1935 (5)

18·412. $\mathrm{Log}_{10}\, i_n(x)$

15 dec.	$n = 20, 21$	$x = 6(·1)20·5$	Thompson (MS.)

18·416. $k_n(x) = x^n K_n(x)$

	n	x		
15 fig.	1(1)20	0(·1)6		Bickley (MS.)
10+ fig.	1(1)20	6(·1)15·5		B.A. (MS., Bickley, etc.)
8–9 fig.	2(1)11 / 12(1)20	0(·01)5 / 0(·1)5	δ^2_m	B.A. (MS.)
5 dec.	1	0(·1)4·5	No Δ	Chandrasekhar 1933 (34)

Chandrasekhar 1933 (27) gives also 4-decimal values of $\frac{2}{\pi} x K_1(x)$ for $x = 0(·5)4·5$. See also Art. 18·44.

18·417. $xK_0(x)$

5 dec.	0(·1)4·5	No Δ	Chandrasekhar 1933 (34)

18·421. $e^{-x} I_n(x)$

	n	x		
10 fig.	2(1)15	4·7(·1)15·3		B.A. (MS., Thompson)
10–11 fig.	2(1)20	14·7(·1)20·3		B.A. (MS., Thompson)
9 dec.	0, 1	20(1)50	No Δ	Badellino 1939 (279)
8 dec.	0, 1	5(·01)10(·1)20	δ^2	**B.A. 6, 1937** (272)
7 dec.	0, 1	0(·02)16	No Δ	Watson 1922 (698)
7 dec.	2(1)5	0(·1)5	No Δ	Watson 1922 (736)
6 dec.	$\frac{1}{3}$	1·2(·05)4(·1)16		Bursian 1935 (6)
4 dec.	$\frac{1}{3}, \frac{2}{3}$	12·1(·1)15	No Δ	Dinnik 1933 (15)

Wagner 1930 (397) gives a 4-figure table of $2xe^{-x^2-y^2} I_0(2xy)$ for $y^2 = 0, \frac{1}{4}, 1, 4, 6, 7, 9$ and $x = 0(·2)6·6$; values $< 10^{-6}$ are omitted. Westgren 1918 (4) gives $e^{-x}\{I_0(x) + I_1(x)\}$ to 4 decimals for $\frac{1}{2}x = 0(·1)6(·2)10(·5)18(1)25$. New York W.P.A. (MS.) has 10-decimal values of $e^{-x} I_{\pm n}(x)$ for $n = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}$; the range of x is not specified (see Lowan, *M.T.A.C.*, 1, 93, 1943).

18·422. $\text{Log}_{10} e^{-x} I_n(x)$

5; 4 d.	$n = 0$	$x = 0(·1)2(·2)4(·5)6$ (1)20; 25(5)65	No Δ	Wagner 1930 (395, 409)
5 dec.	$n = 1(1)6(2)10$	$x = 0(·2)6, 10(10)40$	No Δ	Wagner 1930 (395, 409)

The last figure is often meaningless and there are many larger errors. Miller (MS.) has most of the values to 6 or 7 decimals.

18·426. $e^x K_n(x)$

	n	x		
10+ f.	2(1)20	4·5(·1)20·5		B.A. (MS., Bickley, etc.)
9 d.	0, 1	20(1)50	No Δ	Badellino 1939 (279)
8 d.	0, 1	5(·01)10(·1)20	δ^2	**B.A. 6, 1937** (272)
7 d.	0, 1, $\frac{1}{3}$	0(·02)16	No Δ	Watson 1922 (698, 714)
4 d.	0, 1	·01(·01)·1(·1)3(·2)6(·5) 10(1)20; $\sqrt{x} = ·01(·01)·1, 5(1)10$	No Δ	King 1914 (412)

The first 20 values for $n = 1$, i.e. for $x < ·1$, are to 3 decimals only.

18·43. $M(n+\frac{1}{2}, 2n+1, x) = 2^{2n} n!\, e^{\frac{1}{2}x} x^{-n} I_n(\frac{1}{2}x)$

This is a particular case of the Confluent Hypergeometric Function $M(a, \gamma, x)$ (see Art. 22·55) with $\gamma = 2a = 2n+1$. In particular,

$$M(\tfrac{1}{2}, 1, x) = e^{\frac{1}{2}x} I_0(\tfrac{1}{2}x) \qquad M(1, 2, x) = \frac{e^x - 1}{x} \qquad M(2, 4, x) = 6\left\{\frac{e^x+1}{x^2} - 2\frac{e^x-1}{x^3}\right\}$$

in which the expressions given in Art. 18·21 have been substituted for $I_{1/2}(x)$ and $I_{3/2}(x)$.

22 dec.	$M(1, 2, x)$	$\pm x = 0(1)10(10)50$	Thompson (MS.)
23 dec.	$M(\frac{1}{2}, 1, x)$	$x = 1(1)10$	
24–26 f.	$n = 0(1)50$	$-x = 10(10)100$; $x = 9, 10$	
22 fig.	$n = 47\frac{1}{2}(1)49\frac{1}{2}$	$x = 40, 50$	
22 fig.	$M(25, 50, x)$	$x = 10$	
18 fig.	$n = \frac{1}{2}(1)4\frac{1}{2}$	$x = 10$	

The last one or two figures are unreliable in these tables.

5 d. or 6 f.	$n = 0(\frac{1}{2})1\frac{1}{2}$	$x = 0(\cdot 02)\cdot 1(\cdot 05)1(\cdot 1)2(\cdot 2)3(\cdot 5)8$	No Δ	B.A. 1927 (231, etc., Airey)
4 fig.	$n = 0(\frac{1}{2})3$	$x = 1(1)6(2)10$	No Δ	Webb & Airey 1918 (138)

18·44. $T_m(x) = \dfrac{(\frac{1}{2}x)^m K_m(x)}{\sqrt{\pi}(m-\frac{1}{2})!}$, etc.

When m is half an odd integer, these functions are elementary; see Art. 18·21. In particular,

$$T_{1/2}(x) = \tfrac{1}{2}e^{-x} \qquad T_{3/2}(x) = \tfrac{1}{4}(x+1)e^{-x} \qquad T_{5/2}(x) = \tfrac{1}{16}(x^2+3x+3)e^{-x}$$

6 fig.	$m = 0(\frac{1}{2})5\frac{1}{2}$	$x = 0(\cdot 1)4(\cdot 5)16(2)40(5)120$	Elderton 1929 (195)
6 fig.	$m = 6(\frac{1}{2})11\frac{1}{2}$	$x = 0(\cdot 5)16(2)40(5)120$	Pearson 1931 (138)

Both tables also give, for the same m and x, $\log_{10} T_m(x)$ to 7 decimals and, except for $m = 0$, $\rho = \dfrac{x\,T_{m-1}(x)}{(2m-1)T_m(x)} = \dfrac{K_{m-1}(x)}{K_m(x)}$ to 5 decimals. Neither Elderton nor Pearson gives differences. See also Arts. 20·83.

18·46. $\psi(x) = e^{-x}\{(1+2x)I_0(x) + 2xI_1(x)\}$

3 d.	$\psi(x)$	$\sqrt{2x} = 0(\cdot 1)2$	No Δ	Smart 1938 (43)
2 d.	$\frac{1}{2}\sqrt{\pi}\,\psi(x)$	$2x = 0(1)10(5)20(10)100$ and 150	No Δ	Smart 1938 (219)

18·47. $\sqrt{\frac{1}{2}\pi x}\, e^{-x}\{I_{n-1}(x) + I_n(x)\}$, etc.

4 dec.	$n = 1(\frac{1}{2})2\frac{1}{2}$	$\sqrt{2x} = 0(\cdot 1)2$	No Δ	Smart 1938 (127)
4 dec.	$n = 1(\frac{1}{2})2$	$\sqrt{2x} = 0(\cdot 1)2$	No Δ	Eddington 1908 (592)

Each gives also $|P| = \dfrac{I_{1/2}+I_{3/2}}{I_0+I_1}$ to 4 decimals and $\dfrac{(I_0+I_1)^2}{I_{1/2}+I_{3/2}}$ to 3 decimals.

18·481. $\sqrt{x(1-x^2)\dfrac{I_1(x)}{I_0(x)}}$

4 dec.	$x = 0(\cdot 1)1$	No Δ	Rayleigh 1892 (179), 1896 (361)
4 dec.	$x^2 = \cdot 05, \cdot 1(\cdot 1)\cdot 9$	No Δ	Rayleigh 1878*a* (7)

18·482. $\dfrac{I_{21}(x)}{xI_{20}(x)}$

19 dec.	$x = 1(1)20$	Miller (MS.)

18·5. Auxiliary Functions for the Determination of Modified Bessel Functions. Expressions involving both $I_n(x)$ and $K_n(x)$

Most of the methods described in Arts. 17·5 for the determination of $J_n(x)$ or $Y_n(x)$ are also available, with slight variations, for $I_n(x)$ or $K_n(x)$. A few of these variations, and some additions, are dealt with below. See also Lowan, *M.T.A.C.*, **1**, 93, 1943 and **1**, 163, 1944 for details of New York W.P.A. MS. tables.

18·51. Modified addition formulæ may be used, as suggested in Arts. 17·51, together with the appropriate tables listed in Arts. 18·131 and 18·33.

18·52. Auxiliary Functions for Large x

Suitable functions have been listed in Arts. 18·42. Additional functions have been tabulated as follows.

18·521. $\sqrt{2\pi x}\, e^{-x} I_n(x)$

6 dec.	$n = 0, 1$	$x = 10(\cdot 1)50(1)200(\text{var.})10^6$	Δ	Anding 1911 (45)

Anding also tabulates $\log_{10}\{\sqrt{x}\, I_n(x)\}$.

7 dec.	$n = 0, 1$	$1/x = 0(\cdot 01)\cdot 1$	No Δ	Okaya 1937 (13)
7; 12 dec.	$n = 0, 1$	$1/x = 0(\cdot 001)\cdot 22$; $x = 5, 6, 8, 10$	Δ; no Δ	Okaya 1939 (293)

18·523. $\mathrm{Log}_{10}\{\sqrt{2\pi x}\, e^{-x} I_1(x)\}$

5 dec.	$x = 11(\cdot 1)21(1)109$	Hort 1920 (219)

18·526. $\mathrm{Log}_{10} K_0(x)$

7; 6 dec.	$x = 2(\cdot 001)4$; $4(\cdot 001)12$	No Δ	Topping & Ludlam 1926 (136)

18·53. Auxiliary Functions for Small x

Suitable functions have been listed in Arts. 18·41; others follow.

18·532. $K_0(x) = E_0(x) + F_0(x)\log_{10} x$ and $K_1(x) = \dfrac{E_1(x)}{x} + F_1(x)\log_{10} x$

B.A. **6**, 1937 (265) gives $E_0(x)$, $F_0(x)$, $E_1(x)$ and $F_1(x)$, with δ^2 in each case, to 8 decimals for $x = 0(\cdot 01)\cdot 5$. These functions are fully interpolable. New York W.P.A. has 7-decimal MS. tables, with δ^2, for $x = 0(\cdot 001)\cdot 03$.

18·56. $2\sqrt{n^2+x^2}\, I_n(x)\, K_n(x)$

This function is remarkably close to unity when $n > 0$, for all x.

	n	x		
10 dec.	0(1)5(5)20	0(1)20		Miller (MS.)
7 dec.	0, 1	0(·02)1·2(·1)5		Miller (MS.)
7 dec.	2, 3	0(·1)5		Miller (MS.)
5 dec.	1	·02, ·1, ·3(·3)·9, 1(1)10(3)16	No Δ	Yost, Wheeler & Breit 1936 (185)
5 dec.	3	·1, ·2(·2)1(·5)5	No Δ	Yost, Wheeler & Breit 1936 (185)

This table contains errors.

18·57. $I_1(x)\, K_0(x)$, $I_0(x)\, K_1(x)$, $I_{21}(x)\, K_{20}(x)$ and $I_{20}(x)\, K_{21}(x)$

15–16 dec.	$x = 1(1)20$	Miller (MS.)

18·59. $\dfrac{e^{\pm x}}{\sqrt{2\pi x}}$

These are useful in connection with asymptotic expansions of Bessel Functions. See Arts. 5·153 and 5·156 for tables of $\sqrt{2\pi x}$ and $\sqrt{2/\pi x}$. See also Art. 18·21.

6 fig.	$e^x/\sqrt{2\pi x}$	1(1)20	No Δ	Wrinch & Wrinch 1923 (848)
6 dec.	$e^{-x}/\sqrt{2\pi x}$	1(1)13	No Δ	Wrinch & Wrinch 1923 (848)
6 fig.	$e^x/\sqrt{2\pi x}$	21(1)40	No Δ	Wrinch & Wrinch 1924 (64)

18·61. $\dfrac{J_n(x)}{I_n(x)}$

6 dec.	$n = 2, 3$	$x = 1(1)15$	Wrinch & Wrinch 1924 (64)

18·8. Zeros involving Modified Bessel Functions

18·81. Zero of $I_{-n}(x)$

The function $I_{-n}(x)$ has a real zero when $2m - 1 < n < 2m$ and m is integral. Jahnke & Emde 1933 (282) gives 3-decimal values for $n = 1\frac{1}{4}$, $1\frac{1}{2}$ and $1\frac{3}{4}$. When $n = 1\frac{1}{2}$ the zero satisfies $\coth x = x$; see Art. 5·7117. Jahnke & Emde 1938 (230, Burkhardtsmaier) gives diagrams for $n < 6$. Burkhardtsmaier 1939 (77) gives 4-decimal values of $t = \sqrt{1+(x/n)^2}$ at the zeros for $n = 1·01$, $1·1(·1)1·9$, $1·99$, $3·5$, $5·5$.

18·82. Zeros of $\dfrac{dI_n(x)}{dx}$, etc.

Dinnik 1914 (227) gives 3-decimal values of the zero when $n = -1/3$ and $-2/3$, together with the corresponding minimum of $I_n(x)$ to 4 decimals. Dinnik 1933 (15, 21) gives 2-decimal zeros with 4-decimal minima when $n = -\frac{1}{3}, -\frac{2}{3}, -\frac{1}{4}$ and $-\frac{3}{4}$.

18·85. Zeros of $J_n(x)\dfrac{dI_n(x)}{dx} - I_n(x)\dfrac{dJ_n(x)}{dx} = J_{n+1}(x)\,I_n(x) + J_n(x)\,I_{n+1}(x)$

$$= J_n(x)\,I_{n-1}(x) - J_{n-1}(x)\,I_n(x)$$

4; 3 dec.	$n = 0;\ 1(1)3$	First 10 zeros	Airey 1911*d* (227)

Airey also gives an asymptotic expansion to 6 terms. .

3–4 dec.	$n = 0;\ 1, 2;\ 3$	First 5; 4; 3 zeros	Carrington 1925 (1262)
3 dec.	$n = 0;\ 1, 2;\ 3$	First 5; 4; 3 zeros	Franke 1929 (652)

Also estimates for $n = 4$, 2 zeros, and for $n = 5(1)7$, 1 zero.

3 dec. 4 dec.	$\left.\begin{matrix} x \\ x/\pi \end{matrix}\right\} n = 2$	First 10 zeros	Wrinch 1922 (501)
2–3 dec.	$n = 0;\ 1;\ 2$	First 5; 3; 2 zeros	Jahnke–Emde 1909 (160), 1933 (283), 1938 (234)

From Schulze 1907 (790); the last one or two figures are unreliable.

18·855. $J_{n+p}(2\sqrt{x})I_n(2\sqrt{x}) - J_n(2\sqrt{x})I_{n+p}(2\sqrt{x})$

$p = 1$. Ward 1913 (89, 90, 97, 100) gives 3-decimal zeros for $n = 0$ (4 zeros), 1 (3 zeros), 2 (4 zeros) and 3 (3 zeros).

$p = 2$. Ward 1913 (101, 103) gives 3-decimal zeros for $n = 0$ (3 zeros) and 1 (3 zeros).

18·86.
$$\frac{n^2(\mu-1)\left\{x\dfrac{dJ_n(x)}{dx} - J_n(x)\right\} - x^3\dfrac{dJ_n(x)}{dx}}{n^2(\mu-1)\left\{x\dfrac{dI_n(x)}{dx} - I_n(x)\right\} + x^3\dfrac{dI_n(x)}{dx}}$$

$$= \frac{(\mu-1)\left\{x\dfrac{dJ_n(x)}{dx} - n^2J_n(x)\right\} - x^2J_n(x)}{(\mu-1)\left\{x\dfrac{dI_n(x)}{dx} - n^2I_n(x)\right\} + x^2I_n(x)}$$

When $n = 0$ or 1 (only) this equation becomes

$$\frac{J_n(x)}{J_{n+1}(x)} + \frac{I_n(x)}{I_{n+1}(x)} = \frac{2}{x}(1-\mu)$$

	μ	n		
3 dec.	$\frac{1}{4}$	0, 1; 2, 3	First 9; 10 roots	Airey 1911*d*(231, 232)
3 dec.	$\frac{1}{8}$	0, 1; 2, 3	First 4; 5 roots	Airey 1911*d*(231, 232)

Airey also gives asymptotic expansions to 5 or 6 terms for $\mu = \frac{1}{4}$ and to 3 terms for $\mu = \frac{1}{8}$.

4 dec.	·33(·01)·35	0	First 7 roots	Okaya & Tobiisi 1939 (779, 777)
3 dec.	·33	0, 1; 2(1)10	First 10; 11 roots	Okaya & Tobiisi 1939 (779, 777)
		11; 12	First 2; 1 roots	Okaya & Tobiisi 1939 (779, 777)
3–2 dec.	·33	0, 1; 2(1)6	First 10; 11 roots	Colwell & Hardy 1937 (1044)

Items added in Proof

18·111. $I_0(x)$ and $I_1(x) = I_0'(x)$

12; 10 fig.	0(1)6; 7(1)10(10)100		Takagi 1939 (76)
4–6 fig.	0(1)15		Ollendorff 1929
5 fig.	12(2)20(10)100(100)1000	$I_0(x)$ only	Schelkunoff 1930 (492)

See also Art. 19·6 for derivatives $I_0^{(p)}(x)$.

18·131. $I_n(x)$: General Tables

	n	x	
12 fig.	0(1)11	0(1)6	Takagi 1939 (76)
10 fig.	0(1)23 to 54	7(1)10(10)100	Takagi 1939 (76)

There is an extension for $x > 7$, to fewer figures, for 2 to 10 further values of n; see Miller, *M.T.A.C.*, **1**, 321, 1944.

4–6 fig.	0(1)9	0(1)15	Ollendorff 1929

SECTION 19

BESSEL FUNCTIONS OF COMPLEX ARGUMENT. KELVIN FUNCTIONS

19·0. Introduction

For general complex argument see Arts. 19·6 to 19·8 and 19·96 to 19·98. The greater part of this section is concerned with the Kelvin and allied functions, i.e. Bessel Functions of argument $xi^{3/2} = x(-1+i)/\sqrt{2}$ or $xi^{1/2} = x(1+i)/\sqrt{2}$, etc., with real x. The notation we adopt is as follows:

$$\operatorname{ber}_n x + i \operatorname{bei}_n x = M_n(x) e^{i\theta_n(x)} = J_n(xi^{3/2}) = i^n I_n(xi^{1/2})$$
$$\operatorname{ker}_n x + i \operatorname{kei}_n x = N_n(x) e^{i\phi_n(x)} = i^{-n} K_n(xi^{1/2}) = \tfrac{1}{2}\pi i H_n^{(1)}(xi^{3/2})$$
$$\operatorname{her}_n x + i \operatorname{hei}_n x = H_n^{(1)}(xi^{3/2}) \qquad \operatorname{yer}_n x + i \operatorname{yei}_n x = Y_n(xi^{3/2})$$

The following values, all for $x = 1$, may help in identification, particularly of signs:

$\operatorname{ber} 1 = +0{\cdot}9844$	$\operatorname{ber}' 1 = \dfrac{\operatorname{bei}_1 1 + \operatorname{ber}_1 1}{\sqrt{2}} = -0{\cdot}0624$		$\operatorname{ber}_1 1 = -0{\cdot}3959$
$\operatorname{bei} 1 = +0{\cdot}2496$	$\operatorname{bei}' 1 = \dfrac{\operatorname{bei}_1 1 - \operatorname{ber}_1 1}{\sqrt{2}} = +0{\cdot}4974$		$\operatorname{bei}_1 1 = +0{\cdot}3076$
$M_0(1) = 1{\cdot}0155$	$\theta_0(1) = 14^\circ{\cdot}23$	$M_1(1) = 0{\cdot}5013$	$\theta_1(1) = 142^\circ{\cdot}16$
$\operatorname{ker} 1 = +0{\cdot}2867$	$\operatorname{ker}' 1 = \dfrac{\operatorname{kei}_1 1 + \operatorname{ker}_1 1}{\sqrt{2}} = -0{\cdot}6946$		$\operatorname{ker}_1 1 = -0{\cdot}7403$
$\operatorname{kei} 1 = -0{\cdot}4950$	$\operatorname{kei}' 1 = \dfrac{\operatorname{kei}_1 1 - \operatorname{ker}_1 1}{\sqrt{2}} = +0{\cdot}3524$		$\operatorname{kei}_1 1 = -0{\cdot}2420$
$N_0(1) = 0{\cdot}5720$	$\phi_0(1) = -59^\circ{\cdot}92$	$N_1(1) = 0{\cdot}7789$	$\phi_1(1) = 198^\circ{\cdot}10$

For the theory of Bessel Functions of complex argument see **Watson 1922** and Gray, Mathews & MacRobert 1922. In these, however, the Kelvin Functions receive rather summary treatment. Good lists of formulæ for Kelvin Functions are given in Dwight 1929, 1934 (161) and McLachlan 1934 (168). Many of the references listed below also give useful formulæ.

The section is subdivided as follows:

19·1. Kelvin Functions of the First Kind

See also Arts. 19·6 and 20·15.

19·11. Ber x and bei x

21 dec.	0(·1)6	No Δ	Aldis 1900 (42)
10 dec.	0(·01)10	No Δ	***New York W.P.A. 1943*a* (182)**
9 dec.	0(·1)10	Δ^n	B.A. 1912 (57, Webster)
		No Δ	Dwight 1923 (830), 1929 (813), 1934 (211)
9 dec.	0(·2)6	No Δ	*B.A. 1893 (228); Gray & Mathews 1895 (281); G., M. & MacR. 1922 (302)
6 dec.	0(·02)4·6		*Brasey 1923
5–6 fig.	1(1)10	No Δ	Dwight 1923*a* (858)
5 dec.	0(·2)6		Pedersen 1911 (504)
5 fig.	0(·1)20	No Δ	Dwight 1939 (788), 1941 (214)
3–5 fig.	0(·01)6(·1)10	IΔ to 6	*Jahnke–Emde 1933 (296, 308), 1938 (246, 258)
5; 4 dec.	0(·01)1(·02)2; 2·1(·1)6	No Δ	*Hayashi 1930*b* (112)
From Schleicher 1926.			
4 dec.	0(·1)6	No Δ	*Jahnke–Emde 1909 (137)
4 dec.	0(·2)8	No Δ	*Hayashi 1930*b* (106)
From Dinnik 1922.			
4 dec.	0(·2)6	No Δ	Bisacre 1923 (1037)
4 fig.	0(·01)21	No Δ	*Tölke 1936 (18)
4 fig.	1(1)30	No Δ	Savidge 1910 (53), 1910*a* (109); Russell 1914 (230)
4 fig. or dec.	0(·1)10	No Δ	McLachlan 1934 (177)
4– dec.	0(·5)6(2)10(5)20 and 30	No Δ	Mascart & Joubert 1896 (718)
This table is not given in the first edition.			
4– dec.	0(·5)6(2)10(5)20	No Δ	Kelvin 1889 (36, Maclean), 1890 (493); Jahnke–Emde 1909 (146); Weinreich 1914 (46)

This table contains serious errors. Mascart & Joubert 1896 gives revised values; Russell 1909 (533) or 1909*a* (591) gives further revision.

* These items tabulate ber x and $-$bei x.

19·12. $\mathrm{Ber}'x = \dfrac{\mathrm{bei}_1 x + \mathrm{ber}_1 x}{\sqrt{2}}$ and $\mathrm{bei}'x = \dfrac{\mathrm{bei}_1 x - \mathrm{ber}_1 x}{\sqrt{2}}$

9 dec.	0(·1)10	Δ^n	B.A. 1912 (63, Webster)
		No Δ	Dwight 1923 (830), 1929 (813), 1934 (211)

Savidge (B.A. 1916, p. 122) points out errors in Webster's table of bei$'x$ and gives a revised 6-decimal table for $x = 6{\cdot}5(\cdot1)10{\cdot}5$. Comrie finds all Webster's values from $x = 4{\cdot}6$ to $x = 10$ to be affected, increasingly with increasing x (the *first* decimal is incorrect at $x = 10$). Dwight gives Savidge's corrected values, but does not correct the values from $x = 4{\cdot}6$ to $x = 6{\cdot}4$. A further error: bei$'$ 3·7 should read +0·131 486...

5 dec.	0(·2)6		Pedersen 1911 (504)

5 fig.	0(·1)20	No Δ	Dwight 1939 (788), 1941 (214)
3–5 fig.	0(·01)6(·1)10	1Δ to 6	*Jahnke–Emde 1933 (298, 308), 1938 (248, 258)
5; 4 dec.	0(·01)1(·02)2; 2·1(·1)6	No Δ	Hayashi 1930b (113)

From Schleicher 1926. Ber′x and $-$bei′x tabulated.

4 fig. or dec.	0(·5)6(2)10(5)20 and 30	No Δ	Russell 1914 (231)
4 fig.	0(·01)21	No Δ	*Tölke 1936 (18)
4 fig. or dec.	0(·1)10	No Δ	McLachlan 1934 (178)
4– dec.	0(·5)6(2)10(5)20 and 30	No Δ	Mascart & Joubert 1896 (718)

This table is not given in the first edition.

4± dec.	0(·5)6(2)10(5)20	No Δ	Kelvin 1889 (36, Maclean), 1890 (493); Jahnke–Emde 1909 (146); Weinreich 1914 (46)

This table contains serious errors; see note in Art. 19·11.

* These items tabulate $-$ber′x and bei′x.

19·131. Ber$_1 x$ and bei$_1 x$

10 dec.	0(·01)10	No Δ	***New York W.P.A. 1943a** (182)
5–6 fig.	1(1)10	No Δ	Dwight 1923a (858), 1929 (814), 1934 (215)
4 dec.	0(·2)8	No Δ	*Hayashi 1930b (108)

From Dinnik 1922.

4 dec.	0(·2)6	No Δ	Bisacre 1923 (1037)

$J_1(xi^{1/2}) = -\text{ber}_1 x - i\,\text{bei}_1 x$ tabulated.

4 fig.	1(1)10	No Δ	McLachlan 1934 (181)

* These items tabulate $-$ber$_1 x$ and bei$_1 x$.

19·132. $-\sqrt{2}\,\text{ber}_1 x$ and $\sqrt{2}\,\text{bei}_1 x$

21 dec.	0(·1)6	No Δ	Aldis 1900 (43)
6 dec.	0(·02)4·6		Brasey 1923
4 dec.	0(·1)6	No Δ	Jahnke–Emde 1909 (138)

19·141. Ber$_n x$ and bei$_n x$

5± fig.	$n = 1(1)5$	$x = 1(1)10$	No Δ	Dwight 1923a (858), 1929 (814), 1934 (215)
4– fig.	$n = 1(1)5$	$x = 1(1)10$	No Δ	McLachlan 1934 (181)
4 fig.	$n = 2$	$x = 0(·01)21$	No Δ	Tölke 1936 (18)

Ber$_2 x$ and $-$bei$_2 x$ tabulated.

19·142. $-\dfrac{\text{ber}_3 x \pm \text{bei}_3 x}{\sqrt{2}}$

4 fig.	0(·01)21	No Δ	Tölke 1936 (18)

19·15. Ber$'_n x$ and bei$'_n x$

5± fig.	$n = 1(1)5$	$x = 1(1)10$	No Δ	Dwight 1923a (858), 1929 (814), 1934 (215)
4– fig.	$n = 1(1)5$	$x = 1(1)10$	No Δ	McLachlan 1934 (181)

19·2. Kelvin Functions of the Second Kind

See also Arts. 19·7 and 20·15.

19·211. Ker x and kei x

7; 8; 9 dec.	0(·1)3·8; 3·9(·1)6·4; 6·5(·1)10	No Δ	B.A. 1915 (36, Savidge); Dwight 1923 (830), 1929 (813), 1934 (213)
4 fig.	0(·01)21	No Δ	Tölke 1936 (19)
Ker x and $-$kei x tabulated.			
3–4 fig.	0(·1)10	No Δ	McLachlan 1934 (179)
4 fig.	1(1)30	No Δ	Savidge 1910 (53), 1910*a* (109); Russell 1914 (230)

19·212. $-\operatorname{her} x = -\frac{2}{\pi}\operatorname{kei} x$ and $\operatorname{hei} x = -\frac{2}{\pi}\operatorname{ker} x$

5 dec.	0(·01)1(·02)2(·2)6	No Δ	Hayashi 1930*b* (112)
From Schleicher 1926.			
3–4 fig.	0(·01)6(·1)10	IΔ to 6	Jahnke–Emde 1933 (302, 308), 1938 (252, 258)
4 fig.	0(·2)6	No Δ	Jahnke–Emde 1909 (139); Bisacre 1923 (1038)

Bisacre tabulates $-$her x and $-$hei x; he corrects J.E.'s sign for $-$her 6·0.

19·214. $Y_0(xi^{1/2}) = (\operatorname{yer} x + 2\operatorname{bei} x) - i(\operatorname{yei} x - 2\operatorname{ber} x)$

$$= \left(\operatorname{bei} x - \frac{2}{\pi}\operatorname{ker} x\right) + i\left(\operatorname{ber} x + \frac{2}{\pi}\operatorname{kei} x\right)$$

10 dec.	0(·01)10		New York W.P.A. (in progress)
5; 4 dec.	0(·01)1(·02)2; 2·2(·2)6	No Δ	Hayashi 1930*b* (112)
From Schleicher 1926.			
4 dec.	0(·2)6	No Δ	Jahnke–Emde 1909 (139)

19·221. $\operatorname{Ker}' x = \dfrac{\operatorname{kei}_1 x + \operatorname{ker}_1 x}{\sqrt{2}}$ and $\operatorname{kei}' x = \dfrac{\operatorname{kei}_1 x - \operatorname{ker}_1 x}{\sqrt{2}}$

7; 8; 9 dec.	0(·1)3·8; 3·9(·1)6·4; 6·5(·1)10	No Δ	B.A. 1915 (36, Savidge); Dwight 1923 (830), 1929 (813), 1934 (213)
4 fig.	0(·01)21	No Δ	Tölke 1936 (19)
$-$ker′x and kei′x tabulated.			
4 fig.	0(·1)10	No Δ	McLachlan 1934 (180)

19·222. $-\operatorname{her}' x = -\frac{2}{\pi}\operatorname{kei}' x$ and $\operatorname{hei}' x = -\frac{2}{\pi}\operatorname{ker}' x$

5 dec.	0(·01)1(·02)2(·2)6	No Δ	Hayashi 1930*b* (113)
From Schleicher 1926.			
3–5 fig.	0(·01)6(·1)10	IΔ to 6	Jahnke–Emde 1933 (304, 308), 1938 (254, 258)

Her′x and $-$hei′x tabulated.

19·223. $\dfrac{d}{dx}Y_0(xi^{1/2}) = (\mathrm{yer}'\,x+2\,\mathrm{bei}'\,x) - i(\mathrm{yei}'\,x-2\,\mathrm{ber}'\,x)$

$$= \left(\mathrm{bei}'\,x-\frac{2}{\pi}\,\mathrm{ker}'\,x\right) + i\left(\mathrm{ber}'\,x+\frac{2}{\pi}\,\mathrm{kei}'\,x\right)$$

5; 4 dec.	0(·01)1(·02)2; 2·2(·2)6	No Δ	Hayashi 1930*b* (113)

From Schleicher 1926.

19·225. $Y_1(xi^{1/2}) = -(\mathrm{yer}_1\,x+2\,\mathrm{bei}_1\,x) + i(\mathrm{yei}_1\,x-2\,\mathrm{ber}_1\,x)$

$$= -\left(\mathrm{bei}_1\,x-\frac{2}{\pi}\,\mathrm{ker}_1\,x\right) - i\left(\mathrm{ber}_1\,x+\frac{2}{\pi}\,\mathrm{kei}_1\,x\right)$$

10 dec.	0(·01)10	New York W.P.A. (in progress)

19·226. $\mathrm{Yer}_1\,x = -\mathrm{bei}_1\,x-\dfrac{2}{\pi}\mathrm{ker}_1\,x$ and $\mathrm{yei}_1\,x = \mathrm{ber}_1\,x-\dfrac{2}{\pi}\mathrm{kei}_1\,x$

5 dec.	0(·2)6	Kohler (MS.)

19·241. $\mathrm{Ker}_n\,x$ and $\mathrm{kei}_n\,x$

5± fig.	$n = 1(1)5$	$x = 1(1)10$	No Δ	Dwight 1929 (815, Hastings), 1934 (216)
4 fig.	$n = 1(1)5$	$x = 1(1)10$	No Δ	McLachlan 1934 (181)
4 fig.	$n = 2$	$x = 0(·01)21$	No Δ	Tölke 1936 (19)

$\mathrm{Ker}_2 x$ and $-\mathrm{kei}_2 x$ tabulated.

19·242. $\mathrm{Her}_1\,x = \dfrac{2}{\pi}\mathrm{kei}_1\,x$ and $\mp\mathrm{hei}_1\,x = \pm\dfrac{2}{\pi}\mathrm{ker}_1\,x$

5–6 fig.	0(·2)6	No Δ	Jahnke–Emde 1909 (140)

Bisacre corrects Jahnke & Emde's sign for $\mathrm{her}_1\,2{\cdot}4$.

4 fig.	0(·2)6	No Δ	Bisacre 1923 (1038)

19·243. $-\dfrac{\mathrm{ker}_3\,x\pm\mathrm{kei}_3\,x}{\sqrt{2}}$

4 fig.	0(·01)21	No Δ	Tölke 1936 (19)

19·25. $\mathrm{Ker}'_n\,x$ and $\mathrm{kei}'_n\,x$

5± fig.	$n = 1(1)5$	$x = 1(1)10$	No Δ	Dwight 1929 (815, Hastings), 1934 (216)
4 fig.	$n = 1(1)5$	$x = 1(1)10$	No Δ	McLachlan 1934 (181)

19·3. Kelvin Functions: Modulus and Phase

The quantities tabulated are

$$M_n(x) = \sqrt{\mathrm{ber}_n^2 x+\mathrm{bei}_n^2 x} \qquad \theta_n(x) = \tan^{-1}(\mathrm{bei}_n x/\mathrm{ber}_n x)$$

$$N_n(x) = \sqrt{\mathrm{ker}_n^2 x+\mathrm{kei}_n^2 x} \qquad \phi_n(x) = \tan^{-1}(\mathrm{kei}_n x/\mathrm{ker}_n x)$$

It should be noted that

$$\mathrm{ber}'^2 x+\mathrm{bei}'^2 x = \mathrm{ber}_1^2 x+\mathrm{bei}_1^2 x \qquad \theta_1(x)-45^\circ = \tan^{-1}(\mathrm{bei}'\,x/\mathrm{ber}'\,x)$$

$$\mathrm{ker}'^2 x+\mathrm{kei}'^2 x = \mathrm{ker}_1^2 x+\mathrm{kei}_1^2 x \qquad \phi_1(x)-45^\circ = \tan^{-1}(\mathrm{kei}'\,x/\mathrm{ker}'\,x)$$

The phases are usually given in degrees and decimals.

19·31. $M_n(x)$ and $\theta_n(x)$, etc. for $n = 0$ and 1

No differences are given except in B.A. 1923, which gives δ^4 only.

6–5 dec. 0°·00001	$M_0(x), M_1(x)$ $\theta_0(x), \theta_1(x) - 180°$	0(·1)10	B.A. 1923 (295, Alger)

The changes from 6 to 5 decimals are between $x = 5{\cdot}7$ and $x = 5{\cdot}9$. $M_1(x)$ and $\theta_1(x) - 180°$ are affected by Webster's error, see Art. 19·12.

5 fig. 0°·001	$M_0(x), M_1(x)$ $\theta_0(x), \theta_1(x) - 45°$	0(·1)20	Dwight 1939 (789), 1941 (218)
5 fig. 0°·001	$M_0(x), M_1(x)$ $\theta_0(x), \theta_1(x) - 180°$	0(·1)10	Kennelly, Laws & Pierce 1915*a* (1795), 1915*b* (1999)

There are several errors, up to 50 units in the final digit.

4 fig. 4 dec. 0°·01	$M_0(x)$ $\log_{10}\{\sqrt{x}M_0(x)\}$ $\theta_0(x)$	0(·05)2(·1)4(·5)6(1) 12(2)20(5)45	McLachlan 1934 (182)
4 fig. 4 dec. 0°·01	$M_1(x)$ $\log_{10}\{\sqrt{x}M_1(x)\}$ $\theta_1(x)$	0(·05)2·7(·1)3·8, 4(·5) 8(1)12(2)20(5)50, also 4·25	McLachlan 1934 (183)
5 fig. 1″	$M_0(x), M_1(x)$ $\theta_0(x), \theta_1(x) - 180°$	0(·2)6	Bisacre 1923 (1037)
4–5 fig. 0^g·01	$M_0(x), M_1(x)$ $-\theta_0(x), 200^g - \theta_1(x)$	0(·1)10	Jahnke–Emde 1933 (312), 1938 (262)

There are some errors in $M_0(x)$; one, at $x = 5{\cdot}3$, is a major error.

4 dec. 0^r·0001	$\log_e\{\sqrt{2\pi x}M_0(x)\}$ $\theta_0(x)$	1(1)30	Savidge 1910 (54), 1910*a* (110)

19·361. $N_n(x)$, etc. for $n = 0$ and 1

5 fig.	$N_0(x), N_1(x)$	0(·1)10	No Δ	McLachlan & Meyers 1934 (621)
4 dec.	$\log_e\left\{\sqrt{\frac{2x}{\pi}}N_0(x)\right\}$	1(1)30	No Δ	Savidge 1910 (54), 1910*a* (110)

19·362. $\frac{2}{\pi}N_n(x)$ for $n = 0$ and 1

4–5 fig.	0(·1)10	No Δ	Jahnke–Emde 1933 (312), 1938 (262)
4 fig.	0(·2)6	No Δ	Bisacre 1923 (1038)

19·37. $\phi_n(x)$, etc. for $n = 0$ and 1

No differences are given.

0°·001	$\phi_0(x), \phi_1(x)$	0(·1)10	McLachlan & Meyers 1934 (621)
1″	$\phi_0(x) + 90°, \phi_1(x) + 90°$	0(·2)6	Bisacre 1923 (1038)
0^g·01	$-\phi_0(x) - 100^g, -\phi_1(x) - 100^g$	0(·1)10	Jahnke–Emde 1933 (312), 1938 (262)
0^r·0001	$\phi_0(x)$	1(1)30	Savidge 1910 (54), 1910*a* (110)

19·4. Expressions involving Kelvin Functions

We adopt the notation, mainly from B.A. 1916 (Savidge), for mixed functions as follows:

$$X \equiv Xb(x) = M_0^2(x) = \mathrm{ber}^2 x + \mathrm{bei}^2 x \qquad Xk(x) = N_0^2(x) = \mathrm{ker}^2 x + \mathrm{kei}^2 x$$
$$V \equiv Vb(x) = M_1^2(x) = \mathrm{ber}'^2 x + \mathrm{bei}'^2 x \qquad Vk(x) = N_1^2(x) = \mathrm{ker}'^2 x + \mathrm{kei}'^2 x$$
$$Z \equiv Zb(x) = \mathrm{ber}\, x\, \mathrm{ber}' x + \mathrm{bei}\, x\, \mathrm{bei}' x \qquad Zk(x) = \mathrm{ker}\, x\, \mathrm{ker}' x + \mathrm{kei}\, x\, \mathrm{kei}' x$$
$$W \equiv Wb(x) = \mathrm{ber}\, x\, \mathrm{bei}' x - \mathrm{bei}\, x\, \mathrm{ber}' x \qquad Wk(x) = \mathrm{ker}\, x\, \mathrm{kei}' x - \mathrm{kei}\, x\, \mathrm{ker}' x$$
$$Zb + i\,Wb = M_0 M_1 \exp i(\theta_1 - \theta_0 - \tfrac{1}{4}\pi) \qquad Zk + i\,Wk = N_0 N_1 \exp i(\phi_1 - \phi_0 - \tfrac{1}{4}\pi)$$
$$Vr(x) = \mathrm{ber}' x\, \mathrm{ker}' x + \mathrm{bei}' x\, \mathrm{kei}' x \qquad Vu(x) = \mathrm{bei}' x\, \mathrm{ker}' x - \mathrm{ber}' x\, \mathrm{kei}' x$$
$$Vr + i\,Vu = M_1 N_1 \exp i(\theta_1 - \phi_1)$$

A typical set of values of these, and other, functions is given in B.A. 1916 (117, Savidge) to 6 figures for $x = 6$. Formulæ for similar expressions involving Bessel Functions of order n are given by J. C. Costello (*Phil. Mag.*, (7) **21**, 308–318, 1936).

19·41. $Xb(x)$, $Vb(x)$, $Zb(x)$ and $Wb(x)$

6 fig.	0(·2)10	No Δ	B.A. 1916 (118, Savidge)
2–6 dec.	0(·2)6		Pedersen 1911
4 fig.	0(1)30	No Δ	Savidge 1910 (55), 1910*a* (111); Russell 1914 (509, not *Wb*)

19·42. $Xk(x)$, $Vk(x)$, $Zk(x)$ and $Wk(x)$

6 fig.	All	0(·2)10	No Δ	B.A. 1916 (119, Savidge)
4 fig.	*Xk*, *Vk*	0(1)30	No Δ	Savidge 1910 (57), 1910*a* (113); Russell 1914 (232); Gray 1921 (282)
4 fig.	*Zk*, *Wk*	0(1)5	No Δ	Savidge 1910 (58), 1910*a* (114)

19·43. $Vr(x)$ and $Vu(x)$

8 dec.	$x = 0(\cdot 02)10(\text{var.})20$		Airey (MS.)
6 fig.	0(·2)10 Errors: $x = 2\cdot 2$	No Δ	B.A. 1916 (120, Savidge)
4 fig.	0(1)30	No Δ	Savidge 1910 (57), 1910*a* (113); Russell 1914 (232); Gray 1921 (282)

19·46. $\dfrac{V}{X}$, $\dfrac{Z}{X}$, $\dfrac{W}{X}$, etc.

In addition to the references listed, most of which give formulæ, a useful expansion of $J_{n+1}(xi^{3/2})/J_{n-1}(xi^{3/2})$, to which W/X, etc. are closely related, is given, with numerical coefficients for $n = 1$ and 2, by Butterworth 1913 (297).

5 dec.	$\dfrac{V}{X}, \dfrac{Z}{X}, \dfrac{W}{X}$	0(·2)10, ∞	No Δ	B.A. 1916 (120, Savidge)

Savidge also gives stationary points for V/X and W/X: x to 4 decimals, function values to 5 decimals.

5 dec.	$\frac{x}{4}\frac{W}{X}$	0(·1)10	No Δ	Bloch 1944 (704)
4 dec.	$\frac{V}{X}, \frac{Z}{X}, \frac{W}{X}$	0(1)30, ∞	No Δ	Savidge 1910 (56), 1910a (112)
4 dec.	$\frac{Z}{X}$	3(·2)7	No Δ	Russell 1910 (605), 1910a (447)
6 dec.	$\frac{Z}{X}, \frac{W}{X}$	3(1)6		Butterworth 1913 (297)
4–5 dec.	$\frac{2}{x}\frac{Z}{X}, \frac{2}{x}\frac{W}{X}$	0(·5)6(2 or 5)20		Fischer 1926
4; 3 fig.	$\frac{2}{x}\frac{Z}{X}; \frac{x}{4}\frac{Z}{X}$	0(·5)5	No Δ	Butterworth 1921 (64, 73)

19·47. $\frac{Z}{V}, \frac{W}{V}$, etc.

6 dec.	$\frac{x}{2}\frac{Z}{V}, \frac{x}{2}\frac{W}{V}$	0(·1)10	δ^4 only	B.A. 1923 (296, Alger)

Affected by Webster's error; see Art. 19·12.

5 dec.	$\frac{x}{2}\frac{Z}{V}$	0(·1)10	No Δ	Bloch 1944 (704)
5 dec.	$\frac{Z}{V}, \frac{W}{V}$	0(·1)5(·2)10(·5) 15(var.)100, ∞	Δ^2	Rosa & Grover 1912 (226)
5 dec.	$\frac{4}{x}\frac{Z}{V}, \frac{x}{2}\frac{W}{V}$			
5 dec.	$\frac{4}{x}\frac{Z}{V}, \frac{x}{2}\frac{W}{V}$	0(·1)5(·2)10 (var.)100, ∞	No Δ	Russell 1914 (233) Gray 1921 (283)
5 dec.	$\frac{Z}{V}, \frac{W}{V}$	0(·2)10, ∞	No Δ	B.A. 1916 (120, Savidge)
6 dec.	$\frac{Z}{V}, \frac{W}{V}$	5(1)8		Butterworth 1913 (297)
4 dec.	$\frac{Z}{V}, \frac{W}{V}$	0(1)30, ∞	No Δ	Savidge 1910 (56), 1910a (112)
4 dec.	$\frac{W}{V}, \frac{x}{2}\frac{W}{V}$	0(·5)6(2)10(5)20	No Δ	Kelvin 1889 (36, Maclean), 1890 (493) Gray 1893 (855)

This table contains errors; there is some revision in the two following items.

4 dec.	$\frac{x}{2}\frac{W}{V}$	0(·5)6(2)10(5)20, 30	No Δ	Mascart & Joubert 1896 (718)

This table is not given in the first edition.

4 dec.	$\frac{W}{V}$	·5(·5)2, 5·5, 6(2) 10(5)20(10)50	No Δ	Russell 1909 (534), 1909a (592)

4 fig.	$\frac{4}{x}\frac{W}{V}-\frac{8}{x^2}$	o(·5)5	No Δ	Butterworth 1921 (64, 73)
3 fig.	$\frac{x}{2}\frac{W}{V}-1$			
4 fig.	$\frac{\text{ber}'x}{V}, \frac{\text{bei}'x}{V}$	o(·5)6(2)10(5)20	No Δ	Weinreich 1914 (47)
3 dec.	$\frac{Z}{V}, \frac{W}{V}$			

This table contains errors. Weinreich also gives (p. 55) a number of values of A (to 4 figures) and of ϕ (in angular form to 1′), where

$$Ae^{i\phi} = \frac{(\text{ber}\,y\ \text{ber}'x + \text{bei}\,y\ \text{bei}'x) - i(\text{ber}\,y\ \text{bei}'x - \text{bei}\,y\ \text{ber}'x)}{\text{bei}'^2x + \text{ber}'^2x}$$

2–3; 3–4 d.	$\frac{x}{2}\frac{Z}{V}, \frac{x}{2}\frac{W}{V}$	$x/2\sqrt{2} = 0(\cdot1)\cdot9$; $x = \cdot5(\cdot5)6(2)10(5)20$	No Δ	Jahnke–Emde 1909 (147)

This table contains serious errors.

2–3 dec.	$x\frac{Z}{V}, x\frac{W}{V}$	$x^2 = \frac{1}{2}(\frac{1}{2})3$	No Δ	Thomson 1893 (276) Gray & Mathews 1895 (155) G., M. & MacR. 1922 (170)

19·48. $\frac{1}{2}x\sqrt{\frac{X}{V}} = \frac{1}{2}x\frac{\sqrt{W^2+Z^2}}{V}$ and $\tan^{-1}\frac{Z}{W}$, etc.

6 dec.	$\frac{x}{2}\sqrt{\frac{X}{V}} = \frac{x}{2}\frac{M_0}{M_1}$	o(·1)10	δ^4 only	B.A. 1923 (296, Alger)
0°·00001	$\tan^{-1}\frac{Z}{W} = \theta_0 - \theta_1 + 135°$			

Affected by Webster's error; see Art. 19·12.

4–5 fig.	$\sqrt{\frac{X}{V}} = \frac{M_0}{M_1}, \frac{x}{2}\frac{M_0}{M_1}, \frac{2}{x}\frac{M_0}{M_1}$ $\sqrt{\frac{Xk}{Vk}} = \frac{N_0}{N_1}, \frac{x}{2}\frac{N_0}{N_1}, \frac{2}{x}\frac{N_0}{N_1}$	o(·1)10	No Δ	Jahnke–Emde 1933 (316), 1938 (266)
0g·01	$\theta_1 - \theta_0 - 200^g, \phi_1 - \phi_0$			

19·6. Bessel Functions of the First Kind: General Complex Argument

$$J_n(re^{i\theta}) = U_n(r, \theta) + iV_n(r, \theta)$$

13+ d.	U_0, V_0	$\theta = 0(5°)90°$	$r = 0(\cdot1)10$	D^n	New York W.P.A. (MS.)
13+ d.	U_1, V_1				

See Lowan, *M.T.A.C.*, **1**, 163, 1944 and New York W.P.A. 1943*a* (xxiv) for a description of the tables. The 18 derivatives, with respect to r, are given to a number of places, varying from 8 to 15, adequate for subtabulation to about 12 decimals.

10 d.	U_0, V_0	$\theta = 0(5°)90°$	$r = 0(\cdot01)10$	**New York W.P.A. 1943*a***
10 d.	U_1, V_1			

4 dec.	U_0, V_0	$\theta = \frac{\pi}{16}\left(\frac{\pi}{16}\right)\frac{\pi}{2}$	$r = 0(\cdot 2)8$	Hayashi 1930*b* (106), from Dinnik 1922
4 dec.	U_1, V_1			
4 dec.	V_0, U_1	$\theta = \frac{\pi}{2} - \cdot 001$		

For errors see Lowan & Blanch 1941.

19·7. Bessel Functions of the Second Kind: General Complex Argument

Lowan, *M.T.A.C.*, 1, 165, 1944, describes a 10-place New York W.P.A. table of $Y_0(z)$ and $Y_1(z)$ with $z = re^{i\theta}$ for $\theta = 0(5^\circ)90^\circ$ and $r = 0(\cdot 01)10$. In *M.T.A.C.*, 1, 164, 1944 he describes a table giving 18 derivatives with respect to r of the functions

$$S_0(z) = Y_0(z) - \frac{2}{\pi}(\log_e z + \gamma) J_0(z)$$

$$S_1(z) = Y_1(z) - \frac{2}{\pi}(\log_e z + \gamma) J_1(z) + \frac{2}{\pi z}$$

The derivatives are given to a number of places, varying from 8 to 15, adequate for subtabulation to about 12 decimals.

19·8. Bessel Functions of the Third Kind: General Complex Argument

Jahnke & Emde 1933 (198), 1938 (132) give altitude charts for $H^{(1)}_{3\frac{1}{2}}(x+iy)$ with $-5 < x < 5$, $-4 < y < 3$; also for $H^{(1)}_0(x+iy)$ with $-4 < x \leqslant 2$, $-1\cdot 5 < y < 1\cdot 5$, where $\arg x+iy$ lies between $-\pi$ and $+\pi$.

19·9. Zeros of Kelvin Functions. Complex Zeros of Bessel Functions

19·91. Zeros of Kelvin Functions

5 dec.	First 10 zeros of $\mathrm{ber}\,x$, $\mathrm{bei}\,x$, $\mathrm{ber}'\,x$, $\mathrm{bei}'\,x$, $\mathrm{ker}\,x$, $\mathrm{kei}\,x$, $\mathrm{ker}'\,x$ and $\mathrm{kei}'\,x$	B.A. 1927 (254, Airey)

19·96. Complex Zeros of $J_{-n}(z)$

Jahnke & Emde 1938 (231, Burkhardtsmaier) gives diagrams showing x/n and y/n such that $J_{-n}(x+iy) = 0$. For $2 < n < 4$, one root is shown, and for $4 < n < 5$, two roots, except when n is within about 0·02 of an integer. More complete diagrams are given by Burkhardtsmaier 1939 (78), who gives also 4-decimal values of $\sqrt{1 - z^2/n^2}$ for a complex zero when $n = 2\cdot 1$, $2\cdot 5$ or $3\cdot 5$.

19·97. Zeros of $H^{(1)}_n(z)$ and $H^{(2)}_n(z)$

Jahnke & Emde 1933 (293), 1938 (243) gives a diagram covering all zeros for which $|x| < 4$, $|y| < 4$ where $z = x+iy$.

19·98. Root of $\frac{d}{dz}H^{(2)}_1(z) = 0$ or $H^{(2)}_1(z) = zH^{(2)}_0(z)$

$z = re^{i\theta} = 0\cdot 5012 + i0\cdot 6435$ $\quad r = 0\cdot 8156$ $\quad \theta = 57^{g}\cdot 87 = 52^\circ\cdot 085 = 0^{r}\cdot 9091$

Jahnke–Emde 1909 (142), 1933 (292), 1938 (242)

SECTION 20

MISCELLANEOUS BESSEL AND RELATED FUNCTIONS

20·0. Introduction

In this section we list Bessel Functions of arguments which are not linear functions of the usual argument, and also multiples of such functions. We also list here several functions that are closely related to Bessel Functions, although they do not satisfy differential equations reducible to the standard form. For formulæ, the textbooks mentioned in Art. 17·0 may be consulted.

The subdivision of the section is as follows:

20·1	Bessel–Clifford Functions
20·2	The Airy and Allied Integrals
20·3	Other Bessel Functions of Non-linear Argument
20·5	Lommel–Weber Functions, $\Omega_n(x)$; Struve Functions, $\mathbf{H}_n(x)$; etc.
20·6	The Fresnel Integrals
20·7	Lommel's Functions of Two Variables
20·8	Integrals involving Bessel Functions
20·9	Neumann's Polynomials, $O_n(t)$, and Schläfli's Polynomials, $S_n(t)$

20·1. Bessel–Clifford Functions

$$C_0(x) = J_0(2\sqrt{x}) \qquad C_1(x) = \frac{1}{\sqrt{x}}J_1(2\sqrt{x}) \qquad C_n(x) = (-)^n \frac{d^n}{dx^n}C_0(x) = x^{-\frac{1}{2}n}J_n(2\sqrt{x})$$

The use of these functions was advocated by Greenhill, *Engineering*, **107**, 334, 1919; *Phil. Mag.*, (6) **38**, 501–528, 1919; *Engineering*, **109**, 851, 1920.

6 dec. $\quad n = 0, 1 \quad x = 0(\cdot 02)20 \quad$ No Δ B.A. 1924 (287, Airey)

20·15†. Michell Functions See *Items added in Proof.*

20·2. The Airy and Allied Integrals

The notation, suggested by H. Jeffreys and adopted in a forthcoming B.A. part-volume, now in the press, is

$$\mathrm{Ai}\,x = \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{1}{3}t^3 + xt\right)dt$$

$$\mathrm{Bi}\,x = \frac{1}{\pi}\int_0^\infty \left\{\exp\left(-\tfrac{1}{3}t^3 + xt\right) + \sin\left(\tfrac{1}{3}t^3 + xt\right)\right\}dt$$

These are two independent solutions of the differential equation $y'' = xy$. Both oscillate about zero as $x \to -\infty$, and tend respectively to 0 and ∞ as $x \to +\infty$. These solutions and their derivatives can be expressed in terms of Bessel Functions of order $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$, of argument $\frac{2}{3}x^{3/2}$; explicit formulæ will be given in the B.A. part-volume mentioned above.

† In Section 20, items added in proof are on pages 306–308.

20·21†. Ai x, Bi x and Derivatives

10–11 fig.	Ai x	0(·1)10	v^n	Miller (MS.)
10–12 dec.	Ai x	−20·25(·05)−9·8(·1)0	v^n	Miller (MS.)
10–11 dec.	Ai x, Ai′ x	−7·55(·1)+2·55		Miller (MS.)
8 dec.	Ai x, Ai′ x	−20(·01)+2	δ^2_m, sp.	B.A. (in press)

These have been obtained from the previous items by subtabulation; about 10 decimals are available in machine-script. A 7-decimal table by H. Jeffreys has been incorporated.

14–15 fig.	Bi x	0(·1)2·8	v^n	Miller (MS.)
11–12 fig.	Bi x	−10(·1)0	v^n	Gwyther (MS.)
8 dec.	Bi x	−10(·1)+2·5	v^n	B.A. (in press)

This has been cut down from the previous items.

20·221. Airy's Integral $W(m) = \int_0^\infty \cos \frac{1}{2}\pi(w^3 - mw)\,dw$

$$W(m) = (\tfrac{2}{3}\pi^2)^{1/3}\,\mathrm{Ai}(-\pi^{2/3}m/12^{1/3}) \doteqdot 1{\cdot}87385\ 57776\ \mathrm{Ai}(-0{\cdot}93692\ 78888\,m)$$

8 dec.	−5·6(·2)+5·6	Miller (MS.)
5 dec.	−4(·2)+4	*Airy 1838*a* (390)
4 dec.	−4(·4)+4	*Rayleigh 1879 (405); Mascart 1889 (408)
5 dec.	−5·6(·2)+5·6	*Airy 1849 (598); Hogner 1923 (37)
5 dec.	−5·6, −5(1)−1, −·2(·2)+5·6	*Mascart 1889 (397)

* No differences are given. Airy's tables, from which all these are derived, are subject to end-figure errors up to about 2 units of the fifth decimal. Four exceed this amount:

$W(-3{\cdot}6) = {\cdot}00618$ $\quad W(3{\cdot}4) = {\cdot}78021$ $\quad W(3{\cdot}8) = {\cdot}66045$ $\quad W(4{\cdot}0) = {\cdot}47419$

Airy 1838*a* (402) gives 7-decimal values, the last two figures being practically meaningless.

Taylor 1930 (105) reproduces part of Airy's table, giving $W(m)$ to 4 decimals, but in terms of the argument $\eta = \pi^{2/3}m/12^{1/3}$; the argument m is not given and an incorrect value of $\left(\frac{\pi^2}{12}\right)^{1/3}$ has been used. $\frac{dW}{d\eta}$ and $\frac{dW}{d\eta}\Big/\left(\frac{dW}{d\eta}\right)_{\eta=0}$ are also given, to 3 decimals, for some of the arguments.

20·222. $\left(\frac{3}{\pi^2}\right)^{1/3} W(m) \doteqdot 0{\cdot}67236\ 821\ W(m)$

5 dec.	$m = -4(\cdot 2)+4$	No Δ	Nicholson 1909 (16)

Major errors: T. I, at $m = -0{\cdot}8$; T. II, at $m = 0$, 3·8 and 4.

20·224. $W^2(m)$

5+ dec.	−4(·2)+4	No Δ	Airy 1838*a* (390)
4 dec.	−5·6, −5(1)−1, −·2(·2)+5·6	No Δ	Mascart 1889 (397)
3 dec.	−2(·2)+13·6	No Δ	Pernter 1906 (517); Humphreys 1919 (481), 1920 (474)

20·225†. Hogner's Integral See *Items added in Proof*, page 307.

20·226. The Incomplete Integral $\int_0^x \cos \frac{1}{2}\pi(w^3 - mw)\,dw$

When $x = \infty$ this becomes $W(m)$. See also Steward 1925 (148, 149).

5+ dec.	$x = 1, 1{\cdot}26, 1{\cdot}44, 1{\cdot}58, 1{\cdot}70, 1{\cdot}92, 2$; $m = -4(\cdot 2)+4$	No Δ	Airy 1838*a* (402)
4 dec.	$x = 1, 1{\cdot}26, 1{\cdot}44$ $m = -4(\cdot 4)+4$	No Δ	Rayleigh 1879 (405); Mascart 1889 (408)

Airy gives, in fact, the integrals between limits 0 and 1, 1 and 1·26, etc. from which integrals between 0 and 1·26, etc. can readily be obtained; he gives 7 decimals, the last one or two, or more near $m = +4$, being unreliable. The values of x are approximations to the cube roots of 1(1)5, 7 and 8.

20·23. Log Ai x and log Bi x

12 dec.	$-\log_e \mathrm{Ai}\, x$	20(1)80		Miller, etc. (MS.)
10+ dec.	$-\log_{10} \mathrm{Ai}\, x$	0(·1)25(1)80		Miller, etc. (MS.)
8 dec.	$\log_{10} \mathrm{Ai}\, x$	0(·1)25(1)75	δ^2_m, sp.	B.A. (in press, Miller, etc.)

Both direct and complementary forms are available in MS.

12 dec.	$\log_{10} \mathrm{Bi}\, x$	0(·1)10·1		Miller, etc. (MS.)
8 dec.	$\log_{10} \mathrm{Bi}\, x$	0(·1)10	δ^2_m, sp.	B.A. (in press, Miller, etc.)

20·24. $\dfrac{\mathrm{Ai}'x}{\mathrm{Ai}\,x}$ and $\dfrac{\mathrm{Bi}'x}{\mathrm{Bi}\,x}$

12; 10+ dec.	$\mathrm{Ai}'x/\mathrm{Ai}\,x$	20(1)80; 0(·1)25		Miller, etc. (MS.)
7 dec.	$\mathrm{Ai}'x/\mathrm{Ai}\,x$	0(·1)25(1)75	δ^2_m, sp.	B.A. (in press, Miller, etc.)
12 dec.	$\mathrm{Bi}'x/\mathrm{Bi}\,x$	−1(·1)+10·1		Miller, etc. (MS.)
7 dec.	$\mathrm{Bi}'x/\mathrm{Bi}\,x$	0(·1)10	δ^2_m, sp.	B.A. (in press, Miller, etc.)

20·25. Solutions of $\dfrac{d^2y}{dx^2} = -9ixy$

Rayleigh 1915 (333) gives 3- to 4-figure values, for $x = 0(\cdot 1)2{\cdot}5$, of $S_1 = s_1 + it_1$ and $S_2 = s_2 + it_2$, two solutions of the differential equation $y'' = -9ixy$ such that when $x = 0$, $S_1 = 1$, $S_1' = 0$, $S_2 = 0$, $S_2' = i$. These may be related to $\mathrm{Ai}(3^{2/3}ix)$ and $\mathrm{Bi}(3^{2/3}ix)$. All values for $x = 0{\cdot}7$ are in error; these should be 0·9471, −0·5122, 0·1796, 0·6868 respectively. Also for $x = 1{\cdot}1$, $t_1 = 1{\cdot}865$.

Rayleigh 1915 (337) also gives 3- to 4-figure values, for $x = {\cdot}1(\cdot 2)2{\cdot}5$, of $s_1^2 - t_1^2$, $(s_1^2 + t_1^2)^2$ and $(s_1^2 - t_1^2)/(s_1^2 + t_1^2)^2$.

20·28. Auxiliary Functions

These are defined by

$$\mathrm{Ai}\,x = F(x)\sin\chi(x) \qquad \mathrm{Ai}'x = G(x)\sin\psi(x)$$
$$\mathrm{Bi}\,x = F(x)\cos\chi(x) \qquad \mathrm{Bi}'x = G(x)\cos\psi(x)$$

20·281. $F(x)$ and $G(x)$

12 dec.	$-x = 9(1)83,\ 10(\cdot 1)11\cdot 5$		Miller, etc. (MS.)
11 dec.	$-x = -3\cdot 1(\cdot 1)+30$		Miller, etc. (MS.)
8 dec.	$-x = 0(\cdot 1)30(1)80$	δ^2_m, sp.	B.A. (in press, Miller, etc.)
7 dec.	$x = 0(\cdot 1)2\cdot 5$	δ^2_m, sp.	B.A. (in press, Miller, etc.)

20·282. $\chi(x)$ and $\psi(x)$

9–10 d. of 1°	$-x = -2\cdot 8(\cdot 1)+11\cdot 4,\ 12(1)85$		Miller, etc. (MS.)
9 dec. of 1°	$-x = 10(\cdot 1)30$		Miller, etc. (MS.)
6 dec. of 1°	$-x = -2\cdot 5(\cdot 1)+30(1)80$	δ^2_m, sp.	B.A. (in press, Miller, etc.)

20·29. Zeros and Turning-Values

Let a_s, b_s, a'_s and b'_s denote the sth zeros (counting in the sense $x \to -\infty$) of Aix, Bix, Ai$'x$ and Bi$'x$ respectively. See also Watson 1922 (751), where

$$s_s = \tfrac{2}{3}(-a_s)^{3/2} \qquad d_s = \tfrac{2}{3}(-b_s)^{3/2}$$

are given to 7 decimals, for $s = 1(1)40$ (see Art. 17·753). We also note that

$$s'_s = \tfrac{2}{3}(-b'_s)^{3/2} \qquad d'_s = \tfrac{2}{3}(-a'_s)^{3/2}$$

where s'_s and d'_s are zeros of $J_{-2/3}(x)+J_{2/3}(x)$ and $J_{-2/3}(x)-J_{2/3}(x)$ respectively.

20·291†. a_s, a'_s, Ai$'a_s$ and Aia'_s

10–12 dec.	$s = 1(1)55$	Miller, Gwyther, etc. (MS.)
8 dec.	$s = 1(1)50$	B.A. (in press, Miller, Gwyther, etc.)

20·292. b_s, b'_s, Bi$'b_s$ and Bib'_s

10–12 dec.	$s = 1(1)22$	Miller, Gwyther, etc. (MS.)
8 dec.	$s = 1(1)20$	B.A. (in press, Miller, Gwyther, etc.)

20·293. Zeros of $W(m)$ and $W'(m)$

Stokes 1851 (181) gives the first 50 zeros of Airy's Integral $W(m)$ and the first 10 of its derivative $W'(m)$, to 4 decimals, each with first differences. The only error exceeding a final unit is in the first zero of $W'(m)$, which should read 1·0874.

Mascart 1889 (400) reproduces Stokes's values for the sth zeros of $W(m)$ where $s = 1(1)10(2)14(3)20(5)50$, and his 10 zeros of $W'(m)$. Pernter 1906 (517) and Humphreys 1919 (481), 1920 (474) reproduce Stokes's values for the first 10 zeros of both $W(m)$ and $W'(m)$, and give 3-decimal values of the first 10 maxima of $W^2(m)$.

20·3. Other Bessel Functions of Non-linear Argument

20·31. $\chi(x) = \int_0^\infty e^{-(x-w^2)^2}\,dw$

If $x > 0$

$$\chi(x) = \tfrac{1}{4}\pi\sqrt{x}\,e^{-\frac{1}{2}x^2}[I_{-1/4}(\tfrac{1}{2}x^2) + I_{1/4}(\tfrac{1}{2}x^2)]$$

$$\chi(-x) = \tfrac{1}{4}\pi\sqrt{x}\,e^{-\frac{1}{2}x^2}[I_{-1/4}(\tfrac{1}{2}x^2) - I_{1/4}(\tfrac{1}{2}x^2)]$$

$$= \tfrac{1}{4}\sqrt{2}x\,e^{-\frac{1}{2}x^2}K_{1/4}(\tfrac{1}{2}x^2)$$

We note also that the differential equation $y'' = x^2 y$ has as complete solution $y = \{A\chi(x) + B\chi(-x)\}e^{\frac{1}{2}x^2}$. This function $\chi(x)$ is not to be confused with the function of Arts. 20·28.

4; 5 dec.	$-x = 0(\cdot 1)2\cdot 4;\ 2\cdot 5(\cdot 1)3\cdot 2$	Δ^2	Hartree & Johnston 1939 (188)
4 dec.	$x = 0(\cdot 1)3$	Δ^2	Hartree & Johnston 1939 (188)
4 dec.	$x = 2\cdot 8(\cdot 2)8$	Δ	Hartree & Johnston 1939 (188)

20·5. Lommel–Weber Functions, $\Omega_n(x)$; Struve Functions, $\mathbf{H}_n(x)$; etc.

$$\Omega_n(x) = \frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta - n\theta)\,d\theta$$

$$\mathbf{H}_n(x) = \frac{2(\frac{1}{2}x)^n}{\sqrt{\pi}(n-\frac{1}{2})!}\int_0^{\frac{1}{2}\pi} \sin(x\cos\theta).\sin^{2n}\theta\,d\theta$$

Watson 1922 (308) uses $-\mathbf{E}_n(x)$ for the first integral. He writes also

$$\mathbf{J}_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\,d\theta$$

and calls these Anger's Functions. If n is an integer $\mathbf{J}_n(x)$ reduces to $J_n(x)$; otherwise

$$J_n(x) - \mathbf{J}_n(x) = -\frac{\sin n\pi}{\pi}F_n(x) \quad \text{where} \quad F_n(x) = \int_0^{\infty} \exp(-x\sinh\theta - n\theta)\,d\theta$$

Formulæ for the functions ster x and stei x, i.e. Struve Functions of argument $xi^{3/2}$, are given by McLachlan & Meyers (*Phil. Mag.*, (7) **21**, 425–436, 1936). See Brauer & Brauer 1941 for tables of some incomplete integrals.

20·51. $\Omega_0(x) = \mathbf{H}_0(x)$

7 dec.	$0(\cdot 02)16$	No Δ	Watson 1922 (666)
6 dec.	$0(\cdot 02)16$	No Δ	B.A. 1924 (280, Airey)
4 dec.	$0(\cdot 02)16$		Glazenap 1932
4 dec.	$0(\cdot 01)14\cdot 99$	IΔ	Jahnke–Emde 1938 (212, 218)

This table is given *twice*, on p. 212 as a Lommel–Weber Function and on p. 218 as a Struve Function.

4 dec.	$0(\cdot 1)15\cdot 9$	No Δ	McLachlan 1934 (176)

20·521. $\mathbf{H}_1(x) = \Omega_1(x) + 2/\pi$

7 dec.	$0(\cdot 02)16$	No Δ	Watson 1922 (667)
4 dec.	$0(\cdot 01)14\cdot 99$	IΔ	Jahnke–Emde 1938 (219)
4 dec.	$0(\cdot 02)16$		Glazenap 1932
4 fig.	$0(\cdot 1)15\cdot 9$	No Δ	McLachlan 1934 (176)

20·522. $\frac{2}{x}\mathbf{H}_1(x)$, etc.

4 d.	$2\mathbf{H}_1/x$	$0(\cdot 1)\cdot 2(\cdot 2)4(\cdot 5)10(1)15$	No Δ	Morse 1936 (337)
4 d.	$3\pi\mathbf{H}_1/2x^2$	$0(\cdot 4)18$	No Δ	Selwyn 1943 (287)

Morse also tabulates α_0 and β_0 to 3 decimals for the same arguments, where

$$\tanh\pi(\alpha_0 + i\beta_0) = 1 - \frac{2}{x}J_1(x) - \frac{2i}{x}\mathbf{H}_1(x)$$

20·523. $\Omega_1(x) = \mathbf{H}_1(x) - \dfrac{2}{\pi}$

7 dec.	0(·02)16		Airey (MS.)
6 dec.	0(·02)16	No Δ	B.A. 1924 (280, Airey)
4 dec.	0(·01)14·99	IΔ	Jahnke–Emde 1938 (213)

20·53. $\Omega_{1/2}(x)$ and $\Omega_{-1/2}(x) = -\Omega_{1/2}(-x)$

8 dec.	0(·02)20		Airey (MS.)
6 dec.	0(·02)20	No Δ	B.A. 1925 (244, Airey)

See also Art. 20·666.

20·541. $\Omega_n(n)$ and $\Omega_{n-1}(n)$

6 dec.	$n = 0(\frac{1}{4})10$	No Δ	B.A. 1923 (293, Airey)

20·542. $\Omega_{2m-1}(2m) - \Omega_{2m+1}(2m)$

6 dec.	$m = 1(1)15$	No Δ	Airey 1918*a* (312)

20·55. $F_{\pm n}(n)$ and $F_{\pm(n-1)}(n)$

6 dec.	$n = 0(\frac{1}{4})10$	No Δ	B.A. 1923 (292, Airey)

20·57†. Modified Struve Functions

$$\mathbf{L}_0(x) = -i\,\mathbf{H}_0(ix) = -i\,\Omega_0(ix) \qquad \mathbf{L}_0'(x) = \frac{2}{\pi} - \mathbf{H}_1(ix) = -\Omega_1(ix)$$

6–7 fig.	·02, ·1, ·5, 1(1)12	No Δ	Owen 1924 (736)

20·6. The Fresnel Integrals

$$C(u) = \int_0^u \cos(\tfrac{1}{2}\pi t^2)\,dt = \tfrac{1}{2}\int_0^x J_{-1/2}(t)\,dt$$

$$S(u) = \int_0^u \sin(\tfrac{1}{2}\pi t^2)\,dt = \tfrac{1}{2}\int_0^x J_{1/2}(t)\,dt$$

where $x = \frac{1}{2}\pi u^2$. See Art. 20·811 for other similar integrals involving Bessel Functions of order zero. None of the published tables gives differences, although Lommel 1886 (see Art. 20·62) gives reduced derivatives $v^n(1)$ for $C(\sqrt{2x/\pi})$ and $S(\sqrt{2x/\pi})$.

20·61. $C(u)$ and $S(u)$

9− dec.	0(·05)3·55	Comrie (MS.)
5 dec.	0(·01)3·1	Comrie (MS.)
4 dec.	0(·1)8·5	Jahnke–Emde 1933 (109), 1938 (34)

Errors exceeding 2 final units occur in $C(u)$ when $u = 1·1$ and $5·1$, and in $S(u)$ when $u = 7·3$; the correct values are 0·7638, 0·4998 and 0·5189 respectively. The list of errata given in W. Lash Miller & Gordon 1931 (2872) contains errors.

4 dec.	0(·1)8·5		* Ignatowsky 1907 (895); Jahnke–Emde 1909 (26)

Ignatowsky 1907 also gives 6-decimal values of the logarithms of *his four-decimal values* of $C(u)$ and $S(u)$; these are never reliable to more than 4 decimals, often fewer.

5; 4 dec.	0(·01)·1; ·15(·05)1(·1)5·5		* Gilbert 1863 (51); Mayall 1897 (268)
4 dec.	0(·1)5		* Mascart 1889 (269); Pockels 1906 (1054); Preston 1912 (290); Edwards 1922 (332)
3 dec.	0(·1)5·5		Fresnel 1826 (*Œuvres*, **1**, 319); Airy 1831 (410), 1858 (399)

Fresnel gives a fourth decimal which is systematically 0 to 8 units too large in $C(u)$ and 0 to 5 units too small in $S(u)$.

3 dec.	$\frac{1}{\sqrt{2}}C(u)$, $\frac{1}{\sqrt{2}}S(u)$	$u = 0(\cdot 1)8\cdot 5$	Bouman & de Jong 1931 (35)

Possibly from Ignatowsky 1907 and with corresponding errors; in addition the first value of $C(u)/\sqrt{2}$ should be 0·071. The argument is $2\cdot 5u$, called $100x$.

*All contain errors, agreeing for the main part with Jahnke & Emde 1933, 1938, except that both the latter have $C(u)$ corrected at $u = 1\cdot 8$ to read 0·3337 and $S(u)$ at 1·6 and 2·3 to read 0·6389 and 0·5531 respectively.

20·62. $C\left(\sqrt{\frac{2x}{\pi}}\right)$ and $S\left(\sqrt{\frac{2x}{\pi}}\right)$

The unfortunate notation $C(x)$, $S(x)$ is sometimes used for these functions.

12 dec.	0(1)20		Airey (MS.)
9 dec.	0(·1)20		Airey (MS.)
7 dec.	0(·02)1		Watson 1922 (745)
6 dec.	0(·1)1(·5)50	$v^n(1)$	*Lommel 1886 (648)
6 dec.	0(·5)50		*Watson 1922 (744)
6 dec.	0(·1)20		B.A. 1926 (274, Airey)
5 dec.	0(·02)1(·5)50		*Hayashi 1930*b* (119)
4 dec.	0(·1)1(·5)50		*Jahnke–Emde 1909 (25), 1933 (108), 1938 (35)

*The sixth decimal is unreliable in a considerable number of entries. There is one major error; S at $x = 1\cdot 5$ should read 0·415483. Watson has taken his 6-decimal values on p. 744 from Lommel without modification; the other tables marked also have this error. See also Glazenap 1932.

20·63. Turning-Values of $C\left(\sqrt{\frac{2x}{\pi}}\right)$ and $S\left(\sqrt{\frac{2x}{\pi}}\right)$

These occur at $x = (n-\frac{1}{2})\pi$ for $C\left(\sqrt{\frac{2x}{\pi}}\right)$ and at $x = n\pi$ for $S\left(\sqrt{\frac{2x}{\pi}}\right)$.

6 dec.	$n = 1(1)16$	Lommel 1886 (648); Mascart 1893 (667); Watson 1922 (745)
5 dec.	$n = 1(1)16$	Hayashi 1930*b* (104)

20·64. $C^2(u) + S^2(u)$, etc.

4 d.	$\frac{1}{2}\{C^2(u)+S^2(u)\}$	0(·1)8·5	Ignatowsky 1907 (895)
4 d.	$C^2(u)+S^2(u)$	0(·1)3	Silberstein 1918 (45)
4 d.	$\frac{1}{2}\{\frac{1}{2}+C(u)\}^2+\frac{1}{2}\{\frac{1}{2}+S(u)\}^2$	0(·01)·1(·05)1(·1)5·5	Mayall 1897 (268)
4 d.	$\frac{1}{2}\{\frac{1}{2}-C(u)\}^2+\frac{1}{2}\{\frac{1}{2}-S(u)\}^2$	0(·05)1	Mayall 1897 (268)

All tables contain errors. No differences are given.

20·66. Auxiliary and Related Functions

We express these in terms of the functional notation of Lommel 1886. We may write (cf. Watson 1922, p. 545)

$$C\left(\sqrt{\frac{2x}{\pi}}\right)=\frac{U_{1/2}(2x,0)}{\sqrt{2}}\cos x+\frac{U_{3/2}(2x,0)}{\sqrt{2}}\sin x$$
$$=\tfrac{1}{2}+\frac{V_{1/2}(2x,0)}{\sqrt{2}}\sin x+\frac{V_{3/2}(2x,0)}{\sqrt{2}}\cos x$$

$$S\left(\sqrt{\frac{2x}{\pi}}\right)=\frac{U_{1/2}(2x,0)}{\sqrt{2}}\sin x-\frac{U_{3/2}(2x,0)}{\sqrt{2}}\cos x$$
$$=\tfrac{1}{2}-\frac{V_{1/2}(2x,0)}{\sqrt{2}}\cos x+\frac{V_{3/2}(2x,0)}{\sqrt{2}}\sin x$$

The notation of each author is also indicated, as well as the approximate range of x, which Lommel denotes by z.

Airey 1914*b* (221) gives a useful modification of the asymptotic expansions for $V_{1/2}(2x, 0)$ and $V_{3/2}(2x, 0)$.

20·661. $M(u^2)=\frac{1}{\sqrt{2}}V_{1/2}(u^2\pi,0)$ and $N(u^2)=-\frac{1}{\sqrt{2}}V_{3/2}(u^2\pi,0)$

5 d.	u^2 = 0(·01)·3(·02)2(·1)9(1)30	x = 0 to 47+	Δ	Gilbert 1863 (47)
4 d.	u = 0(·05)1·5(·1)8·5(·5)13	x = 0 to 265+	No Δ	Miller & Gordon 1931 (2873)

Miller & Gordon denote these functions by $(K+H)/\sqrt{\pi}$ and $(K-H)/\sqrt{\pi}$ and their x is our u. Gilbert uses μ for u.

20·662. $N(v)=\frac{1}{2}\sqrt{\pi}\,V_{1/2}(2v^2,0)$ and $M(v)=-\frac{1}{2}\sqrt{\pi}\,V_{3/2}(2v^2,0)$

5 dec.	v = 0(0·1)6	x = 0 to 36	No Δ	Struve 1882*a* (80)

20·663. $N(\beta)=\frac{1}{2}\sqrt{\pi}\,V_{1/2}\{(2\beta+1)\pi,0\}$ and $M(\beta)=-\frac{1}{2}\sqrt{\pi}\,V_{3/2}\{(2\beta+1)\pi,0\}$

6 dec.	β = 0(·1)8·9	x = 1·6− to 29·5+	Δ	Lindstedt 1882 (725)

20·664. $\sqrt{\frac{\pi}{2y}}V_{1/2}(y,0)$, $\sqrt{\frac{\pi}{2y}}V_{3/2}(y,0)$ and $M_1^2=\frac{\pi}{2y}\{V_{1/2}^2(y,0)+V_{3/2}^2(y,0)\}$.

6 dec.	y = 0(1)60	x = 0 to 30	No Δ	Lommel 1886 (658)

20·665. $\frac{1}{2}V_{1/2}(y, 0)$, $\frac{1}{2}V_{3/2}(y, 0)$, M_2^2, etc.

$$M_2^2 = \tfrac{1}{4}\{V_{1/2}^2(y, 0) + V_{3/2}^2(y, 0)\} = \tfrac{1}{2}\left\{\tfrac{1}{2} - C\left(\sqrt{\tfrac{y}{\pi}}\right)\right\}^2 + \tfrac{1}{2}\left\{\tfrac{1}{2} - S\left(\sqrt{\tfrac{y}{\pi}}\right)\right\}^2$$

6 d. $y = 2x = 0(1)60(2)100$, $x = 0$ to 50 No Δ Lommel 1886 (663)

4 d. $u = \sqrt{\frac{2x}{\pi}} = 0(\cdot 1)5\cdot 5$, $x = 0$ to $47\cdot 5+$ No Δ Fresnel 1826 (*Œuvres*, 1, 341)

Only $2M_2^2$ is given. When $u = -U$, $U > 0$, $2M_2^2$ becomes

$$\{\tfrac{1}{2} + C(U)\}^2 + \{\tfrac{1}{2} + S(U)\}^2$$

Fresnel (*Œuvres* only, 1, facing p. 382) gives 4-decimal values for $U = 0(\cdot 1)5\cdot 5$ and also (p. 322) the first fourteen turning-values with corresponding values of U. Gilbert 1863 (52) gives slightly discrepant values for the same turning-points. Fresnel's last digit contains systematic errors up to 4 final units.

4 d. $u = \sqrt{\frac{2x}{\pi}} = 0(\cdot 1)8\cdot 5$ $x = 0$ to $113+$ No Δ Ignatowsky 1907 (895)

Ignatowsky gives $4M_2^2$, denoted by $\mathfrak{C}_0'$.

20·666. $\sqrt{\frac{\pi}{2y}}\,U_{1/2}(y, 0)$, $\sqrt{\frac{\pi}{2y}}\,U_{3/2}(y, 0)$ and M^2

$$M^2 = \frac{\pi}{2y}\{U_{1/2}^2(y, 0) + U_{3/2}^2(y, 0)\} = \frac{\pi}{y}\left\{C^2\left(\sqrt{\tfrac{y}{\pi}}\right) + S^2\left(\sqrt{\tfrac{y}{\pi}}\right)\right\}$$

It may be noted (see Art. 20·53) that

$$\sqrt{\frac{\pi}{2y}}\,U_{1/2}(y, 0) = \tfrac{1}{4}\pi\{\Omega_{-1/2}(\tfrac{1}{2}y) - \Omega_{1/2}(\tfrac{1}{2}y)\}$$

$$\sqrt{\frac{\pi}{2y}}\,U_{3/2}(y, 0) = \tfrac{1}{4}\pi\{\Omega_{-1/2}(\tfrac{1}{2}y) + \Omega_{1/2}(\tfrac{1}{2}y)\}$$

6 dec. $y = 0(1)60$ $x = 0$ to 30 No Δ Lommel 1886 (652)

Lommel also tabulates 8 non-trivial extreme values of M^2, together with the corresponding values of the two constituent functions, and of y, these being zeros of $\sqrt{\pi/2y}\,U_{1/2}(y, 0) - M^2$. In addition he gives 4 zeros (> 0) of $U_{3/2}(y, 0)$.

Lamb 1905 (393) tabulates 4-decimal values, derived from Lommel's, of $\sqrt{\frac{1}{2}\pi y}\,U_{1/2}(y, 0)$, which he denotes by M, for $y = 0(1)60$. He also gives (in terms of his M) My, $\frac{dM}{dy}$, $\sqrt{y}\frac{dM}{dy}$ and $y^2\frac{dM}{dy}$ to 5 or 6 decimals or figures. It may be noted that

$$2\frac{dM}{dy} = 1 - \sqrt{\tfrac{1}{2}\pi y}\,U_{3/2}(y, 0) + \sqrt{\frac{\pi}{2y}}\,U_{1/2}(y, 0)$$

Terazawa 1916 (77) gives 5-, 6-, 5-decimal values of $N(y) = \sqrt{\frac{\pi}{2y}}\,U_{3/2}(y, 0)$, $\frac{N(y)}{y}$ and $\frac{N(y)}{\sqrt{y}}$ respectively for $y = 0(1)60$.

20·667. $\frac{1}{2}\cos(\frac{1}{2}y+\frac{1}{4}\pi) - \frac{1}{2}U_{3/2}(y, 0)$, $\frac{1}{2}\sin(\frac{1}{2}y+\frac{1}{4}\pi) + \frac{1}{2}U_{1/2}(y, 0)$ **and** M_1^2

$$M_1^2 = \{\tfrac{1}{2}\cos(\tfrac{1}{2}y+\tfrac{1}{4}\pi) - \tfrac{1}{2}U_{3/2}(y, 0)\}^2 + \{\tfrac{1}{2}\sin(\tfrac{1}{2}y+\tfrac{1}{4}\pi) + \tfrac{1}{2}U_{1/2}(y, 0)\}^2$$
$$= \tfrac{1}{2}\left\{\tfrac{1}{2}+C\left(\sqrt{\tfrac{y}{\pi}}\right)\right\}^2 + \tfrac{1}{2}\left\{\tfrac{1}{2}+S\left(\sqrt{\tfrac{y}{\pi}}\right)\right\}^2$$

6 dec. $y = 0(1)60(2)100$ $x = 0$ to 50 No Δ Lommel 1886 (661)

Lommel also tabulates 16 turning-values of M_1^2. These occur at zeros of $\frac{1}{2}\sin(\frac{1}{2}y+\frac{1}{4}\pi) + \frac{1}{2}U_{1/2}(y, 0)$; these zeros and the corresponding values of $\frac{1}{2}\cos(\frac{1}{2}y+\frac{1}{4}\pi) - \frac{1}{2}U_{3/2}(y, 0)$ are also given. Mascart 1889 (280) gives 4-decimal values of $u = \sqrt{\frac{y}{\pi}}$ for the first 10 of these zeros together with the corresponding values of M_1^2 to 3 decimals; he also gives 4-decimal values of $\frac{u_p^2}{p} = \frac{y_p}{p\pi}$ and of $\frac{u_p^2}{(p-\frac{1}{4})} \doteq 2$, where y_p is the abscissa of the pth turning-value. Fresnel 1826 (*Œuvres*, **1**, 322) gives, to 4 decimals, the first 14 values of u_p and of $2M_1^2$; the final digit in the latter is unreliable; he also gives (*Œuvres*, **1**, on diagram facing p. 382, not in the original memoir) $2M_1^2$ to 4 decimals for $u = \sqrt{\frac{y}{\pi}} = 0(\cdot 1)5\cdot 5$, the last figure or two being unreliable; the first 6 lines of the former table are also given in Rayleigh 1888 (443).

20·668. $\frac{1}{2}\{1 - C(\sqrt{m-q}) - C(\sqrt{m+q})\}^2 + \frac{1}{2}\{1 - S(\sqrt{m-q}) - S(\sqrt{m+q})\}^2$

4 dec.	$m = q = 0(\cdot 1)4\cdot 2$		Ignatowsky 1907 (895)
4 dec.	$m = \cdot 5$	$q = 0(\cdot 3)1\cdot 8,\ 2(\cdot 1)8$	Ignatowsky 1907 (899)
	$m = 1\cdot 5$	$q = 0(\cdot 1)7$	
	$m = 3$	$q = 0(\cdot 1)5\cdot 5$	

All these tables are very inaccurate.

20·67. Repeated Fresnel Integrals

$$C_n(u) = \int_0^u C_{n-1}(t)\,dt \qquad C_1(u) = \int_0^u \cos(\tfrac{1}{2}\pi t^2)\,dt = C(u)$$
$$c_n(u) = \int_u^\infty c_{n-1}(t)\,dt \qquad c_1(u) = \int_u^\infty \cos(\tfrac{1}{2}\pi t^2)\,dt = \tfrac{1}{2} - C(u)$$
$$S_n(u) = \int_0^u S_{n-1}(t)\,dt \qquad S_1(u) = \int_0^u \sin(\tfrac{1}{2}\pi t^2)\,dt = S(u)$$
$$s_n(u) = \int_u^\infty s_{n-1}(t)\,dt \qquad s_1(u) = \int_u^\infty \sin(\tfrac{1}{2}\pi t^2)\,dt = \tfrac{1}{2} - S(u)$$

Baumann 1935 (78) tabulates, to 4; 3 decimals for $u = 0(\cdot 1)3\cdot 9;\ 4(\cdot 1)5$

$$F_1(u) = -\pi\{c_2(u) - s_2(u)\} = 1 - \pi\{C_2(u) - S_2(u)\}$$
$$= \cos\tfrac{1}{2}\pi u^2 + \sin\tfrac{1}{2}\pi u^2 - \pi u\{C(u) - S(u)\}$$

and, to 4; 3 decimals for $u = 0(\cdot 1)1\cdot 4;\ 1\cdot 5(\cdot 1)2$

$$F_2(u) = -3\pi^2\{c_4(u) + s_4(u)\} = \tfrac{1}{2}\pi^2 u^3 - \tfrac{3}{2}\pi u^2 + 1 - 3\pi^2\{C_4(u) + S_4(u)\}$$

The latter function can also be expressed in terms of $C(u)$ and $S(u)$, etc. Baumann uses suffix $n-1$ where we use n.

20·7. Lommel's Functions of Two Variables

We take the definitions of these functions from Watson 1922 (537).

$$U_n(y, z) = \left(\frac{y}{z}\right)^n J_n(z) - \left(\frac{y}{z}\right)^{n+2} J_{n+2}(z) + \left(\frac{y}{z}\right)^{n+4} J_{n+4}(z) - \dots$$

$$V_n(y, z) = \cos \tfrac{1}{2}\left(y + \frac{z^2}{y} + n\pi\right) + U_{-n+2}(y, z)$$

When n is an integer the latter is equivalent to

$$V_n(y, z) = \left(\frac{y}{z}\right)^{-n} J_{-n}(z) - \left(\frac{y}{z}\right)^{-n-2} J_{-n-2}(z) + \left(\frac{y}{z}\right)^{-n-4} J_{-n-4}(z) - \dots$$

We note that

$$U_1(z, z) = -V_1(z, z) = \tfrac{1}{2} \sin z$$

$$U_2(z, z) = V_2(z, z) = \tfrac{1}{2}\{J_0(z) - \cos z\} \qquad U_0(z, z) = V_0(z, z) = \tfrac{1}{2}\{J_0(z) + \cos z\}$$

Lommel 1885 (but not 1886) and Gray, Mathews & MacRobert 1922 give $V_n(y, z)$ an additional sign $(-)^n$.

All the tables listed in this article are to 6 decimals, unless otherwise stated.

Lommel 1885 (317 ff.) gives

$$\frac{2}{y} U_1(y, z), \quad \frac{2}{y} U_2(y, z) \quad \text{and} \quad M^2 = \frac{4}{y^2}\{U_1^2(y, z) + U_2^2(y, z)\}$$

for $y = \pi(\pi)10\pi$, $z = 0(1)12$. He also gives 6 to 4 lines of extreme values of M^2 with corresponding values of z, $\frac{2}{y} U_1(y, z)$ and $\frac{2}{y} U_2(y, z)$; these extremes correspond to the first 3 zeros of $J_1(z)$ and to the first three, two or one zeros of $U_2(y, z)$. Gray & Mathews 1895 (182) and Gray, Mathews & MacRobert 1922 (194) reproduce the tables of extreme values, etc., for $y = \pi, 5\pi, 9\pi$. Lommel gives a collected table of the first three, two or one zeros of $U_2(y, z)$ on p. 322 for $y = 0(\pi)10\pi$ and $12(1)14$, $20{\cdot}5$, $26{\cdot}2$ and tabulates the corresponding values of $\tan^{-1} \frac{\partial y}{\partial z}$ to 1″; for $y = 0$, the zeros are those of $J_2(x)$ and are affected by the errors referred to in Art. 17·7411.

On pages 323 ff. Lommel 1885 gives values of $-\frac{2}{y} V_1(y, z)$, $\frac{2}{y} V_0(y, z)$ and $M_1^2 = \frac{4}{y^2}\{V_1^2(y, z) + V_0^2(y, z)\}$ for $y = \pi(\pi)8\pi$, $z = 0(1)12$; tables giving 11 to 7 lines of extreme values of M_1^2 follow, for each $y = \pi(\pi)8\pi$, corresponding to zeros of $J_1(z)$ and $V_0(y, z)$; corresponding values of z, $-\frac{2}{y} V_1(y, z)$ and $\frac{2}{y} V_0(y, z)$ are also given. Zeros of $V_0(y, z)$ are collected in a table on p. 327, 8 to 4 being given for each $y = \pi(\pi)8\pi$, together with corresponding values of $\tan^{-1} \frac{\partial y}{\partial z}$ to 1″.

Rayleigh 1891 (94) tabulates 4-decimal values, derived from Lommel's (pp. 317 ff.), of pM^2 for $y = p\pi$, $p = 1(1)4$, $z = 0(1)12$, together with the corresponding extreme values; he also gives $\frac{1}{2}M^2$ for $y = \frac{1}{2}\pi$, $z = 0(1)4$.

H. H. Hopkins has 3-decimal tables in MS. of U_1, U_2, V_0 and V_1 for $z = 1(1)50$ and $y/z = 0(\cdot05)1$.

See also Conrady 1919 and Buxton 1921.

Lommel 1886 (653 ff.) gives

$$\sqrt{\frac{\pi}{2y}}\,U_{1/2}(y, z),\quad \sqrt{\frac{\pi}{2y}}\,U_{3/2}(y, z)\quad\text{and}\quad M^2 = \frac{\pi}{2y}\{U_{1/2}^2(y, z) + U_{3/2}^2(y, z)\}$$

for $y = 3(3)30$, $z = 0(1)12$; for each y there are also 7 to 4 lines of entries (including z itself, which is a multiple of π or a zero of $U_{3/2}(y, z)$) corresponding to extreme values of M^2 for which $z < 4\pi$. On pages 659 ff. the functions

$$\sqrt{\frac{\pi}{2y}}\,V_{3/2}(y, z),\quad \sqrt{\frac{\pi}{2y}}\,V_{1/2}(y, z)\quad\text{and}\quad M_1^2 = \frac{\pi}{2y}\{V_{3/2}^2(y, z) + V_{1/2}^2(y, z)\}$$

are similarly tabulated for $y = 3(3)12$, $z = 0(1)12$, with 13 to 8 lines of entries corresponding to extreme values of M_1^2; the values of z are here multiples of π or zeros of $V_{1/2}(y, z)$.

20·8. Integrals involving Bessel Functions

20·811. $\int_0^x J_0(t)\,dt$ and $\int_0^x Y_0(t)\,dt$

See Art. 20·6 for similar integrals involving functions of order $\pm\frac{1}{2}$. It may be noted that

$$\tfrac{1}{2}\int_0^x J_0(t)\,dt = \tfrac{1}{4}\pi x\{J_0(x)\mathbf{H}_0'(x) + J_1(x)\mathbf{H}_0(x)\}$$

$$\tfrac{1}{2}\int_0^x Y_0(t)\,dt = \tfrac{1}{4}\pi x\{Y_0(x)\mathbf{H}_0'(x) + Y_1(x)\mathbf{H}_0(x)\}$$

$$\mathbf{H}_0'(x) = \frac{2}{\pi} - \mathbf{H}_1(x) = -\Omega_1(x)$$

Tables of the Struve Functions are listed in Arts. 20·5.

10 dec.	0(·01)10	No Δ	Lowan & Abramowitz 1943 (3)
7 dec.	0(·02)1	No Δ	Watson 1922 (752)
4 dec.	0(·02)1		Glazenap 1932

Watson tabulates $\frac{1}{2}\int_0^x J_0(t)\,dt$ and $\frac{1}{2}\int_0^x Y_0(t)\,dt$ and also tabulates the first 16 extreme values of each function to 7 decimals.

20·813. Havelock's Integrals $P_n(x)$

If
$$P_n(x) = \int_0^{\frac{1}{2}\pi} \cos^n\phi \sin(x\sec\phi - \tfrac{1}{2}n\pi)\,d\phi$$

then
$$P_n(x) = -\int_x^\infty P_{n-1}(t)\,dt = (-)^n\,\tfrac{1}{2}\pi\,G\int_x^\infty \frac{(t-x)^n}{n!}Y_0(t)\,dt$$

where G denotes a generalized integral in Hardy's sense—that is, with an implied factor $e^{-\lambda t}$ to secure convergence, in which the constant λ is subsequently made to tend to zero.

We have
$$P_0(x) = -\frac{\pi}{2}\int_0^x Y_0(t)\,dt = \frac{\pi}{2}\int_x^\infty Y_0(t)\,dt$$

4 dec.	$n = 3(1)5$	$x = 0(·4)4·8,\ 1(1)10$	No Δ	Havelock 1925 (82)

20·815. $\mathrm{Ji}_0(x) = \int_x^\infty \frac{J_0(t)}{t}\,dt$ and $F(x) = \mathrm{Ji}_0(x) + \log_e \tfrac{1}{2}x$

10 dec.	$\mathrm{Ji}_0(x)$	0(·1)10(1)22	δ^4_m, $3 \leqslant x \leqslant 10$	Lowan, Blanch & Abramowitz 1943 (54)
12 dec.	$F(x)$	0(·1)3; 10(1)21	δ^6; v^n, $n \leqslant 13$	

20·816. $\int_0^x \frac{J_1(t)}{t}\,dt$

4 dec.	0(·1)14, 15	Gans 1925 (22)

Gans also tabulates certain related functions.

20·818. $n\int_0^1 J_n(nx)\,dx$

7 dec.	n = 1(2)23	Watson 1918*b* (369)

20·82. $\int_0^x I_0(x)\,dx$ and $e^{-x}\int_0^x I_0(x)\,dx$

Bursian & Fock 1931 tabulates these functions to 7 decimals for x = 0(·1)6 and x = 0(·1)16 respectively. Müller 1939 (52) gives the former to 6 figures, with δ^2, for x = 0(·1)5·2(·2)16.

20·831. The Integrals $\mathrm{Ki}_n(x)$

$$\mathrm{Ki}_n(x) = \int_0^\infty \frac{e^{-x\cosh u}\,du}{\cosh^n u} = \int_x^\infty \mathrm{Ki}_{n-1}(t)\,dt = \int_x^\infty \frac{(t-x)^{n-1}}{(n-1)!}K_0(t)\,dt$$

$$\mathrm{Ki}_0(x) = K_0(x)$$

9 dec.	n = 1(1)16	x = 0(·05)·2(·1)2, 3	No Δ	Bickley & Nayler 1935 (344)
4 dec.	n = 2	x = ·25(·01)1(·05)7	Δ	Seeliger 1899 (447)

Bursian & Fock 1931 gives $\mathrm{Ki}_1(x)$ and $e^x\mathrm{Ki}_1(x)$ to 7–8 figures for x = 0(·1)12 and 0(·1)16 respectively.

20·832. $\int_0^x K_0(t)\,dt = \tfrac{1}{2}\pi - \mathrm{Ki}_1(x)$

7 fig.	x = ·02, ·1, ·5, 1(1)12	No Δ	Owen 1924 (736)
6+ fig.	x = 0(·02)·04, ·1(·1)2·6(·2)13·6(·4)16	δ^2	Müller 1939 (53)
6 dec.	x = 0(·1)4(·5)18	No Δ	Pearson, Stouffer & David 1932 (344)

$\tfrac{1}{2} - \mathrm{Ki}_1(x)/\pi$ is given (column m = 0); computed by David.

20·833. $\int_0^x tK_1(t)\,dt = \int_0^x K_0(t)\,dt - xK_0(x)$

5 dec.	x = 0(·1)4·5, ∞	No Δ	Chandrasekhar 1933 (34)
6 dec.	x = 0(·1)4(·5)18	No Δ	Pearson, Stouffer & David 1932 (344)

$\tfrac{1}{2} - \{\mathrm{Ki}_1(x) + xK_0(x)\}/\pi$ is given (column m = 1); computed by Fieller.

See Art. 18·417 for $xK_0(x)$ or Art. 18·31 for $K_0(x)$.

$$20{\cdot}834.\quad S_m(x) = \int_0^x T_m(t)\,dt = \int_0^x \frac{(\tfrac{1}{2}t)^m K_m(t)}{\sqrt{\pi}\,(m-\tfrac{1}{2})!}\,dt$$

For $T_m(x)$ see Art. 18·44. If m is a positive integer

$$S_m(x) = \int_0^x \frac{k_m(t)\,dt}{\pi(2m-1)!!} \quad \text{with} \quad k_n(x) = x^n K_n(x)$$

If m is half an odd positive integer, $S_m(x)$ is elementary; in particular

$$S_{1/2}(x) = \tfrac{1}{2}(1-e^{-x}) \qquad S_{3/2}(x) = \tfrac{1}{2} - \tfrac{1}{4}(2+x)e^{-x}$$

6 dec.	$m = 0(\tfrac{1}{2})5\tfrac{1}{2}$	$x = 0(\cdot1)4(\cdot5)18$	No Δ	Pearson, Stouffer & David
6 dec.	$m = 6\tfrac{1}{2}(\tfrac{1}{2})11\tfrac{1}{2}$	$x = 0(\cdot5)18$		1932 (346, David)

$$20{\cdot}835.\quad \int_0^x e^t K_0(t)\,dt$$

6; 3 d.	$\sqrt{x} = \cdot01(\cdot01)\cdot1;\ 5(1)10$	No Δ	King 1914 (412)
4 dec.	$x = \cdot02(\cdot01)\cdot1(\cdot1)3(\cdot2)6(\cdot5)10(1)20$		

King also gives 4-decimal values of $x\Big/\Big\{\int_0^x e^t K_0(t)\,dt\Big\}$.

$$20{\cdot}84.\quad \textbf{Struve's Integral}\ I(z) = \frac{1}{\pi}\int_z^\infty \frac{\mathbf{H}_1(2t)}{t^2}\,dt$$

4 dec.	$0(\cdot1)4(\cdot2)7(\cdot4)15$	No Δ	Struve 1882*b* (1016)
4 dec.	$0(\cdot5)3(1)7(2)11$ and 15	No Δ	Rayleigh 1888 (434)
4 dec.	$0(\cdot5)2(1)5(2)11$ and 15	No Δ	Gray & Mathews 1895 (207); G., M. & MacR. 1922 (218)

$$20{\cdot}851.\quad \int \frac{J_1^2(x)}{x}\,dx$$

Steiner 1894 deals with this integral. His tables should be used with caution since his values for the zeros of $J_1(x)$ are rarely correct in the second decimal, although he gives five; his values for these zeros are multiples $(n+\tfrac{1}{4}) \times 3{\cdot}1415900$ when n exceeds 11, and earlier values are erratic. The corresponding 6-decimal values of the integral between successive zeros are probably more reliable, however, since the integrand touches the x-axis at its zeros.

$$20{\cdot}852.\quad \left|\int_0^v e^{\frac{1}{2}i\pi t^2} J_0(\alpha'\sqrt{t})\,dt\right|^2$$

When $\alpha' = 0$ this becomes $C^2(v)+S^2(v)$; see Art. 20·64.

4 dec.	$\alpha'\dfrac{10\sqrt{10}}{2\pi} = 1, 10, 20, 30$	$v = 0(\cdot1)3$	No Δ	Silberstein 1918 (45)

20·855. $\mathrm{Jc}(\lambda, x) = \int_0^x J_0(\lambda t)\cos t\,dt, \qquad \mathrm{Js}(\lambda, x) = \int_0^x J_0(\lambda t)\sin t\,dt$

$$\mathrm{Yc}(\lambda, x) = \int_0^x Y_0(\lambda t)\cos t\,dt, \qquad \mathrm{Ys}(\lambda, x) = \int_0^x Y_0(\lambda t)\sin t\,dt$$

Schwarz 1944 gives values (without differences) as follows:

6; 8 d.	$\mathrm{Jc}(\lambda, x)$, $\mathrm{Js}(\lambda, x)$	$\lambda = \cdot 1(\cdot 1)1$	$x = 0(\cdot 02)2;\ 0(\cdot 1)5$	pp. 351, 366
6 dec.	$\mathrm{Yc}(\lambda, x)$, $\mathrm{Ys}(\lambda, x)$	$\lambda = \cdot 1(\cdot 1)1$	$x = 0(\cdot 02)2(\cdot 1)5$	pp. 357, 370

Auxiliary functions

$$\mathrm{Cc}(\lambda, x) = \mathrm{Yc}(\lambda, x) - \frac{2}{\pi}\log_e \lambda x . \mathrm{Jc}(\lambda x)$$

$$\mathrm{Cs}(\lambda, x) = \mathrm{Ys}(\lambda, x) - \frac{2}{\pi}\log_e \lambda x . \mathrm{Js}(\lambda x)$$

are also given (p. 362), to 6 decimals, for $\lambda = 0(\cdot 1)1$, $x = 0(\cdot 02)X$. where X varies from $X = 2$ when $\lambda = 0$ to $X = 0{\cdot}2$ when $\lambda = 1$. Note (see Section 13) that

$$\mathrm{Cc}(0, x) = \frac{2}{\pi}\{(\gamma - \log_e 2)\sin x - \mathrm{Si}\, x\}$$

$$\mathrm{Cs}(0, x) = \frac{2}{\pi}\{(\gamma - \log_e 2)(1 - \cos x) - \gamma - \log_e x + \mathrm{Ci}\, x\}$$

20·859. $\displaystyle\int_0^\infty \frac{e^{-qt} J_\mu(t) J_\nu(t)}{t}\,dt$

Nomura 1940 (316) gives 7-decimal values of these integrals for half-odd-integer μ and ν satisfying $\mu + \nu = 2(1)8$, $\mu < \nu$, and with $q = \cdot 4(\cdot 2)1{\cdot}2,\ 1{\cdot}5(\cdot 5)3,\ 5,\ 10,\ 20.$ Nomura 1941 (173) gives the same, except that $(2\mu, 2\nu) = (1, 1)$ is included, while $(2\mu, 2\nu) = (3, 13)$ and $(1, 15)$ are excluded. Nomura 1940*a* (32) gives similar tables for non-negative integer μ and ν, $\mu < \nu < 4$; $\mu < 3$, $\nu = 5$ and $\mu < 1$, $\nu = 6$ ($\mu = \nu = 0$ excluded). Nomura 1940 and 1940*a* give 5-decimal values of certain multiples of some of the integrals. For half-odd-integer μ and ν, the integrals are elementary although they may be complicated. For integer μ and ν, the integrals may be expressed in terms of complete elliptic integrals.

20·861. Toronto Functions $T(m, n, r)$, etc.

$$T(m, n, r) = 2r^{n-m+1} e^{-r^2} \int_0^\infty t^{m-n} e^{-t^2} I_n(2rt)\,dt$$

$$= r^{2n-m+1} e^{-r^2} \frac{(\frac{1}{2}m - \frac{1}{2})!}{n!} M(\tfrac{1}{2}m + \tfrac{1}{2}, n + 1, r^2)$$

5 dec.	$m = -\frac{1}{2}(\frac{1}{2}) + 1$, $n = -2(\frac{1}{2}) + 2$, $r = 0(\cdot 2)4, 5, 6, 10, 25, 50$	Heatley 1943 (26)

Table incomplete when $r > 2$. See Art. 22·91.

3 dec.	$2T(2, 0, x)$, $\dfrac{4x}{\sqrt{\pi}} T(2, 0, x)$	$x = 0(\cdot 1)1(\cdot 2)3$	Wagner 1930 (399, 400)

Fisher 1928 (665) gives 4-decimal values of B such that $f = 0{\cdot}95$, for $n_1 = 1(1)7$, $\beta = 0(\cdot 2)5$, where

$$f = T_{B/\sqrt{2}}(n_1-1, \tfrac{1}{2}n_1-1, \beta/\sqrt{2}) = \int_0^B \left(\frac{t}{\beta}\right)^{\frac{1}{2}n_1-1} e^{-\frac{1}{2}(t^2+\beta^2)} I_{\frac{1}{2}n_1-1}(t\beta)\, t\, dt$$

in which the upper limit ∞ in the integral for $T(m, n, r)$ is replaced by $B/\sqrt{2}$.

$$20{\cdot}865. \quad \frac{1}{\pi}\int_0^\infty \frac{t^m dt}{I_n^2(t)}$$

7 dec.	$m = 2, 4$	$n = 0, 1$	Watson 1930*a* (37)

$$20{\cdot}87. \quad I_{2n} = \int_0^\infty \frac{K_0(m).m^{2n}}{I_0(m)} dm$$

5 fig. or dec.	$n = 0(1)6$	Knight 1936 (130)

$$20{\cdot}88. \quad \int_0^\infty \frac{e^{-xu^2}}{J_0^2(u)+Y_0^2(u)} \frac{du}{u}$$

3 dec.	$0(\cdot 01)1(\cdot 1)10(1)100(10)1000$	No Δ	Jaeger & Clarke 1942 (229)

20·9. Neumann's Polynomials, $O_n(t)$, and Schläfli's Polynomials, $S_n(t)$

Neumann's Polynomials, $O_n(t)$, occur in the expansion

$$\frac{1}{t-z} = J_0(z)O_0(t) + 2J_1(z)O_1(t) + \ldots + 2J_n(z)O_n(t) + \ldots$$

They are polynomials in $1/t$. Watson 1922 (274) gives the coefficients for $n = 0(1)5$, and Otti 1898 (4, 5) for $n = 0(1)15$.

Schläfli's Polynomials, $S_n(t)$, are given by

$$\tfrac{1}{2}nS_n(t) = tO_n(t) - \cos^2 \tfrac{1}{2}n\pi$$

Watson 1922 (286) gives the coefficients for $n = 1(1)6$, and Otti 1898 (13, 14) for $n = 1(1)12$.

Items added in Proof

20·15. Michell Functions

These are four independent solutions of the differential equation

$$\frac{d^2}{dx^2}\left(x^3 \frac{d^2u}{dx^2}\right) + xu = 0$$

and may be expressed in terms of Bessel–Clifford Functions of semi-imaginary argument, or in terms of Kelvin Functions thus:

$$u_1 = \frac{1}{\sqrt{x}} \operatorname{bei}'(2\sqrt{x}) \qquad u_2 = -\frac{2}{\sqrt{x}} \operatorname{ber}'(2\sqrt{x})$$

$$u_3 = \frac{1}{\sqrt{x}}\{2(1-\gamma)\operatorname{bei}'(2\sqrt{x}) - \tfrac{1}{2}\pi \operatorname{ber}'(2\sqrt{x}) - 2\operatorname{kei}'(2\sqrt{x})\}$$

$$u_4 = \frac{1}{\sqrt{x}}\{-2\gamma \operatorname{ber}'(2\sqrt{x}) + \tfrac{1}{2}\pi \operatorname{bei}'(2\sqrt{x}) - 2\operatorname{ker}'(2\sqrt{x})\}$$

see *M.T.A.C.*, **1** (7), 253 and 308, 1944.

9 dec.	u_n, $n = 1(1)4$	$x = 0(\cdot 5)10(1)30$	D^3	Hollister 1937 (A 14)
8 dec.	u_n, $n = 1(1)4$	$x = 0(\cdot 5)2(1)30$	D^3	B. A. Smith 1920 (400), 1921 (2052)

20·21. Ai x, Bi x and Derivatives, etc.

Kramers 1926 (840, van der Held) gives $\sqrt{\pi}\,\mathrm{Ai}\,x$ to 3 decimals for $x = -2\cdot 4(\cdot 1)+2\cdot 5$, with 2-figure values for $x = 3$, $3\frac{1}{2}$, 4, 5 and 9.

Gans 1915 (729) gives 4-figure values, for $x = 0(\cdot 2)2$, of

$$c_1P(x) = \tfrac{1}{2}(\sqrt{3}\,\mathrm{Bi}\,x - 3\,\mathrm{Ai}\,x) \quad \text{and} \quad c_2Q(x) = \tfrac{1}{2}(\sqrt{3}\,\mathrm{Bi}\,x + 3\,\mathrm{Ai}\,x)$$

He gives (p. 732) values, mainly to 4 decimals and for $x = 0(\cdot 2)2$, of

$$\begin{aligned} W + iW' &= e^{-\pi i/3}c_1P(-x) + c_2Q(-x) \\ &= \tfrac{3}{4}\{\sqrt{3}\,\mathrm{Bi}(-x) + \mathrm{Ai}(-x)\} - i\tfrac{3}{4}\{\mathrm{Bi}(-x) - \sqrt{3}\,\mathrm{Ai}(-x)\} \end{aligned}$$

Gans also gives 4-figure values of $4\pi(\mathrm{Ai}\,x)^2$ and 4-decimal values of W^2, W'^2 and $W^2 + W'^2 = \frac{9}{4}\{\mathrm{Ai}(-x)\}^2 + \frac{9}{4}\{\mathrm{Bi}(-x)\}^2 = \frac{9}{4}\{F(-x)\}^2$, see Art. 20·28.

There are MS. tables by P. M. Woodward and Mrs. A. M. Woodward giving 4-decimal values, with provision for interpolation, of Ai z, Bi z, Ai′ z and Bi′ z for $z = x+iy$, with $x = -2\cdot 4(\cdot 2)+2\cdot 4$ and $y = -2\cdot 4(\cdot 2)0$.

20·225. Hogner's Integral

$$\begin{aligned} I_1(\sigma) \equiv I_{1r}(\sigma) + iI_{1i}(\sigma) &= \int_{-\infty+ic}^{-i\infty} \exp\{i(\sigma w^2 + w^3)\}\,dw \\ &= 3^{-1/3}\pi e^{2i\sigma^3/27}\{\mathrm{Ai}(-3^{-4/3}\sigma^2) - i\,\mathrm{Bi}(-3^{-4/3}\sigma^2)\} \end{aligned}$$

$$I_2(\sigma) = \int_{-i\infty}^{\infty+ic} \exp\{i(\sigma w^2 + w^3)\}\,dw = I_1^*(-\sigma) = I_{1r}(-\sigma) - iI_{1i}(-\sigma)$$

In these expressions c is a positive constant, and $I_1^*(-\sigma)$ denotes the complex conjugate of $I_1(-\sigma)$. These are solutions of the differential equation

$$\frac{d^2I}{d\sigma^2} - \left(\frac{1}{\sigma} + i\frac{4\sigma^2}{9}\right)\frac{dI}{d\sigma} - i\frac{2\sigma}{9}I = 0$$

5 dec.	I_{1r}, I_{1i}	$\sigma = -3\cdot 2(\cdot 2)+3\cdot 2$	No Δ	Hogner 1923 (34)

20·291. a_s and a_s'

6 dec.	$s = 1(1)5$	Bates 1938

Only a_s given, last figure unreliable; a_2 should read 4·087949.

3 dec.	$s = 1(1)3$	Bell 1944 (585)

20·57. Modified Struve Functions

Müller 1939 (54) gives 3- to 5-decimal values, without differences, of

$$\tfrac{1}{2}\pi\{I_0(x) - \mathbf{L}_0(x)\} \qquad \text{and of} \qquad -\tfrac{1}{2}\pi\{I_0'(x) - \mathbf{L}_0'(x)\} = 1 - \tfrac{1}{2}\pi\{I_1(x) - \mathbf{L}_1(x)\}$$

for $x = 0(\cdot 1)1(\cdot 2)13\cdot 6(\cdot 4)16$.

Goldstein 1929*a* (446) gives 4-decimal values, without differences, of $F_n(x)$ with $n = 1, 3, 5$ and $x = \cdot 2(\cdot 2)2(\cdot 5)5(1)9$, where

$$F_n(x) = -\frac{1}{1+x^2} + \tfrac{1}{2}n\pi(-1)^{\frac{1}{2}n+\frac{1}{2}}\{I_n(nx) - i^{-n-1}\Omega_n(inx)\}, \quad n = 1, 3, 5$$

In terms of $\mathbf{L}_n(nx) = i^{-n-1}\,\mathbf{H}_n(inx)$, these become

$$F_1(x) = \frac{x^2}{1+x^2} - \tfrac{1}{2}\pi\{I_1(x) - \mathbf{L}_1(x)\}$$

$$F_3(x) = \frac{x^2}{1+x^2} + \tfrac{3}{2}\pi\{I_3(3x) - \mathbf{L}_3(3x)\} - \frac{(3x)^2}{1\cdot 5}$$

$$F_5(x) = \frac{x^2}{1+x^2} - \tfrac{5}{2}\pi\{I_5(5x) - \mathbf{L}_5(5x)\} - \frac{(5x)^2}{3\cdot 7} + \frac{(5x)^4}{1\cdot 3\cdot 7\cdot 9}$$

Goldstein also gives 4-decimal values of

$$G(x) = \pi\left\{\frac{x^2}{1+x^2} - \frac{8}{\pi^2}\sum_{m=0}^{\infty}\frac{F_{2m+1}(x)}{(2m+1)^2}\right\}$$

SECTION 21

ELLIPTIC INTEGRALS, ELLIPTIC FUNCTIONS, THETA FUNCTIONS

21·0. Introduction

We use k for modulus and θ for modular angle, so that $k = \sin\theta$. The only exception is in Art. 21·4, where we have thought it best to keep the notation of the paper under consideration, namely α = modular angle.

The most useful tables in general are those with argument θ or k^2, either of which facilitates the tabulation of functions of the complementary modulus,

$$k' = \cos\theta = \sqrt{(1-k^2)}$$

In dealing with elliptic integrals, we shall describe

$$\int_0^\phi (1-k^2\sin^2\phi)^{-1/2}d\phi \quad\text{and}\quad \int_0^\phi (1-k^2\sin^2\phi)^{1/2}d\phi$$

as $F(k^2, \phi)$ and $E(k^2, \phi)$ respectively when k^2 is a tabular argument, but as $F(\theta, \phi)$ and $E(\theta, \phi)$ when θ is a tabular argument; in the same way we shall use $K(\theta)$, $K(k^2)$, $K(k)$, etc. for the complete integral of the first kind, and similarly with E. The slight variations of notation will cause no confusion.

We use M for $\pi/2K$, the arithmetic-geometric mean of 1 and k'.

The notation for theta functions which we shall use is:

$$\begin{aligned}
\vartheta_1(x) &= 2q^{1/4}(\sin x - q^2\sin 3x + q^6\sin 5x - q^{12}\sin 7x + \ldots)\\
\vartheta_2(x) &= 2q^{1/4}(\cos x + q^2\cos 3x + q^6\cos 5x + q^{12}\cos 7x + \ldots)\\
\vartheta_3(x) &= 1 + 2q\cos 2x + 2q^4\cos 4x + 2q^9\cos 6x + \ldots\\
\vartheta_4(x) &= 1 - 2q\cos 2x + 2q^4\cos 4x - 2q^9\cos 6x + \ldots
\end{aligned}$$

where, as usual, $q = e^{-\pi K'/K}$. The tables of other authors will be indexed in our notation, with some description of the notation of the table when the difference is important.

If $x = Mu = \pi u/2K$, we have

$$\operatorname{sn} u = \frac{1}{\sqrt{k}}\frac{\vartheta_1(x)}{\vartheta_4(x)} \qquad \operatorname{cn} u = \sqrt{\frac{k'}{k}}\frac{\vartheta_2(x)}{\vartheta_4(x)} \qquad \operatorname{dn} u = \sqrt{k'}\frac{\vartheta_3(x)}{\vartheta_4(x)}$$

In discussing Jacobian elliptic functions and theta functions, x and u will be used without exception in the above senses, as the natural theta function and elliptic function arguments. With this understanding, Jacobi's Θ and H functions satisfy

$$\begin{aligned}
\Theta(u) &= \vartheta_4(Mu) = \vartheta_4(x) \qquad & \Theta(u+K) &= \vartheta_4(x+\tfrac{1}{2}\pi) = \vartheta_3(x)\\
\mathrm{H}(u) &= \vartheta_1(Mu) = \vartheta_1(x) \qquad & \mathrm{H}(u+K) &= \vartheta_1(x+\tfrac{1}{2}\pi) = \vartheta_2(x)
\end{aligned}$$

We use ϑ_2, ϑ_3, ϑ_4 for $\vartheta_2(0)$, $\vartheta_3(0)$, $\vartheta_4(0)$, and ϑ_1' foı $\vartheta_1'(0)$, etc.

The general arrangement of the section is as follows:

21·1. Tables connected with the Modulus

21·11. θ, k^2 and k as Functions of One Another

A table of θ against k^2 in Hayashi 1930*b* (126) contains serious errors.

k^2 is given to 5 decimals without differences for

$$\theta = 0(1^\circ)70^\circ(0^\circ{\cdot}5)80^\circ(0^\circ{\cdot}2)89^\circ(0^\circ{\cdot}1)90^\circ$$

in Hayashi 1930*b* (145). See also Arts. 7·72, 7·9.

For k^2 as function of k, see squares in Art. 2·11.

For k as function of k^2, see square roots in Art. 2·61. Hayashi 1930*a* (72) gives k to 12 decimals without differences for $k^2 = 0(\cdot 001)1$.

For k as function of θ, see sines in Section 7. When θ is the argument in elliptic tables, values of k are sometimes printed alongside.

For θ as function of k, see inverse sines in Arts. 9·1 ff.

21·12. $\mathrm{Log}_e(4/k')$

4 dec.	$k^2 = \cdot 7(\cdot 01)1$	$\mathrm{I}\Delta$	Jahnke–Emde 1938 (80)

21·15. $(k_1')^2 = 4k'/(1+k')^2$

This table is for use in applying Landen's transformation, where

$$k_1 = (1-k')/(1+k')$$

7 dec.	$k^2 = 0(\cdot 001)1$	Δ	Nagaoka & Sakurai 1922

21·16. Log k_n and log k_n'

Suppose k_n and k_n' are the *n*th Landen modulus and complementary modulus, defined by

$$k_0 = k, \quad k_0' = \sqrt{(1-k_0^2)}, \quad k_n = (1-k_{n-1}')/(1+k_{n-1}') \quad \text{and} \quad k_n' = \sqrt{(1-k_n^2)}$$

Legendre 1816, 1826 (T. 6) gives 14-decimal logarithms of these quantities for $\theta = 0(0^\circ{\cdot}1)15^\circ(0^\circ{\cdot}5)45^\circ$. As θ increases, more moduli are given; up to k_2, k_1' at the beginning of the table, and up to k_4, k_3' at the end.

21·2. Complete Elliptic Integrals of the First and Second Kinds

For notation see Art. 21·0.

For New York W.P.A. tables of K in progress, see Art. 21·53.

For $\varpi = 2A = K_0\sqrt{2}$, $B = M_0/\sqrt{2}$, where zero suffix denotes modular angle 45°, see Art. 5·82.

21·21. $K(\theta)$ and $E(\theta)$

For ninetieths of K, see Arts. 21·59 ff.

15 dec.	$0(1^\circ)90^\circ$		Spenceley (MS. 1943)
12 dec.	$0(1^\circ)90^\circ$	Δ^6 to 45°, then Δ	**Legendre 1816, 1826** (T. 8)

Reproduced in facsimile in Legendre–Emde 1931. The 12-decimal values, but not the differences, are given in Moseley 1845. Values to 9–10 decimals may also be extracted from Legendre's great tables of $F(\theta, \phi)$ and $E(\theta, \phi)$, see Art. 21·31.

9–10 d.	$0(1°)10°(5°)80°(1°)90°$	No Δ	Adams & Hippisley 1922
7 dec.	$0(1°)90°$, K only	No Δ	Láska 1888–1894
6 dec.	$0(1°)90°$	Δ^2	Rosa & Cohen 1908 Rosa & Grover 1912
6 dec.	$0(1°)90°$	No Δ	Russell 1914 (117)
5 dec.	$0(1°)70°(0°{\cdot}5)80°(0°{\cdot}2)89°(0°{\cdot}1)90°$	No Δ	Lévy 1898 Hancock 1917 (from Lévy) Hayashi 1930*b*
5 dec.	$0(1°)90°$	No Δ	Dale 1903
5 dec.	$86°(1')90°$, K only (See also Art. 21·211)	Δ	Dwight 1935
4 dec.	$0(1°)45°(0°{\cdot}5)80°(0°{\cdot}2)89°(0°{\cdot}1)90°$	No Δ	Silberstein 1923
4 dec.	$0(1°)65°(0°{\cdot}5)80°(0°{\cdot}2)89°(0°{\cdot}1)90°$	No Δ	Allen 1941
4 dec.	$0(1°)70°(0°{\cdot}5)80°(0°{\cdot}2)89°(0°{\cdot}1)90°$	No Δ	Jahnke–Emde 1909–1938
4 dec.	$0(1°)90°$	No Δ	Peirce 1910 Hütte 1931 Fowle 1933

21·211. Special Tables for $K(\theta)$ as θ approaches 90°

Dwight 1935 tabulates $K/\log_e(4 \sec \theta)$ to 7–11 decimals with Δ for $\theta = 86°(1')90°$. He also tabulates K itself, see Art. 21·21.

Heuman 1941 writes

$$\frac{2}{\pi}K(\theta°) = G_0(\theta°) - a \log_{10}(90-\theta), \quad \text{where} \quad a = \frac{2}{\pi}\log_e 10 = 1{\cdot}46587\ 1$$

and tabulates G_0 to 6 decimals with first differences for $\theta° = 65°(0°{\cdot}1)90°$. See also Art. 21·231.

21·215. $\mathrm{Log}_{10} K(\theta)$ and $\log_{10} E(\theta)$

12 dec.	$0(0°{\cdot}1)90°$	Δ^3 to 70°, then Δ^4	**Legendre 1816, 1826** (T. 1)

From 0 to 15° and from 75° to 90°, 14-decimal values of the functions are given, but the differences throughout are those of the 12-decimal values. The 1826 table is reproduced in facsimile in Legendre–Pearson 1934. On errors see Heuman 1941.

8 dec.	$45°(0°{\cdot}1)90°$	Δ^2	Rosa & Cohen 1908 Rosa & Grover 1912
8 dec.	$80°(0°{\cdot}1)90°$	Δ^2	Russell 1914 (118)
7 dec.	$0(0°{\cdot}5)90°$	No Δ	Bertrand 1870 (714) Potin 1925 Greenhill 1892 (10, 177)
7 dec.	$0(1°)90°$	No Δ	Legendre 1811 (118)
6 dec.	$0(1°)90°$	No Δ	Fowle 1933
5 dec.	$0(0°{\cdot}1)90°$	No Δ	Bohlin 1900 Witkowski 1904
4 dec.	$0(1^g)100^g$	No Δ	Hoüel 1901

21·216. Special Table for $\log_{10} K$ as θ approaches $\frac{1}{2}\pi$

Hoüel 1901 tabulates $\log_{10} K - \log_{10}\log_e(4/k')$ to 4 decimals without differences for $\theta = 50^g(1^g)100^g$. The quantity tabulated vanishes at $\theta = 100^g$.

21·221. $M(\theta) = \pi/2K(\theta)$

7 dec.	$0(0°{\cdot}5)90°$	No Δ	Gauss 1866*a*

21·222. $1/M(\theta) = 2K/\pi$

6 dec.	$0(0°{\cdot}1)90°$	Δ to 65°	Heuman 1941

21·223. Log $M(\theta) = \log(\pi/2K)$

7 dec.	$0(0°{\cdot}5)90°$	No Δ	Gauss 1866*a*

21·224. Log $\{1/M(\theta)\} = \log(2K/\pi)$

14 dec.	$0(0°{\cdot}1)15°(0°{\cdot}5)45°$	No Δ	Legendre 1816, 1826 (T. 6)
4 dec.	$0(1^g)100^g$	No Δ	Hoüel 1901

21·226. $2E(\theta)/\pi$

6 dec.	$0(0°{\cdot}1)90°$	Δ	Heuman 1941

21·228. Functions of tan θ

Rosa & Cohen 1908 (113) and Rosa & Grover 1912 (193) give K, E, log K, log E, all to 7 decimals without differences, for tan $\theta = {\cdot}1({\cdot}1)1({\cdot}5)3(1)5(2{\cdot}5)12{\cdot}5$.

21·23. $K(k^2)$ and $E(k^2)$

Both K and E are given unless the contrary is indicated.

16 dec.	K	$k^2 = 0({\cdot}01)1$		New York W.P.A. (MS.)
10; 12 d.	K	$k^2 = 0({\cdot}001){\cdot}834;\ {\cdot}835({\cdot}001)1$	No Δ	Hayashi 1930*a* (72)
10 dec.	K	$k^2 = 0(10^{-7})10^{-5}(10^{-5}){\cdot}00249$	No Δ	Hayashi 1930*a* (62)
8 dec.	K	$k^2 = {\cdot}00250(10^{-5}){\cdot}003$		
8 dec.	K	$1-k^2 = 0(10^{-7})10^{-5}(10^{-5}){\cdot}003$		
10 dec.	E	$k^2 = 0({\cdot}001)1$	No Δ	Hayashi 1933 (62)
9 dec.		$0({\cdot}01)1$	Δ	Milne-Thomson 1931*a*
7 dec.		$0({\cdot}001)1$	Δ	Samoilova-Yakhontova 1935
7 dec.		$0({\cdot}01)1$	No Δ	Milne-Thomson 1931*b*
6 dec.		$0({\cdot}001)1$	Δ	Nagaoka & Sakurai 1922
5 dec.		$0({\cdot}001)1$	No Δ	Hayashi 1930*b*
4 dec.	K	$0({\cdot}001){\cdot}997(10^{-5}){\cdot}99999(10^{-7})1$	IΔ	Dwight 1941
5 dec.	E	$0({\cdot}001)1$		
4 dec.		$0({\cdot}01)1$	IΔ	Jahnke–Emde 1938

21·231. Special Tables for $K(k^2)$ as k^2 approaches 1

Airey 1935*a* puts $K = K_1 \log_e(4/k') - K_2$, where K_1 and K_2 are power series in k'^2, and tabulates K_1 and K_2 with δ^2 to

13 dec. for $k'^2 = 0({\cdot}00001){\cdot}00010$
12 dec. for $k'^2 = 0({\cdot}0001){\cdot}0010$
10 dec. for $k'^2 = 0({\cdot}001){\cdot}170$

Samoilova-Yakhontova 1935 (106) writes $K = \beta \log_{10}(4/k')$, so that β approaches $\log_e 10 = 2{\cdot}30258\ 51$ as k^2 approaches 1, and tabulates β to 7 decimals with first differences for $k^2 = {\cdot}95({\cdot}001)\,1$. Higher differences are not negligible.

Jahnke & Emde 1938 (78) puts $K = \log_e(4/k') + h$, and tabulates h to 4 decimals with IΔ for $k^2 = {\cdot}7({\cdot}01)\,1$. See also Art. 21·12.

See also Art. 21·211.

21·232. Special Tables for $E(k^2)$ as k^2 approaches 1

Airey 1935*a* puts $E = E_1 \log_e(4/k') + E_2$, where E_1 and E_2 are power series in k'^2, and tabulates E_1 and E_2 with δ^2 to

12 dec. for $k'^2 = 0({\cdot}00001){\cdot}00010$
11 dec. for $k'^2 = 0({\cdot}0001){\cdot}0010$
10 dec. for $k'^2 = 0({\cdot}001){\cdot}100$

21·241. $M(k^2) = \pi/2K(k^2)$

12 dec.	$k^2 = {\cdot}835({\cdot}001)\,1$		No Δ	Hayashi 1930*a* (82)
8 dec.	$k^2 = {\cdot}997(10^{-5}){\cdot}99999(10^{-7})\,1$		No Δ	Hayashi 1930*a* (62)
10 dec.	$k^2 = 0({\cdot}01)\,1$	$(M = 1/\vartheta_3^2)$	Δ	Milne-Thomson 1931*a*

21·242. $(K-E)/k^2$, etc.

Jahnke & Emde 1938 (82) tabulates to about 4 decimals with IΔ for $k^2 = 0({\cdot}01)\,1$

$$D = (K-E)/k^2 \qquad B = K - D \qquad C = (D-B)/k^2$$

as well as $C + 2 - \log_e(4/k')$ and $D + 1 - \log_e(4/k')$ to about 4 decimals with IΔ for $k^2 = {\cdot}7({\cdot}01)\,1$. All these tables are from Emde 1936.

21·245. K'/K as a Function of k^2

The following give K'/K and K/K':

10 dec.	$k^2 = 0({\cdot}001){\cdot}5$	No Δ	Hayashi 1930*a* (72)
8 dec.	$k^2 = 0(10^{-7})\,10^{-5}(10^{-5}){\cdot}003$	No Δ	Hayashi 1930*a* (62)

21·246. E/K as a Function of k^2

7 dec.	$k^2 = 0({\cdot}1)\,1$	No Δ	Milne-Thomson 1932 (238)

21·25. Functions of k

All the functions of k mentioned below occur in connection with the expansion of the disturbing function in dynamical astronomy. If we put

$$(1 + k^2 - 2k\cos S)^{-1/2} = \tfrac{1}{2}b_0 + \sum_1^\infty b_n \cos nS$$

we have $b_0 = 4K/\pi$, $b_1 = 4(K-E)/\pi k$. For the remaining b_n, which are obtainable from b_0 and b_1 by recurrence relations, see Laplace coefficients (Section 22). Tables of the derivatives of Laplace coefficients involve dK/dk, dE/dk, etc.

21·251. $K(k)$, $E(k)$ and $M(k)$

10 dec.		$k = 0(\cdot 01)1$	No Δ	Fletcher 1940
12 dec.	K, M	$k = 0(\cdot 01)\cdot 7(\cdot 005)1$	Δ^n	Fletcher (MS.)
	E	$k = 0(\cdot 01)\cdot 9(\cdot 005)1$		

21·252. $b_0 = 4K/\pi$

11 dec.	$k = 0(\cdot 01)\cdot 75(\cdot 005)1$	Fletcher 1939

δ_m^{2n} of 8-decimal values given up to $k = \cdot 9$.

21·253. Log $b_0 = \log(4K/\pi)$

7 dec.	$k = 0(\cdot 005)\cdot 75$	Runkle 1856

Logarithms of variations for ·001 are given to 4–5 decimals.

See also Brown & Brouwer 1932, where the argument is $k^2/(1-k^2)$.

21·254. $b_1 = 4(K-E)/\pi k$

11 dec.	$k = 0(\cdot 01)\cdot 9(\cdot 005)1$	Fletcher 1939

δ_m^2 of 8-decimal values given up to $k = \cdot 75$.

21·255. Log $(b_1/k) = \log\left(\dfrac{4}{\pi}\cdot\dfrac{K-E}{k^2}\right)$

7 dec.	$k = 0(\cdot 005)\cdot 75$	Runkle 1856

Logarithms of variations for ·001 are given to 4–5 decimals. On errors, see Witt 1932.

See also Brown & Brouwer 1932.

21·256. $4(K-E)/\pi$

11 dec.	$k = 0(\cdot 01)\cdot 9(\cdot 005)1$	Fletcher (MS.)

21·26. K and E with Argument $\log_{10} k'^{-2}$

In April 1943 Mr. E. L. Kaplan, writing to us from Washington, kindly reported that he had a 10-decimal table (completed but as yet unpublished) of K and E for $\log_{10} k'^{-2} = 1(\cdot 005)2(\cdot 01)6$.

21·27. Auxiliary Tables and Methods for the Calculation of K and E

Witt 1904 gives auxiliary tables for 8-decimal logarithmic work (for details see Mises 1928, p. 28).

Robbins 1909 tabulates N_1 to 8 decimals without differences for

$$k = \cdot 4(\cdot 01)\cdot 6(\cdot 001)\cdot 849, \quad \text{where} \quad N_1 = \log\frac{K}{2\pi} + 2\log(1+\sqrt{k'})$$

Newcomb 1884 (69) tabulates $\log N$ to 7 decimals without differences for

$$k = \cdot 45(\cdot 01)\cdot 75, \quad \text{where} \quad N = \frac{4K}{\pi}\cos\frac{\theta}{2}\sqrt[4]{(\cos\theta)}$$

The last figure is approximate.

Idelson 1927 comments on the above tables and gives an auxiliary table for the 8-decimal logarithmic calculation of K.

See also Witt 1932.

21·28. Tables of Functions involving K and E

The following are of the nature of applications of complete elliptic integrals. Most of the tables are of little use except in connection with the particular purpose for which they were made. See also Arts. 16·29 and 21·331.

21·281. $f = \left(\frac{2}{k} - k\right)K - \frac{2}{k}E$

This occurs in connection with the mutual inductance of two circles with a common axis. Tables must be very common.

Much information on inductance problems is given in Rosa & Grover 1912, which is an extended version of Rosa & Cohen 1908. See also later papers (including some mentioned below) by Grover and others in the publications of the Bureau of Standards.

7 dec. $\log_{10} f$ $\theta = 60°(0°{\cdot}1)90°$ No Δ Maxwell 1892 (Niven)

Reproduced (from 1881 edition) in A. Gray 1893 and in Mascart & Joubert 1897.

7 dec. $\log_{10} f$ $\theta = 60°(0°{\cdot}1)90°$ Δ Rosa & Cohen 1908; Rosa & Grover 1912

Recalculation of table in Maxwell. Considerable changes in 7th decimal.

5 dec. f $\theta = 60°(0°{\cdot}1)90°$ No Δ Jahnke–Emde 1909 (77)

Deduced from logarithms in Maxwell (1881 edition, table as above).

5–6; 4 d. $4\pi f$ $\theta = 0(6')90°$; $89°(1')90°$ No Δ Pidduck 1925 (186)

5–6 fig. $4\pi f$; 6 dec. $\log_{10}(4\pi f)$ $k^2 = 0(\cdot001)1$ Δ^2 Nagaoka & Sakurai 1927 (140)

See also Nagaoka 1909, 1911*a*, 1912. Values from the first paper are reproduced in Rosa & Grover 1912.

5–6 fig. $4\pi f/1000$; 6 dec. $\log_{10}(4\pi f/1000)$ $k^2 = 0(\cdot005)1$ Δ^n Grover 1924

There are auxiliary tables for $k^2 < \cdot1$ and $k'^2 < \cdot1$, and special tables for equal circles.

4 fig. $2\pi k f$ $k^2 = 0(\cdot01)1$ Δ Curtis & Sparks 1924

Various other tables and charts.

21·282. $f = k\left(\frac{2-k^2}{1-k^2}E - 2K\right) = \sin\theta\{(1+\sec^2\theta)E - 2K\}$

This occurs in connection with the attraction of two circular currents with a common axis. See Grover 1916, where various q-series, etc. are tabulated.

8 dec. $\log_{10} f$ $\theta = 55°(0°{\cdot}1)70°$ Δ^2 Rosa, Dorsey & Miller 1912

7 dec. $\log_{10} f$ $\theta = 55°(0°{\cdot}1)69°{\cdot}9$ No Δ Rayleigh & Sidgwick 1884

The recomputation by Rosa, etc. shows that the 7th decimal is valueless. The table is reproduced in Mascart & Joubert 1897.

5; 4 d. f $\theta = 55°(0°{\cdot}1)57°{\cdot}1$; $57°{\cdot}2(0°{\cdot}1)69°{\cdot}9$ No Δ Jahnke–Emde 1909 (79)

From logarithms of Rayleigh & Sidgwick 1884.

5–6 fig. πf; 6 dec. $\log_{10}(\pi f)$ $k^2 = 0(\cdot001)1$ Δ^2 Nagaoka & Sakurai 1927 (161)

$$21\cdot283.\quad f = \frac{4}{3}\left\{\frac{K+(\tan^2\theta-1)E}{\sin\theta} - \tan^2\theta\right\}$$

This occurs in connection with the inductance of thin-walled cylindrical coils. See also Art. 21·228.

4 d.	$\frac{1}{2}f$	$\tan\theta = 0(\cdot01)1$, $\cot\theta = 0(\cdot01)1$		Emde 1912
4–5 f.	πf, $\pi f\tan\theta$	$\tan\theta = 0(\cdot01)1$, $\cot\theta = 0(\cdot01)1$	No Δ	Jahnke–Emde 1933 (152), 1938 (88)

Further references are given.

4 d.	$2\pi f$	$\tan\theta = \cdot2(\cdot1)1(\cdot2)4$	No Δ	Russell 1914 (114)
6 d.	$\frac{f}{\pi}\cot\theta$, $\log_{10}(\pi f\cot\theta)$	$k^2 = 0(\cdot001)1$	Δ^2	Nagaoka & Sakurai 1927 (77, 98, 118)
6 d.	,, ,,	$\tan\theta$, $\cot\theta = 0(\cdot001)1$	Δ^2	
6 d.	$\frac{f}{\pi}\cot\theta$, $\log_{10}\left(\frac{f}{\pi}\cot\theta\right)$	$\theta = 0(1°)90°$	Δ^2	Nagaoka 1909
6 d.	$\frac{f}{\pi}\cot\theta$	$\tan\theta = 0(\cdot01)1(\cdot05)2(\cdot1)5(\cdot5)10$	Δ	

21·284. Other Applications

Hill 1881 tabulates, logarithmically to 8 decimals with Δ^2 for $\theta = 0(0°\cdot1)50°$, $2E/\pi k'^2$ and two more complicated functions of the complete integrals.

Shook 1927 tabulates the arithmetic-geometric mean of 1 and x to 4 figures without differences for $x = 1(\cdot01)20(1)100$.

Spielrein 1915 tabulates

$$\frac{16\pi}{3(1-a)^2}\left\{\int_a^1 (K-E)\,dk - a^3\int_a^1 \frac{K-E}{k^3}\,dk\right\}$$

Values are given to about 4 figures for $a = 0, \cdot05(\cdot01)\cdot95, \cdot99$ and to about 7 figures for $a = 0(\cdot05)\cdot90$.

See also Smekal 1918, Levy & Forsdyke 1927 and Watson 1939.

21·3. Incomplete Elliptic Integrals of the First and Second Kinds

In addition to his main table, Legendre's volumes give much auxiliary numerical matter, which does not call for indexing in detail.

21·31. $F(\theta, \phi)$ and $E(\theta, \phi)$

12 dec.	$\theta = 0(1°)90°$, $\phi = 45°$	Δ^5 or Δ^6	Legendre 1816, 1826 (T. 8)

Reproduced in facsimile in Legendre–Emde 1931.

10 dec.	$\theta = 89°$, $\phi = 0(0°\cdot5)90°$	Δ	Legendre 1816 (84), 1826 (77)
10 dec.	$\theta = 0(1°)45°$, $\phi = 0(1°)90°$	No Δ	**Legendre 1816, 1826** (T. 9)
9 dec.	$\theta = 45°(1°)90°$, $\phi = 0(1°)90°$	No Δ	**Legendre 1816, 1826** (T. 9)

On errors in these fundamental tables see Samoilova-Yakhontova 1935 (6) and particularly Heuman 1941. A few of the errors in Legendre 1816 are corrected in Legendre 1826. The 1816 table is reproduced in facsimile in Potin 1925 and Legendre–Emde 1931, while the 1826 table is reproduced in facsimile in Legendre–Pearson 1934.

5 dec.	$\theta = 0(5^\circ)90^\circ$, $\phi = 0(1^\circ)90^\circ$	No Δ	Lévy 1898 Bohlin 1900 Dale 1903 Hancock 1917 (from Lévy) Hayashi 1930*b*

Smaller 5-decimal tables are contained in Bertrand 1870 and Láska 1888–1894.

4 dec.	$\theta = 0(5^\circ)90^\circ$, $\phi = 0(1^\circ)90^\circ$	No Δ	Jahnke–Emde 1909–1938

The values of $F(\theta, \phi)$ for $\phi = 1^\circ(1^\circ)5^\circ$ are given to 5 decimals (4 figures). Also $F(89^\circ, \phi)$ is given for $\phi = 0$(var.)$65^\circ(1^\circ)90^\circ$. Errata, *M.T.A.C.*, **1**, 198, 1944.

4 dec.	$\theta = 0(5^\circ)90^\circ$, $\phi = 0(1^\circ)90^\circ$	No Δ	Silberstein 1923

The values of $F(\theta, \phi)$ for $\phi = 1^\circ(1^\circ)5^\circ$ are given to 5 decimals (4 figures). Smaller 4-decimal tables are given in Peirce 1910 and Hütte 1931.

4 dec.	$\theta = 0(10^g)100^g$, $\phi = 0(10^g)100^g$	No Δ	Hoüel 1901

21·315. $\mathrm{Log}_{10} F(\theta, \phi)$ and $\log_{10} E(\theta, \phi)$

4 dec.	$\theta = 0(10^g)100^g$, $\phi = 0(10^g)100^g$ Also log F for $\theta = 90^g(1^g)100^g$, $\phi = 90^g(1^g)100^g$	No Δ	Hoüel 1901

21·317. $F(\theta, \phi)/K$

4 dec.	$\theta = 0(10^g)100^g$, $\phi = 0(10^g)100^g$	No Δ	Hoüel 1901

21·32. $F(k^2, \phi)$ and $E(k^2, \phi)$

5 dec.	$k^2 = 0(\cdot01)1$, $\phi = 0(1^\circ)90^\circ$	IΔ in k^2, Δ in ϕ	Samoilova-Yakhontova 1935

21·331. $E(k, \phi)$ with Arguments k' and $\sin\phi$

11 dec.	$k' = 0(\cdot01)1$, $\sin\phi = 0(\cdot01)1$	Δ^2	Schmidt 1842

These are arcs of ellipses with semi-axes $a = 1$, $b = k'$ between the points $(0, k')$ and $(\sin\phi, k'\cos\phi)$. Verdam 1842 deals with Schmidt's table of elliptic arcs.

The corresponding complete circumferences, $4E$, are given to 4 decimals for $k' = 0(\cdot01)1$ in Moore 1674. The quadrants, E, are given to 5 decimals for $k' = 0(\cdot01)1$ in Schlömilch 1866.

21·35. Lemniscate Case ($\theta = 45^\circ$ or $k^2 = \frac{1}{2}$)

Legendre 1816, 1826 (T. 2) gives $F(45^\circ, \phi)$ and $E(45^\circ, \phi)$ to 12 decimals with Δ^n for $\phi = 0(0^\circ{\cdot}5)90^\circ$.

Greenhill 1892 (11) gives from Legendre 1826 $F(45^\circ, \phi)$ to 5 decimals without differences for $\phi = 0(0^\circ{\cdot}5)90^\circ$.

Any tables of $F(\theta, \phi)$ and $E(\theta, \phi)$ may of course be used at $\theta = 45^\circ$. The special point of the above is the $0^\circ{\cdot}5$ interval in ϕ, and, in the case of Legendre, the 12 decimals. See also MacGregor 1942.

21·39. Numerical Calculation of Elliptic Integrals

On the numerical calculation of elliptic integrals of various kinds, see, apart from several of the usual textbooks on elliptic functions:

L. V. King, *On the Direct Numerical Calculation of Elliptic Functions and Integrals*, Cambridge 1924
F. W. Grover 1933 (and a paper by King immediately preceding this)
Runge 1891
Schlömilch 1879

21·4. Elliptic Integrals of the Third Kind

The incomplete elliptic integral of the third kind, say

$$\int_0^\phi \frac{1}{1-p\sin^2\phi}\frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}$$

is apparently a function of three variables, but when $0 < p < k^2$ it may be expressed in terms of theta and Jacobian zeta functions of real argument. When p lies outside the range 0 to k^2, the arguments become complex, and the integral cannot be made to depend on functions of two real variables; it has apparently not been tabulated as a function of three variables.

But the complete integral with upper limit of integration $\frac{1}{2}\pi$ may be expressed in terms of complete and incomplete integrals of the first and second kinds, and in the case $k^2 < p < 1$ (to which the case $p < 0$ may also be reduced) it has been tabulated in Heuman 1941. The function tabulated is

$$\Lambda_0(\alpha, \beta) = \frac{2}{\pi}\int_0^{\pi/2} \frac{\cos^2\alpha\sin\beta\cos\beta\sqrt{(1-\cos^2\alpha\sin^2\beta)}}{\cos^2\alpha\cos^2\beta+\sin^2\alpha\cos^2\phi}\frac{d\phi}{\sqrt{(1-\sin^2\alpha\sin^2\phi)}}$$

It is given to 6 decimals without differences for $\alpha = 0(1^\circ)90^\circ$, $\beta = 0(1^\circ)90^\circ$. There is also a supplementary table to 6 decimals for $\alpha = 0(0^\circ\cdot1)5^\circ\cdot9$, $\beta = 80^\circ(1^\circ)89^\circ$.

See also Art. 16·29.

21·5. Jacobian Elliptic Functions

21·51. $\phi = \mathrm{am}\, u$

Legendre 1816, 1826 (T. 7) gives ϕ to 7 decimals of 1″ with Δ^3 for $\theta = 0(0^\circ\cdot1)45^\circ$, $u = \frac{1}{10}K$ only.

1′	$\theta = 0(5^\circ)80^\circ(1^\circ)89^\circ$, $x = \pi u/2K = 0(1^\circ)90^\circ$	No Δ	Adams & Hippisley 1922
0°·01	$\theta = 15^\circ$, $x = 0(3^\circ)90^\circ$	No Δ	B.A. 1912
0°·01	$\theta = 45^\circ, 75^\circ$, $x = 0(1^\circ)90^\circ$ [b]		
0°·001	$\theta = 45^\circ$, $x = 0(1^\circ)90^\circ$	No Δ	B.A. 1919

Small provisional tables for $\theta = 45^\circ$ and 75° in B.A. 1911 are disregarded here and in other articles.

21·511. $\phi = \mathrm{am}\, u$ for Singular Moduli

Values of ϕ are also given for cases in which the modulus is such that the period-ratio K/K' has certain definite values, as follows:

$0°{\cdot}01$	$K/K' = 2, x = 0(1°)90°$	No Δ	B.A. 1912
$0°{\cdot}01$	$\begin{cases} K/K' = 1/\sqrt{2}, \sqrt{2}, 3/\sqrt{2}, 2\sqrt{2}, 3, 2\sqrt{3}, 4, \\ \quad 3\sqrt{2}, 5 (\text{to } 0°{\cdot}1 \text{ only}), 3\sqrt{3}, x = 0(1°)90° \end{cases}$	No Δ	B.A. 1913
$1'$	$K/K' = 2, 4, x = 0(1°)90°$	No Δ	B.A. 1919

These values of the period-ratio in the various B.A. reports will be referred to again, principally in Arts. 21·581, 21·591, 21·821. The information about modular angle and complete integrals given in the headings is often very rough; parts at least of the main body of the tables must be affected. The information about the cases $K/K' = 2$ and 4 in B.A. 1919 supersedes that in B.A. 1912 and B.A. 1913 for these cases.

For extensive calculations on singular moduli, together with references to earlier work, see the following series of six papers by G. N. Watson:

(1) *Q.J. Math.*, (Oxf.) **3**, 81–98, 1932
(2) *Q.J. Math.*, (Oxf.) **3**, 189–212, 1932
(3) *Proc. London Math. Soc.*, (2) **40**, 83–142, 1935
(4) *Acta Arithmetica*, **1**, 284–323, 1936
(5) *Proc. London Math. Soc.*, (2) **42**, 377–397, 1937
(6) *Proc. London Math. Soc.*, (2) **42**, 398–409, 1937

21·52. sn u, cn u and dn u

15 dec.	$\theta = 0(1°)89°,\ u/K = 0(\tfrac{1}{90})1$		Spenceley (MS. 1943)
5 dec.	$\begin{cases} k^2 = 0(\cdot 1)\cdot 5, & u = 0(\cdot 01)2 \\ k^2 = \cdot 6(\cdot 1)\cdot 8, & u = 0(\cdot 01)2{\cdot}5 \\ k^2 = \cdot 9, & u = 0(\cdot 01)3 \end{cases}$	Δ in u	Milne-Thomson 1931*b*

When $k^2 = 1$ the tables give, also to 5 decimals with differences,

$$\operatorname{sn} u = \tanh u \text{ for } u = 0(\cdot 01)3(\cdot 1)6{\cdot}5$$
$$\operatorname{cn} u = \operatorname{dn} u = \operatorname{sech} u \text{ for } u = 0(\cdot 01)4(\cdot 1)12{\cdot}9$$

See also Reno (MS. 1944) and Art. 21·53 below.

21·53. sn $(u+iv)$

2–3 d.	$\theta = 15°, 20°, 75°,\ u = 0(K/10)K,\ v = 0(K'/10)K'$	Bergmann 1925

New York W.P.A. reported in May 1942 that computations were in progress for "tables of the complete elliptic integrals K of the first kind, the nome $q = \exp(-\pi K'/K)$, and the elliptic functions am u, sn u, cn u, dn u, for real and imaginary arguments". For later information see A. N. Lowan, *M.T.A.C.*, **1**, 125–126, 1943.

21·58. Jacobian Zeta Function Z(u)

This is defined by $E(u) = \int_0^u \operatorname{dn}^2 u\, du = \dfrac{E}{K}u + \mathrm{Z}(u)$, and we have

$$\mathrm{Z}(u) = \Theta'(u)/\Theta(u) = M\vartheta_4'(x)/\vartheta_4(x), \quad \text{where} \quad x = Mu = \pi u/2K$$

7 dec.	$\begin{cases} k^2 = 0(\cdot 1)\cdot 6, & u = 0(\cdot 01)2 \\ k^2 = \cdot 7, \cdot 8, & u = 0(\cdot 01)2{\cdot}5 \\ k^2 = \cdot 9, 1, & u = 0(\cdot 01)3 \end{cases}$	δ^2 in u	Milne-Thomson 1932

Note that when $k^2 = 1$ we have $\mathrm{Z}(u) = \tanh u$.

In the following tables, $E(r)$ is used for $Z(rK/90)$, i.e. for the value of Z when $x = r°$; we shall describe the tables in our notation.

10 dec.	$\theta = 0(5°)80°(1°)89°$, $x = \pi u/2K = 0(1°)90°$	No Δ	Adams & Hippisley 1922
7 dec.	$\theta = 15°$, $x = 0(3°)90°$	No Δ	B.A. 1912
6 fig.	$\theta = 45°, 75°$, $x = 0(1°)90°$		
10 dec.	$\theta = 45°$, $x = 0(1°)90°$	No Δ	B.A. 1919

See also Art. 21·88.

21·581. Zeta Function for Singular Moduli

Values of the zeta function are also given for the cases mentioned in Art. 21·511 as follows:

6–7 dec.	Same arguments as in Art. 21·511	No Δ	B.A. 1912
6–7 dec.	" " " "	No Δ	B.A. 1913
	Only 4 decimals in case $K/K' = 5$. Some tables incomplete.		
10 dec.	Same arguments as in Art. 21·511	No Δ	B.A. 1919

21·59. $u = 2Kx/\pi$

Conversion from x, the natural argument of theta functions, to u is facilitated by the following tables, in which when $x = r°$ the tables give $rK/90$.

10 dec.	$\theta = 0(5°)80°(1°)89°$, $x = 0(1°)90°$	Adams & Hippisley 1922
7 dec.	$\theta = 15°$, $x = 0(3°)90°$; $\theta = 45°, 75°$, $x = 0(1°)90°$	B.A. 1912
10 dec.	$\theta = 45°$, $x = 0(1°)90°$	B.A. 1919

See also Art. 21·88.

21·591. $u = 2Kx/\pi$ for Singular Moduli

Values are also given for the cases mentioned in Art. 21·511 as follows:

7 dec.	Same arguments as in Art. 21·511	B.A. 1912
6–7 dec.	" " " "	B.A. 1913
10 dec.	" " " "	B.A. 1919

21·6. Weierstrassian Elliptic Functions

Greenhill 1889 (Hadcock) gives a table of $\wp'(u)$, $\wp(u)$, $\zeta(u)$, $\sigma(u)$ to 5 decimals (but not more than 7–8 figures) without differences for the "equianharmonic" case in which $\wp'^2 = 4\wp^3 - 1$. The real half-period ω_2 is 1·52995, and the argument is $r = \dfrac{180u}{\omega_2} = 0(1)240$. The table is reproduced to not more than 4 decimals in Jahnke & Emde 1909 (73), 1933 (168), 1938 (102). For two errors see *M.T.A.C.*, **1**, 109, 1943.

Greenhill 1886 (Hadcock) gives a table of $\wp(u)$ to 5–7 figures without differences for the case in which $\wp'^2 = 4\wp^3 - 4$, where $u = r\omega_2/180$ and $u = r\omega_2'/180$, and in both cases $r = 0(1)180$. Here ω_2 is the real half-period, and $\omega_2' = \omega_3 - \omega_1 = i\omega_2\sqrt{3}$. This case differs trivially from that of Greenhill 1889; the successive values of $\wp(u)$ for real u in the 1886 paper are $2^{2/3}$ times those in the 1889 paper; in both cases the corresponding Jacobian modulus is sin 15°.

See also Innes 1907 (case of real roots), Nagaoka & Sakurai 1922 (56) and Rosenhead 1940 (numerical application).

21·7. Jacobi's Nome q, Theta Functions of Zero, etc.

q is defined as $e^{-\pi K'/K}$. If $q = \Sigma a_n k^{2n}$, the exact values of a_n, $n \doteq 1(1)12$, are given in Hermite 1897 (Tisserand). The first 14 terms of the series obtained by the usual Weierstrass inversion of

$$\epsilon = \frac{q+q^9+q^{25}+\dots}{1+2q^4+2q^{16}+\dots}$$

are given in Lowan, Blanch & Horenstein 1942.

For New York W.P.A. tables of q in progress, see Art. 21·53.

21·71. $q(\theta)$

14–15 d.	$0(5°)80°(1°)90°$	No Δ	Adams & Hippisley 1922
8 dec.	$0(1°)90°$	No Δ	Glaisher 1877*a*
7; 6; 5 d.	$0(1°)27°$; $28°(1°)79°$; $80°(1°)90°$	No Δ	Láska 1888–1894

$e^{-\pi}$ and various rational powers thereof are given to 20–21 figures in B.A. 1919. See also Arts. 5·52 and 10·61.

21·711. $1/q(\theta)$

6–7 fig.	15°, 45°, 75°	B.A. 1912

Values to 4–7 figures are given in B.A. 1912, 1913 for the other moduli considered (see Art. 21·511).

21·715. $\text{Log}_{10}\, q(\theta)$ and $\log_{10} 1/q(\theta)$

10 dec.	$0(0°·1)45°(1°)90°$	$-\log q$	No Δ	**Plana 1848**
10 dec.	$0(1°)45°$	$\log q$	No Δ	Innes 1902
8 dec.	$0(1')90°$	$\log q$	No Δ	Meissel 1860
8 dec.	$0(1°)90°$	$\log q$	No Δ	Glaisher 1877*a*
5 dec.	$0(0°·1)90°$	$\log q$	Δ	**Jacobi 1843**
5 dec.	$0(5')90°$	$\log q$	No Δ	Bertrand 1870; Lévy 1898; Potin 1925
5 dec.	$0(1°)90°$	$\log q$	No Δ	Schlömilch 1879
5 dec.	$0(1°)90°$	$\log q$	No Δ	Silberstein 1923
5 dec.	$0(1°)90°$	$-\log q$	No Δ	Láska 1888–1894
4 dec.	$0(0°·1)90°$	$\log q$	No Δ	Bohlin 1900
4 dec.	$0(5')90°$	$\log q$	No Δ	Jahnke-Emde 1909–1938
4 dec.	$0(1^g)100^g$	$\log q$	No Δ	Hoüel 1901

21·716. $\text{Log}_{10}(16q/k^2)$

This is an auxiliary quantity for small k, since $16q/k^2 \to 1$ as $k \to 0$.

4 dec.	$0(1^g)50^g$	No Δ	Hoüel 1901

21·717. $\text{Log}_{10} \log_{10} \dfrac{1}{q(\theta)}$

This is $C + \log_{10}(K'/K)$, where $C = \log_{10}(\pi \log_{10} e)$, so that the expression is easily derivable from Legendre's table of $\log_{10} K$. See Art. 5·511.

14; 12 dec.	$0(0°·1)15°$; $15°(0°·1)45°$	No Δ	Verhulst 1841 (Loxhay)
10 dec.	$0(1°)45°$	No Δ	Innes 1902 (from Verhulst)
7 dec.	$0(0°·5)90°$	No Δ	Bertrand 1870; Potin 1925

21·718. $\mathrm{Log}_{10}\{q(\theta)\cot^2\tfrac{1}{2}\theta\}$

10 dec.	$\theta = 0(1^\circ)45^\circ$	$\log v^4$	Innes 1902

21·72. $q(k^2)$

10 dec.	$k^2 = 0(\cdot01)1$	Δ	Milne-Thomson 1930*b*
8 dec.	$k^2 = 0(\cdot01)1$	No Δ	Milne-Thomson 1931*b*
8 dec.	$k^2 = 0(\cdot001)1$	Δ	Samoilova-Yakhontova 1935
8 dec.	$k^2 = 0(\cdot001)\cdot5$	No Δ	Hayashi 1930*a* (84)
5 dec.	$k^2 = 0(\cdot001)\cdot5$	No Δ	Hayashi 1930*b* (148)

21·724. Powers of q

11 dec.	$q^{25/4}$	$k^2 = \cdot3(\cdot001)\cdot5$	No Δ	Hayashi 1930*a* (96)
10; 8 d.	$q^4, q^{9/4}; q^{1/4}$	$k^2 = 0(\cdot001)\cdot5$	No Δ	Hayashi 1930*a* (84)

On extensive MS. tables see A. N. Lowan, *M.T.A.C.*, **1**, 125–126, 1943.

21·725. $\mathrm{Log}_{10}\, q(k^2)$

10 dec.	$k^2 = 0(\cdot001)1$	No Δ	Hayashi 1930*a* (72)
8 dec.	$k^2 = 0(10^{-7})10^{-5}(10^{-5})\cdot003$, $1-k^2 =$ same	No Δ	Hayashi 1930*a* (62)
7 dec.	$k^2 = 0(\cdot001)1$	Δ	Nagaoka & Sakurai 1922

21·726. $\mathrm{Log}_{10}(q/k^2)$ and $\log_e(k^2/q)$

These are auxiliary tables for small k.

7 dec.	$k^2 = 0(\cdot001)\cdot05$	Δ	Nagaoka & Sakurai 1922

21·728. $\mathrm{Log}_{10}\log_e(1/q)$

7 dec.	$k^2 = 0(\cdot001)1$	Δ	Nagaoka & Sakurai 1922

21·75. Theta Functions of Zero

See also Arts. 21·8 ff., case $x = 0$.

For $1/\vartheta_3^2 = M$, see Arts. 21·2 ff. If $M = 1 - 4g_1q + 4g_2q^2 - 4g_3q^3 + \ldots$, the g's are all integral, and are given up to g_{27} in Glaisher 1885.

21·751. ϑ_2 and ϑ_4 as Functions of θ

10–12 dec.	$\theta = 0(5^\circ)80^\circ(1^\circ)89^\circ$	No Δ	Adams & Hippisley 1922

21·752. $\mathrm{Log}_{10}\,\vartheta_3$ as a Function of θ

9 dec.	$\theta = 0(0^\circ\!\cdot5)90^\circ$	No Δ	Bertrand 1870 (714); Potin 1925

21·755. ϑ_2, ϑ_3 and ϑ_4 as Functions of k^2

8 dec.	$k^2 = 0(\cdot001)\cdot5$	No Δ	Hayashi 1930*a* (84)
5 dec.	$k^2 = 0(\cdot001)\cdot5$	No Δ	Hayashi 1930*b* (148)

21·756. Common Logarithms of ϑ_2, ϑ_3 and ϑ_4 as Functions of k^2

7 dec. $k^2 = 0(\cdot 001)\cdot 5$ Δ Nagaoka & Sakurai 1922

21·76. Derivatives of Theta Functions at Zero

The powers of π in the following articles are due to definitions of the theta functions with angle πx in place of our x.

21·761. $\pi\vartheta_1'$, etc.

Hayashi 1930*a* (84) gives $\pi\vartheta_1'$, $\pi^2\vartheta_1'''/\vartheta_1'$, $\pi^2\vartheta_2''/\vartheta_2$, $\pi^2\vartheta_3''/\vartheta_3$, $\pi^2\vartheta_4''/\vartheta_4$ to 8 decimals without differences for $k^2 = 0(\cdot 001)\cdot 5$.

Hayashi 1930*b* (148) gives $\pi\vartheta_1'$ to 5 decimals without differences for $k^2 = 0(\cdot 001)\cdot 5$.

21·762. $\text{Log}_{10}(\pi\vartheta_1')$, etc.

Nagaoka & Sakurai 1922 gives logarithms of the 5 functions mentioned under Hayashi 1930*a* in Art. 21·761, to 7 decimals for $k^2 = 0(\cdot 001)\cdot 5$ with first differences.

21·763. Logarithms of $\dfrac{\pi\vartheta_1'}{\sqrt{k}}$, $\dfrac{-\pi^2\vartheta_3''}{k^2\vartheta_3}$ and $\dfrac{\pi^2\vartheta_4''}{k^2\vartheta_4}$

These are auxiliary tables for small k.

7 dec. $k^2 = 0(\cdot 001)\cdot 05$ Δ Nagaoka & Sakurai 1922

21·78. Diffusion Functions

McKay 1930 tabulates the "simple diffusion function"

$$1 - \frac{8}{\pi^2}\left(e^{-x} + \tfrac{1}{9}e^{-9x} + \tfrac{1}{25}e^{-25x} + \dots\right)$$

to 4 decimals without differences for $x = 0(\cdot 01)3\cdot 59$.

A very brief table of

$$\frac{8}{\pi^2}\left(e^{-\pi^2 x/4} + \tfrac{1}{9}e^{-9\pi^2 x/4} + \tfrac{1}{25}e^{-25\pi^2 x/4} + \dots\right)$$

and

$$\frac{96}{\pi^4}\left(e^{-\pi^2 x/4} + \tfrac{1}{81}e^{-9\pi^2 x/4} + \tfrac{1}{625}e^{-25\pi^2 x/4} + \dots\right)$$

is given in Sherwood 1932. Somewhat larger tables relating to these and similar functions are given to 3–4 decimals in A. B. Newman 1931, 1932. Newman 1932 also tabulates functions of two variables which occur in the solution of special problems; these reduce to functions of the type we are considering when a parameter h tends to infinity. See also Newman & Green 1935.

Newman 1934 (607) tabulates the four functions

$$\frac{2}{\pi^2}\sum_{n=1}^{\infty}(\pm)^{n-1}\frac{e^{-n^2P}}{n^2} \quad \text{and} \quad \frac{2}{\pi^4}\sum_{n=1}^{\infty}(\pm)^{n-1}\frac{e^{-n^2P}}{n^4}$$

to 6 decimals for $P = 0(\cdot 1)1(\cdot 5)12$. Olson & Schultz 1942 (875) gives

$$(4/\pi)\left(e^{-\pi^2\theta} - \tfrac{1}{3}e^{-9\pi^2\theta} + \tfrac{1}{5}e^{-25\pi^2\theta} - \dots\right)$$

and its common logarithm to 5 decimals for $\theta = 0(\cdot 001)\cdot 2$.

21·8. Theta Functions

For definitions see Art. 21·0.

21·81. $\vartheta_1(x)$, $\vartheta_2(x)$, $\vartheta_3(x)$ and $\vartheta_4(x)$

The common logarithms of these functions are given to 4 decimals without differences for $\theta = 0(10^g)90^g$, $x = 0(10^g)100^g$ in Hoüel 1901.

Natural values deduced (with correction) from these are given to 4 figures without differences for the same arguments in Jahnke & Emde 1909 (71), 1933 (117), 1938 (45).

Natural values of $\vartheta_n(ix)$, $n = 1(1)4$, in the case $\theta = 9^\circ$ are given to 4 decimals without differences for $x/\pi = 0(\cdot05)1$ in Rosenhead 1929.

21·82. $A(x) = \dfrac{\vartheta_1(x)}{\vartheta_2}$, $B(x) = \dfrac{\vartheta_2(x)}{\vartheta_2}$, $C(x) = \dfrac{\vartheta_3(x)}{\vartheta_4}$ and $D(x) = \dfrac{\vartheta_4(x)}{\vartheta_4}$

15 dec.	$\theta = 0(1^\circ)89^\circ$, $x = 0(1^\circ)90^\circ$		Spenceley (MS. 1943)
10 dec.	$\theta = 0(5^\circ)80^\circ(1^\circ)89^\circ$, $x = 0(1^\circ)90^\circ$	No Δ	Adams & Hippisley 1922

For a few errata, see *M.T.A.C.*, **1**, 325, 1944.

6 fig.	$\theta = 15^\circ$, $x = 0(3^\circ)90^\circ$	No Δ	B.A. 1912
About 6 f.	$\theta = 45^\circ$ (errors), 75°, $x = 0(1^\circ)90^\circ$		
10 dec.	$\theta = 45^\circ$, $x = 0(1^\circ)90^\circ$	No Δ	B.A. 1919

21·821. $A(x)$, etc. for Singular Moduli

Values of $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are given for the cases mentioned in Art. 21·511 as follows:

6 fig.	Same arguments as in Art. 21·511	No Δ	B.A. 1912
About 6 f.	,, ,, ,, ,,	No Δ	B.A. 1913

Only about 4 figures in the case $K/K' = 5$.

10 dec.	Same arguments as in Art. 21·511	No Δ	B.A. 1919

21·83. Glaisher's Tables

$$\Theta_1(x) = \frac{\vartheta_3\vartheta_1(x)}{\vartheta_2} = \frac{\vartheta_1(x)}{\sqrt{k}} \qquad \Theta_2(x) = \frac{\vartheta_4\vartheta_2(x)}{\vartheta_2} = \sqrt{\frac{k'}{k}}\,\vartheta_2(x)$$

$$\Theta_3(x) = \frac{\vartheta_4\vartheta_3(x)}{\vartheta_3} = \sqrt{k'}\,.\,\vartheta_3(x) \qquad \Theta_4(x) = \vartheta_4(x)$$

These four functions, which are three numerators and a common denominator for sn, cn, dn (see Art. 21·0), were computed in 1872–1875 for the British Association under the superintendence of J. W. L. Glaisher. They were tabulated to 8 decimals, both naturally and logarithmically, for $\theta = 0(1^\circ)89^\circ$, $x = 0(1^\circ)90^\circ$.

The tables were set in type from $\theta = 0$ to $\theta = 89^\circ$ (360 pages), but have never been published, and no printed copy appears to have survived. The volumes containing the computations are now in the possession of the British Association.

See B.A. 1873 (171); *Mess. Math.*, **6**, 111–112, 1876, and B.A. 1930 (250); also *Coll. Math. Papers* of H. J. S. Smith, **1**, lxiii, 1894.

21·85. Derivatives of Theta Functions

Values of $\pi\vartheta_n'(ix)$, $n = 1(1)4$, in the case $\theta = 9°$ are given to 4 decimals without differences for $x/\pi = 0(\cdot05)1$ in Rosenhead 1929.

21·86. Logarithmic Derivatives of Theta Functions

The functions $\vartheta_n'(x)/\vartheta_n(x)$, $n = 1(1)4$, are tabulated logarithmically to 4 decimals without differences for $\theta = 0(10^g)90^g$, $x = 0(10^g)100^g$ in Hoüel 1901.

Natural values deduced (with correction) from these are given to 4 figures without differences for the same arguments in Jahnke & Emde 1909 (71), 1933 (117), 1938 (45).

21·88. Lemniscate Case (Special Sections)

Apart from the tables already listed for $\theta = 45°$ or $k^2 = \frac{1}{2}$, special tables with intervals $\frac{1}{7}$ and $\frac{1}{17}$ of a quarter-period are provided for this modulus in B.A. 1919. The tables give 15-decimal values, without differences, of

$$\frac{\Theta(u)}{\Theta(0)}, \quad \frac{\mathrm{H}(u)}{\mathrm{H}(K)}, \quad Z(u) \quad \text{and} \quad u$$

for $u = 0(K/7)K$ and for $u = 0(K/17)K$, i.e. for $x = 0(\pi/14)\pi/2$ and for $x = 0(\pi/34)\pi/2$. The theta-function ratios are $\vartheta_4(x)/\vartheta_4$ and $\vartheta_1(x)/\vartheta_2$ respectively, i.e. $D(x)$ and $A(x)$.

See also Wilton 1915.

SECTION 22

MISCELLANEOUS HIGHER MATHEMATICAL FUNCTIONS

Most of this section is concerned with functions of well-known types, but in Arts. 22·7 ff. we have taken the opportunity to include a few completely miscellaneous functions which have not found a place elsewhere in the *Index*.

Some of the functions dealt with in this section depend on a number of variables or parameters, and it has not always seemed profitable to go into full details about range and interval of the arguments, number of decimals or figures tabulated, etc.

The arrangement of Section 22 is as follows:

22·1. Riemann Zeta Function

This is the function $\zeta(s)$ of the complex variable $s = \sigma + it$ which may be defined when $\sigma > 1$ by $\zeta(s) = \sum_1^\infty n^{-s}$ or by $\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}dx}{e^x - 1}$. For the definition for other values of s, see Whittaker & Watson, *Modern Analysis*, Chapter XIII, or E. C. Titchmarsh, *The Zeta-Function of Riemann*, Cambridge, 1930. We may mention that when $\sigma > 0$, $\zeta(s)$ is the limit of $\left(\sum_1^n n^{-s}\right) - \frac{n^{1-s}}{1-s}$ as $n \to \infty$, and that when $\sigma < 0$, $\zeta(s)$ may be found from Riemann's functional equation

$$\cos \tfrac{1}{2}\pi s \Gamma(s) \zeta(s) = 2^{s-1}\pi^s \zeta(1-s)$$

The function has been the subject of much numerical investigation, but some of this has been concerned with locating zeros without calculating them precisely. The function is zero when s is an even negative integer, and all its non-trivial zeros are known to lie in the critical strip $0 < \sigma < 1$. Riemann's hypothesis, that they all lie on the line $\sigma = \frac{1}{2}$, has never been proved or disproved. Use is made of the functions $\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$ and $\frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$, which are real on $\sigma = \frac{1}{2}$. Diagrams of $\zeta(s)$ in $-6 < \sigma < 5$, $0 < t < 28$ are given in Jahnke & Emde 1933 (320), 1938 (270).

Most of the tabulations mentioned in the first few articles refer only to real values of s. The useful tables in Gram 1925 were made in 1903–1904 under Gram's direction by H. S. Nielsen, found in 1924 by C. Burrau among the papers left by Gram at his death, and edited for publication by N. E. Nörlund. For values when s is a positive integer, see Arts. 4·6 ff. Graphs of $\zeta(s)$, $(s-1)\zeta(s)$, $\zeta(s) - \frac{1}{s-1}$ and $\zeta'(s)/\zeta(s)$ for real values of s between -6 and $+5$ are given in Walther 1926.

22·11. $\zeta(s)$

10 fig.; 10 dec.	$s = -24(\cdot 1)-16\cdot 5;\ -16\cdot 4(\cdot 1)+24$	No Δ	Gram 1925

Reproduced in Dwight 1941 (222).

Tables in which either ζ or $\zeta - 1$ has about 4 significant figures are given for $s = -23\cdot 9(\cdot 1)+23\cdot 9$ in Jahnke & Emde 1933 (323), 1938 (273).

Gram 1895 gives $\zeta(\frac{1}{2}) = -1\cdot 46035\ 45088\ 09586\ 81288\ 94991\ 52512\ 5$. This was verified to 26 decimals by a previous calculation due to Stieltjes. Glaisher 1872*a* (55) gives $\zeta(1/m)$ and $\zeta(1+1/m)$ to 6 decimals for $m = 1(1)20$; cf. Art. 4·6112.

22·12. $(s-1)\zeta(s)$

This equals 1 at $s = 1$, the only pole of $\zeta(s)$.

10; 11 dec.	$s = -2(\cdot 1)0;\ 0(\cdot 1)+4$	Δ^4	Gram 1925

Reproduced (omitting the odd differences) in Dwight 1941 (226).

22·121. Coefficients in the Expansion of $(s-1)\zeta(s)$

If $(s-1)\zeta(s) = 1 + \sum_{n=1}^{\infty} C_n(s-1)^n$, the values of C_n are given as follows. Each author gives all coefficients not zero to the number of decimals tabulated.

16 dec.	$n = 1(1)16$	Gram 1895
9 dec.	$n = 1(1)9$	Jensen 1887

22·13. $\zeta'(s)/\zeta(s)$

7 dec.	$s = -6(\cdot 1)+7$	No Δ	Walther 1926

Reproduced in Dwight 1941 (227).

22·14. Coefficients in the Expansion of $\log_e\{\frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)\}$

Gram 1895 gives to 16 decimals the coefficients of $s^{0(1)13}$, i.e. all the coefficients which do not vanish to 16 decimals.

22·15. Coefficients in Expansions relating to $\xi(t)$

The function $\xi(t) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$, where $s = \frac{1}{2}+it$, is real when t is real, and is an even function of t.

The coefficients of t^{2n} in the expansion of $\log_e \xi(t)$ are given as follows. (In the notation of Art. 22·16, the coefficient of t^{2n} is $-n^{-1}\Sigma\alpha^{-2n}$.)

22–30 dec.	$n = 0(1)9$	Gram 1902, 1903
16 dec.	$n = 0(1)6$	Gram 1895

The coefficients of t^{2n} in the expansion of $\xi(t)/\xi(0)$ are given to 16 decimals for $n = 0(1)7$ in Gram 1895; the value of $\xi(0)$ is also given to 16 decimals.

22·16. The Zeros of $\xi(t)$

These are the non-trivial zeros of $\zeta(\frac{1}{2}+it)$, and Riemann's hypothesis may be stated in the form that all the zeros of $\xi(t)$ are real. We shall denote the real zeros of $\xi(t)$ by $\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \ldots$

The 10 zeros, a_1 to a_{10}, which lie in $0 < t < 50$ are given to 6 decimals in Gram 1902, 1903, and reproduced in Jahnke & Emde 1933 (324), 1938 (274). Gram 1895 contains an early determination of a_1, a_2, a_3.

The 19 zeros, a_{11} to a_{29}, which lie in $50 < t < 100$ are given to 3 decimals in Hutchinson 1925, and reproduced in Jahnke & Emde (*loc. cit.*). Gram 1902, 1903 gives a_{11} to a_{15} to 1 decimal.

Titchmarsh 1936 reports calculations supervised by Comrie which show that there are 1041 zeros in $0 < t < 1468$, all lying on the line $\sigma = \frac{1}{2}$. If $N(t)$ denotes the number of zeros in $0 < I(s) < t$, we may note the following values:

t	$t/2\pi$	$N(t)$	
50	7·96	10	See above
100	15·92	29	
200	31·83	79	Backlund 1912
300	47·75	138	Hutchinson 1925
390	62·06	195	Titchmarsh 1935
500	79·58	269	Hutchinson 1925
1468	233·64	1041	Titchmarsh 1936

22·18. Real Zeros of Real or Imaginary Part of $\zeta(\frac{1}{2}+it)$

If $\zeta(\frac{1}{2}+it) = \rho(t)e^{i\phi(t)}$, the zeros α_n of $\rho(t)$ are those discussed above. The zeros β_n of $\cos\phi(t)$ and the zeros γ_n of $\sin\phi(t)$ may also be investigated. See Hutchinson 1925, where all the γ's from $\gamma_{82} = 201·5$ to $\gamma_{140} = 300·5$ are given, and Titchmarsh 1935. See also Gram 1902, 1903, Lindelöf 1903 and Backlund 1912.

22·2. Mathieu Functions, Spheroidal Wave Functions, Lamé Functions

The Mathieu functions are solutions of the equation $\dfrac{d^2y}{dx^2} + (a - 16q\cos 2x)y = 0$ with period 2π or π. See Whittaker & Watson, *Modern Analysis*, Chapter XIX, and most of the papers referred to below. Various notations exist. The coefficient of y may be taken as $a + 16q\cos 2x$ instead of $a - 16q\cos 2x$; in any case the solutions for values of q differing only in sign are simply connected. The form $a - 16q\cos 2x$ is mainly used below, and we need consider only positive values of q. Increasing use is made of quantities $\theta = 8q$ and $\alpha = \frac{1}{4}a$.

Solutions with period 2π or π exist only when a has certain values ("characteristic numbers"). Such numbers $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ give rise to periodic solutions

$$ce_0(x, q), \quad ce_1(x, q), \quad ce_2(x, q), \quad \ldots, \quad se_1(x, q), \quad se_2(x, q), \quad \ldots$$

or $ce_0(x, \theta), \ldots$ which tend to a constant, $\cos x$, $\cos 2x, \ldots, \sin x$, $\sin 2x, \ldots$ as q tends to zero; the constant to which ce_0 reduces is usually taken to be 1 (Goldstein) or $\dfrac{1}{\sqrt{2}}$ (Ince). Ince and Goldstein use identical normalized values of the other functions (mean of square of function equals $\frac{1}{2}$). Stratton, Morse, Chu & Hutner 1941 use the equation $y'' + (b - c^2\cos^2 x)y = 0$, so that

$$c^2 = 32q = 4\theta \quad \text{and} \quad b = a + 16q = a + 2\theta$$

using our q. Their normalization is equivalent to taking

$$ce_n(0, q) = 1 \qquad se_n'(0, q) = 1$$

where the prime denotes an x-derivative.

Solutions of half-integral order may be defined, and some numerical results for the corresponding characteristic numbers are given in Ince 1927 and Goldstein 1929.

Tables relating to the case of pure imaginary q are given in Mulholland & Goldstein 1929. See also Jahnke & Emde 1938 (294).

Arts. 22·21–22·25 refer to the ordinary elliptic-cylinder functions of integral order for real q. The most comprehensive tables are those in **Ince 1932.**

22·21. Characteristic Numbers

With the equation $y'' + (a - 16q\cos 2x)y = 0$, the characteristic numbers are such that $a_0 - b_1$, $a_1 - b_2$, $a_2 - b_3$, ... tend to zero as $q \to +\infty$. See the diagram in Jahnke & Emde 1938 (286), which may be supplemented by those in Ince 1925, 1928 and Lubkin & Stoker 1943 (221). The numbers are sometimes tabulated as $\alpha = \frac{1}{4}a$ and $\beta = \frac{1}{4}b$. It should be noted that Ince 1925, 1926, 1927 use the opposite convention for the sign of q from that now usual. A similar remark applies to Lubkin & Stoker 1943, which uses the equation $\dfrac{d^2F}{d\vartheta^2} + (\alpha + \beta\cos\vartheta)F = 0$; on putting $F = y$, $\vartheta = 2x$ and $\beta = 4q$, this becomes $y'' + (4\alpha + 16q\cos 2x)y = 0$, and the table is indexed below as though this equation had been used.

$a_{0(1)3}$, $b_{1(1)4}$	to 10 dec.	$q = 2, 4{\cdot}5, 8, 12{\cdot}5, 18$ i.e. $\sqrt{32q} = 8(4)24$	Ince 1927
$a_{0(1)7}$, $b_{1(1)7}$	to 10 dec.	$\sqrt{\theta} = \sqrt{8q} = 1(1)10$	Bickley (MS. 1944)
$a_{0(1)5}$, $b_{1(1)6}$	to 7 dec.	$\theta = 8q = 0(1)10(2)20(4)40$	**Ince 1932**
$a_{1(2)7}$, $b_{1(2)7}$	to 7 dec.	$\theta = 8q = 0(\cdot 1)2{\cdot}3$	Hidaka 1936
$a_{0(1)5}$, $b_{1(1)6}$	to 6 dec.	$q = 0(\cdot 5)5$	Ince 1926
$\alpha_0, \alpha_1, \beta_1, \beta_2$	to 5 dec.	$4q = 0(\cdot 2)4(\cdot 4)10(1)16(2)20$	Lubkin & Stoker 1943
$\alpha_{2(1)5}$, $\beta_{3(1)6}$	to 5 dec.	$4q = 0(\cdot 2)4(\cdot 4)6(2)20$	Lubkin & Stoker 1943
$\alpha_{0(1)2}$, β_1, β_2	to 5 dec.	$q = 0(\cdot 1)1(\cdot 2)4,\ 5(5)30(\text{var.})200$	Goldstein 1927
$a_{0(1)2}$, b_1, b_2	to 5 dec.	$q = 0(\cdot 1)1$	Ince 1925

Stratton, Morse, etc. 1941 tabulate the characteristic values of their b (see Art. 22·2 above) which are equal to $a_{0(1)4} + 2\theta$ and $b_{1(1)4} + 2\theta$. The results are given to 5 decimals or 5 figures for $c = 2\sqrt{\theta} = 0(\cdot 2)4{\cdot}4$, also $\cdot 1$ and $\cdot 5(1)4{\cdot}5$.

Numerical results concerning the asymptotic behaviour of the characteristic numbers are given in Goldstein 1929.

22·22. Coefficients in Fourier Expansions of Mathieu Functions

Both Ince and Goldstein continue the sequences of coefficients until they vanish to the number of decimals considered.

7 dec. coefs. in $ce_{0(1)5}$, $se_{1(1)6}$	$\theta = 8q = 0(1)10(2)20(4)40$	**Ince 1932**
5 dec. coefs. in $ce_{0(1)2}$, se_1, se_2	$q = 0(\cdot 1)1(\cdot 2)4,\ 5(5)30(\text{var.})200$	Goldstein 1927

Stratton, Morse, etc. 1941 tabulate Fourier coefficients in the expansions of $ce_{0(1)4}$ and $se_{1(1)4}$ for $c = 2\sqrt{\theta} = 0(\cdot 2)4{\cdot}4$, also $\cdot 1$ and $\cdot 5(1)4{\cdot}5$. The values are to 5 decimals or 5 figures, all coefficients which are not zero to 5 decimals being given. If $ce_n(x) = \sum_k A_{k,n}\cos kx$ and $se_n(x) = \sum_k B_{k,n}\sin kx$, where both $A_{k,n}$ and $B_{k,n}$ vanish when k and n are of different parity, the normalization mentioned in Art. 22·2 above gives that $\sum A_{k,n} = 1$ and $\sum kB_{k,n} = 1$.

Hidaka 1936 gives 7-decimal Fourier coefficients $A_{k,n}$ and $B_{k,n}$ for $n = 1(2)7$, $k = 1(2)17$, $\theta = 0(\cdot 1)2\cdot 3$. Values of $kA_{k,n}$ and $kB_{k,n}$ are also given to 7 decimals for use in calculating the x-derivatives of $ce_{1(2)7}$ and $se_{1(2)7}$.

22·23. $ce_n(x)$ and $se_n(x)$

The Mathieu functions $ce_{0(1)5}$, $se_{1(1)6}$ themselves are tabulated to 5 decimals with second differences in x for $\theta = 8q = 0(1)10$, $x = 0(1^\circ)90^\circ$ in **Ince 1932.**

Graphs of $ce_{0(1)2}$, se_1, se_2, and diagrams (not to scale) of ce_4, ce_5, se_5, se_6, are given in Jahnke & Emde 1938 (284–293).

22·24. Zeros of $ce_n(x)$ and $se_n(x)$

The zeros of $ce_{2(1)5}$ and $se_{3(1)6}$ are given to $0^\circ\cdot 0001$ for $\theta = 8q = 0(1)10(2)20(4)40$ in Ince 1932.

22·25. Turning-Points of $ce_n(x)$ and $se_n(x)$

Ince 1932 gives the turning-points of $ce_{1(1)5}$ and $se_{2(1)6}$ to $0^\circ\cdot 0001$ for

$$\theta = 8q = 0(1)10(2)20(4)40$$

22·27. Non-Periodic Solutions of Mathieu's Equation

Goldstein 1928 gives tables relating to the second solution in some of the cases in which the first is periodic.

Ince 1928 solves $y'' + (3 - 4\cos 2x)y = 0$ in the form

$$y = A\sum_{-\infty}^{\infty} e_r \cos(2r+\rho)x + B\sum_{-\infty}^{\infty} e_r \sin(2r+\rho)x$$

ρ being found to 8 decimals and the coefficients e_r to 6 decimals. An exploratory table related to the characteristic exponents of $y'' + \epsilon(1 + k\cos x)y = 0$ is given in Brainerd & Weygandt 1940 (469).

The New York W.P.A. workers have had under consideration the computation of a "table of the non-periodic solutions of the Mathieu differential equation".

22·28. Spheroidal Wave Functions

Quantities relating to both prolate and oblate spheroidal wave functions are tabulated in Stratton, Morse, Chu and Huther 1941. A few equations occurring in the prolate case may serve to give some idea of the nature of the work. Further information may be found in a paper by Chu and Stratton, *J. of Math. and Phys.*, **20,** 259–309, 1941. This paper, which contains no tables, is reproduced as part of Stratton, Morse, etc. 1941.

The equation $\nabla^2 W + k^2 W = 0$ is to be solved in spheroidal co-ordinates (ψ, θ, ϕ) or (ξ, η, ϕ) connected with rectangular co-ordinates (x, y, z) and cylindrical co-ordinates (r, ϕ, z) by the equations

$$x = f\sinh\psi\sin\theta\cos\phi, \quad y = f\sinh\psi\sin\theta\sin\phi \quad z = f\cosh\psi\cos\theta$$

$$z + ir = f\cosh(\psi + i\theta) \quad \xi = \cosh\psi, \quad \eta = \cos\theta$$

$$r = f\sinh\psi\sin\theta = f\sqrt{(\xi^2-1)(1-\eta^2)} \quad z = f\cosh\psi\cos\theta = f\xi\eta$$

Normal solutions are found to be $W = R(\xi)S(\eta)\sin(m\phi+\epsilon)$, where m is a positive integer or zero, and R and S satisfy

$$(\xi^2-1)R'' + 2\xi R' + \left(A + c^2\xi^2 - \frac{m^2}{\xi^2-1}\right)R = 0 \qquad (1)$$

$$(1-\eta^2)S'' - 2\eta S' - \left(A + c^2\eta^2 + \frac{m^2}{1-\eta^2}\right)S = 0 \qquad (2)$$

where $c = kf$, and the characteristic values of A have to be determined from conditions such as finiteness and periodicity over the surface of a spheroid. Putting $R = (\xi^2-1)^{\frac{1}{2}m}U(\xi)$ and $S = (1-\eta^2)^{\frac{1}{2}m}V(\eta)$, the differential equations become

$$(\xi^2-1)U'' + 2(m+1)\xi U' + \{A+m(m+1)+c^2\xi^2\}U = 0 \qquad (3)$$

$$(1-\eta^2)V'' - 2(m+1)\eta V' - \{A+m(m+1)+c^2\eta^2\}V = 0 \qquad (4)$$

It will be noticed that the radial functions R or U and the angular functions S or V satisfy the same differential equations, which we have written in slightly different forms in recognition of the fact that $\xi > 1$, $|\eta| < 1$.

We shall describe the tabulations as far as they relate to the finite prolate angular solutions. When $c = 0$ the co-ordinates are spherical, and the characteristic numbers $A_{m,l}$ are $-(m+l)(m+l+1)$, l being a positive integer or zero. In this special case, finite solutions of (2) and (4) are $S = P^m_{l+m}(\eta)$ and $V = T^m_l(\eta)$, where

$$P^m_{l+m}(\eta) = (1-\eta^2)^{\frac{1}{2}m}T^m_l(\eta)$$

is an associated Legendre function and $T^m_l(\eta) = (d/d\eta)^m P_{l+m}(\eta)$ is the authors' form of Gegenbauer function. (The authors' $P^m_{l+m}(\eta)$ would be called $P_{l+m,\,m}(\eta)$ in the notation of our Section 16.) When c is not zero, S and V are expanded in series of associated Legendre functions and Gegenbauer functions respectively.

The tabulations are for $c = 0(\cdot 2)5$, also $\cdot 1$ and $\cdot 5(1)4{\cdot}5$, and for $m = 0(1)3$, $l = 0(1)3-m$. In each case the characteristic value $A_{m,l}$ is given to 5 decimals, and coefficients d_n are given to 5 figures for the corresponding solutions of the form $S = \sum_{n=0}^{\infty} d_n P^m_{n+m}(\eta)$ and $V = \sum_{n=0}^{\infty} d_n T^m_n(\eta)$. (For tables of associated Legendre functions, see Arts. 16·4 and 16·5; for tables of Gegenbauer functions or derivatives of Legendre functions, see Art. 16·3.)

Further quantities are tabulated, and the various values may be used for radial functions also. The tables for oblate functions have the same arguments and are parallel to those for prolate functions. The work itself should be consulted for the complicated formulæ involved. It has been convenient to consider it here, since in the case $m = -\frac{1}{2}$ the solutions of the differential equations involve Mathieu functions. The tables relating to these functions contained in the same work have been fully described in Arts. 22·2–22·22 above.

22·29. Lamé Functions

Four-decimal tables of characteristic numbers h for periodic solutions of Lamé's equation

$$\frac{d^2y}{du^2} = \{n(n+1)k^2 \operatorname{sn}^2 u - h\}y$$

where sn u has modulus k, are given for $n = -\frac{1}{2}$, $0(1)25$, $k^2 = \cdot 1, \cdot 5, \cdot 9$ in Ince 1940*a* and for $n = \frac{1}{2}(1)\frac{9}{2}$, $k^2 = \cdot 1, \cdot 5, \cdot 9$ in Ince 1940*b*.

The New York W.P.A. workers have had under consideration the computation of tables of Lamé polynomials, which occur for integral n.

22·3. Solutions of Emden's Equation, etc.

Emden's differential equation, which occurs in connection with stellar constitution, is $y'' + \frac{2}{x}y' + y^n = 0$, where dashes denote derivatives with respect to x. Except when $n = 0$ or 1, the equation is non-linear. We discriminate between Emden functions, which are solutions such that $y' = 0$ when $x = 0$, and other solutions of Emden's equation.

22·31. Emden Functions

The solution usually tabulated is that for which $y = 1, y' = 0$ when $x = 0$. When the index n has the values 0, 1, 5, the solution may be expressed in finite terms, as follows:

$$\begin{array}{ll} n = 0 & y = 1 - \frac{1}{6}x^2 \\ n = 1 & y = (\sin x)/x \\ n = 5 & y = (1 + \frac{1}{3}x^2)^{-1/2} \end{array}$$

When $n < 5$, y becomes zero for some value x_0 of x (e.g. $x_0 = \sqrt{6}$ when $n = 0$, $x_0 = \pi$ when $n = 1$), and the solution is usually continued to the first such zero.

All solutions for which $y' = 0$ when $x = 0$ are homologous. If $y = f(x)$ is a solution of the above kind, $y = A^{\frac{2}{n-1}}f(Ax)$ is another, vanishing at $x = x_0/A$. A here denotes any positive constant. Thus the solution for which $y = 0$ at $x = 1$ and $y' = 0$ at $x = 0$ is $y = x_0^{\frac{2}{n-1}}f(x_0 x)$, and a table of this solution is also useful.

The most important and accurate tables are those in **B.A. 2, 1932** (Miller & Sadler). These tabulate the solutions for which $y = 1, y' = 0$ at $x = 0$. They give x/x_0, y, y' to 7 decimals when $n = 1(\cdot5)4\cdot5$, and y, y' to 5 decimals when $n = 5$. The intervals in x are various (most often ·1). Except when $n = 5$, higher derivatives of y are given to facilitate interpolation. A number of auxiliary functions are also tabulated. The solution for $n = 3$ continues to the third zero of y, the others to the first zero. Chandrasekhar 1939*a* gives a table for $n = 3\cdot25$.

Solutions for $n = 0(\cdot5)3, 4, 4\cdot5, 4\cdot9, 5, 6$ are given in Emden 1907 (77–85). Those for $n = 0, 1, 2, 2\cdot5, 3, 4, 5$ are reproduced in Milne 1930 (187), and those for $n = 2, 2\cdot5, 3$ in Eddington 1926 (82), but the B.A. tables show that the values of y and y' are not accurate to the number of decimals given, even in Milne and Eddington, who curtail some of Emden's values. Solutions are also given for $n = 2$ in Miller 1929 and for $n = 1\cdot5$ in Fairclough 1932, but the B.A. tables must be regarded as superseding these.

Solutions for $n = 5(\cdot5)8$, as well as others taken from Emden 1907, are given in von Zeipel 1913. The quantity tabulated is y^n.

Solutions for which $y = 0$ when $x = 1$ and $y' = 0$ when $x = 0$ are given for $n = 3$ in Fairclough 1931, 1933.

On putting $x = 1/z$, Emden's equation becomes $d^2y/dz^2 = -y^n/z^4$. Solutions with $y = 1$, $dy/dz = 0$ at $z = \infty$ are given for $n = 1\cdot5, 2\cdot5, 3, 4$ in Kelvin & Green 1908. References to earlier work by Lane, Ritter, etc. may be found here and elsewhere. None of this early work compares in accuracy with the B.A. tables.

22·32. Other Solutions of Emden's Equation

A solution of Emden's equation for which $y = 0$ when $x = 1$ and $y' = 0$ when $x = 0$ will have a certain gradient, say $y' = y'_c$, when $x = 1$. Other solutions may be obtained in which, when $x = 1$, y is zero and y' has any value which is not a critical value y'_c. Various such solutions are given for $n = 3$ in Fairclough 1931, 1933.

Alternatively the solution of Emden's equation for which $y = 1$, $y' = 0$ when $x = 0$ determines a critical gradient when $x = x_0$, and other solutions may be started with zero y and any other gradient at $x = x_0$. Several such solutions are given for $n = 1\cdot5$ in Fairclough 1932.

Two solutions of a transformed form of the equation $y'' + 2y'/x - y^2 = 0$ are tabulated in Hartree 1937*a*. Putting $y = -Y$, we see that the equation becomes Emden's equation of order 2. Compare Art. 22·86.

22·35. Isothermal Equation

The thermodynamical analysis from which Emden's equation arises requires modification when $n = \infty$, and the resulting differential equation is

$$y'' + \frac{2}{x} y' + e^y = 0$$

The solution for which $y = 0, y' = 0$ when $x = 0$ is reproduced from Emden 1907 (135) in Eddington 1926 (90) and in Milne 1930 (196).

On putting $x = 1/z$, the equation becomes $d^2y/dz^2 = -e^y/z^4$. Values of e^y and $\dfrac{dy}{dz}$ in the solution for which $y = 0$, $dy/dz = 0$ at $z = \infty$ are given (with further references) in Kelvin & Green 1908 (23).

22·36. White-Dwarf Functions

Chandrasekhar 1939 tabulates solutions of

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2 \frac{dy}{dx}\right) = -(y^2 - C)^{\frac{3}{2}}$$

with initial conditions $y = 1$ and $dy/dx = 0$ at $x = 0$, for $C = \cdot01, \cdot02, \cdot05, \cdot1(\cdot1)\cdot6, \cdot8$. See also Chandrasekhar 1935.

22·4. Laplace Coefficients and Related Functions

These are the coefficients $b_s^{(i)}$ which occur in the expansion

$$(1 + \alpha^2 - 2\alpha \cos x)^{-s} = \tfrac{1}{2} b_s^{(0)} + \sum_{i=1}^{\infty} b_s^{(i)} \cos ix$$

where $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$. It may be shown that, in terms of the hypergeometric function,

$$b_s^{(i)} = 2\,\frac{s(s+1)\ldots(s+i-1)}{i!}\,\alpha^i F(s, s+i, i+1, \alpha^2)$$

These coefficients, their derivatives with respect to α, and other related functions satisfy innumerable recurrence relations, which may be found in the literature of dynamical astronomy (see H. von Zeipel, *Enc. d. Math. Wiss.*, Band VI 2, 1. Hälfte, 557–665, 1912). The coefficients are connected with the complete elliptic integrals with modulus α; in particular,

$$b_{\frac{1}{2}}^{(0)} = 4K/\pi \qquad b_{\frac{1}{2}}^{(1)} = 4(K - E)/\pi\alpha$$

See Arts. 21·25 ff.

The principal tables do not tabulate the Laplace coefficients and their derivatives directly; various integral powers of α and β, where $\beta^2 = \alpha^2/(1-\alpha^2)$, may be taken out. The argument is usually α, $\log \alpha$ or β^2, and to facilitate comparison we consider together tables with the same argument.

In practice values of α greater than ·75 occur rarely.

22·41. Tables with Argument α

Extensive tables are given in Runkle 1856; we consider also small tables given in Brown & Brouwer 1932 (see Art. 22·43 for their main tables) and elsewhere.

Runkle 1856 tabulates to 5–7 decimals the logarithms of $\alpha^{\cdots} D^n b_s^{(i)}/\beta^{2n+4s-2}$, where $D = d/d\alpha$ and $\alpha^{\cdots}$ is some convenient integral power of α. The parameters take the values

$$\begin{cases} s = \tfrac{1}{2}, & i = 0(1)9, \quad n = 0(1)5 \\ s = \tfrac{3}{2}, & i = 0(1)9, \quad n = 0(1)2 \\ s = \tfrac{5}{2}, & i = 0, 2, \quad n = 0 \end{cases}$$

The argument α goes up to ·75 at interval ·005 or ·01. The main functions are tabulated to 7 decimals for $\alpha = 0(·005)·75$; but functions for finding coefficients which depend on high powers of α, and are thus very small for small α, are not necessarily tabulated to 7 decimals, nor for small α, nor at the closer interval. Logarithms of first variations for a change of ·001 in α are given for the more important functions to facilitate interpolation. In spite of various drawbacks, including inaccuracy, Runkle's tables are of considerable use. On errors affecting the important coefficient $b_{\frac{1}{2}}^{(1)}$, see Witt 1932.

Brown & Brouwer 1932 (134) tabulates to 6 figures with first differences $K_{\frac{1}{2}}^{(i)} = (1-\alpha^2)b_{\frac{1}{2}}^{(i)}$ and $K_{\frac{3}{2}}^{(i)} = (1-\alpha^2)^2 b_{\frac{3}{2}}^{(i)}$ for $i = 0(1)4$, $\alpha = ·900(·001)·950$.

Robbins 1909 tabulates the logarithms of $p_i = b_{\frac{1}{2}}^{(i)}/\alpha b_{\frac{1}{2}}^{(i-1)}$ to 7 decimals without differences for $i = 1(1)8$, $\alpha = 0(·02)·86$.

Fletcher 1939 tabulates $b_{\frac{1}{2}}^{(0)}$ and $b_{\frac{1}{2}}^{(1)}$. See Arts. 21·25 ff.

Norén & Raab 1900 tabulates mainly $\frac{1}{4}\alpha^{\frac{1}{2}} b_{\frac{3}{2}}^{(1)}$ and $\frac{1}{4}\alpha^{\frac{1}{2}} b_{\frac{3}{2}}^{(2)}$. Natural values to 5 decimals are given for $\alpha = 0(·001)·200$, and logarithmic values to 6 decimals for $\alpha = ·200(·001)·830$, in both cases with first differences.

An extensive set of coefficients for $\alpha = ·75$ is given as an example of their calculation in Andoyer 1923 (419). See also Legendre 1815 (289), 1826 (551).

22·42. Tables with Argument $\log \alpha$

Masal 1891 tabulates to 4–7 decimals, with first differences, the logarithms of

$$\beta_n^{(s)} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin^{2n}\phi\, d\phi}{(1-\alpha^2\sin^2\phi)^{\frac{1}{2}s}}$$

for

$$\begin{cases} s = 1, 3, 5, & n = \dfrac{s-1}{2}(1)12, & \log \alpha = \bar{1}·30(·01)\bar{1}·60(·001)\bar{1}·85 \\ s = 7, & n = 3(1)12, & \log \alpha = \bar{1}·60(·001)\bar{1}·85 \\ s = 9, & n = 4(1)13, & \log \alpha = \bar{1}·60(·001)\bar{1}·85 \end{cases}$$

We have $\beta_n^{(s)} = \dfrac{(2n-1)!!}{(2n)!!} F(\tfrac{1}{2}s, n+\tfrac{1}{2}, n+1, \alpha^2)$ and $b_{\frac{1}{2}}^{(i)} = 2\alpha^i \beta_i^{(1)}$. The upper limit $\log \alpha = \bar{1}·850$ corresponds to $\alpha = ·7079$ approximately.

Gyldén 1896 tabulates to 4–7 decimals, with first differences, the logarithms of

$$\gamma_i^{m,n} = \frac{a^{m+n+2i}}{i!}\left(\frac{d}{da^2}\right)^i \left(\tfrac{1}{2}a^{-n}b_{m/2}^{(n)}\right)$$

for $\log a = \bar{1}\cdot620(\cdot001)\bar{1}\cdot800$ and

$$\begin{cases} m=1, & i=0(1)4, & n=0(1)9,\ 10,\ 11 \text{ or } 12 \\ m=3, & i=0,\ 1,\ 2, & n=0(1)6 \\ m=5, & i=0,\ 1, & n=0(1)3 \end{cases}$$

We have

$$\gamma_i^{m,n} = \frac{\Gamma(\frac{1}{2}m+i)\Gamma(\frac{1}{2}m+n+i)}{i!(n+i)!\{\Gamma(\frac{1}{2}m)\}^2}\, a^{m+n+2i}F(\tfrac{1}{2}m+i,\ \tfrac{1}{2}m+n+i,\ n+i+1,\ a^2)$$

$$\gamma_i^{1,n} = \frac{(2i-1)!!}{(2i)!!}\, a^{n+2i+1}\beta_{n+i}^{(2i+1)} \qquad \gamma_0^{1,n} = a^{n+1}\beta_n^{(1)} = \tfrac{1}{2}ab_{\frac{1}{2}}^{(n)}$$

22·43. Tables with Argument $p = \beta^2 = \alpha^2/(1-\alpha^2)$

p is the argument used in the important tables of **Brown & Brouwer 1932,** which give logarithms of $G_s^{(i)} = (1-a^2)^s b_s^{(i)}/a^i$ for $s = \frac{1}{2}(1)\frac{7}{2}$, $i = 0(1)11$, $p = 0(\cdot01)2\cdot50$. The logarithms are to 8 decimals for $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ and to 7 decimals for $s = \frac{7}{2}$. First differences are provided. The upper limit $p = 2\cdot50$ corresponds to $a = \cdot8452$ approximately. See also Art. 22·41.

22·46. Coefficients in the Expansions of Laplace Coefficients, etc.

Coefficients in the expansions of various Laplace coefficients and related functions in powers of a are given in Runkle 1856 (vi, 42), Leverrier 1856 (see also Arts. 3·56 and 3·562 of this *Index*) and Leverrier, *Annales Obs. Paris* (*Mém.*), **10,** 18, 1874 ($b_{\frac{1}{2}}^{(0)}$, $b_{\frac{1}{2}}^{(1)}$ only). Coefficients in similar expansions in powers of β or of (β^2 – constant) are given in Wellmann 1891, E. W. Brown 1928, 1932, K. P. Williams 1930 and Brown & Brouwer 1932 (133).

22·5. Hypergeometric Functions

The tabulation of the general hypergeometric function $F(\alpha, \beta, \gamma, x)$ would call for tables with 4 arguments, and we have not encountered any direct tabulation. Many of the functions dealt with elsewhere in this *Index* may be expressed in terms of hypergeometric functions with particular values of α, β, γ. The extensive tables of Laplace coefficients, etc. which have been made (see Arts. 22·4 ff. above) probably come nearest to a general tabulation of hypergeometric functions.

Wang 1938 tabulates $\frac{2\pi}{u}\{(1+\sigma)^3F(\frac{3}{2}, \frac{3}{2}, 1, \sigma^2)-1\}$, where $\sigma = (u-\sqrt{u^2-1})^2$, to 4 decimals without differences for $u = 1\cdot4(\cdot01)3\cdot1(\cdot1)5$. The function $F(\frac{3}{2}, \frac{3}{2}, 1, \sigma^2)$ which occurs here is $\frac{1}{2}b_{\frac{3}{2}}^{(0)}(\sigma)$, see Art. 22·4.

Pearson & Elderton 1923 (Henderson) gives $2F(-n, 1, n+1, x)-1$ to 6 decimals without differences for $n = 0(1)10$, $x = -1(\cdot1)+1$.

Pearson 1931*a* tabulates quantities involving

$$F(1, 1, \tfrac{1}{2}N+\tfrac{1}{2}, x^2) \quad\text{and}\quad F(2, 2, \tfrac{1}{2}N+\tfrac{3}{2}, x^2)$$

for $N = 3(1)25$, 50, 100, 200, 400. The tables are reproduced in Pearson 1931 (254).

Logarithms of $F(\frac{1}{8}, \frac{5}{8}, 2, \sin^2 \frac{1}{2}x)$ and of $F(-\frac{1}{8}, \frac{7}{8}, 2, \sin^2 \frac{1}{2}x)$ are given to 7 decimals with Δ^2 for $x = 0(1°)90°$ in Robbins 1907 and for $x = 90°(1°)180°$ in Merfield 1908.

For roots of hypergeometric functions with a numerator and four denominators, see Wrinch & Wrinch 1926.

We now proceed to consider confluent hypergeometric functions, which have been fairly extensively tabulated.

22·55. Confluent Hypergeometric Functions

Various forms are in use. We consider first $M(\alpha, \gamma, x)$, then the second solution of the differential equation satisfied by $M(\alpha, \gamma, x)$, and finally solutions of other forms of equation.

$$22·56. \quad M(\alpha, \gamma, x) = 1 + \frac{\alpha}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \dots$$

In general, $M(\alpha, \gamma, x)$ satisfies the differential equation $xy'' + (\gamma - x)y' - \alpha y = 0$. The substitution $y = x^{1-\gamma}z$ transforms this into $xz'' + (2-\gamma-x)z' - (\alpha-\gamma+1)z = 0$, so that $y = x^{1-\gamma}M(\alpha-\gamma+1, 2-\gamma, x)$ also satisfies the original equation; it is a second independent solution if γ is not integral. Collections of formulæ involving $M(\alpha, \gamma, x)$ are given in Webb & Airey 1918 (but see Art. 22·57 below), B.A. 1926 (276) and Airey 1937, as well as in a manuscript by A. J. Thompson. Other notations, such as $F(\alpha, \gamma, x)$, are also in use for $M(\alpha, \gamma, x)$.

Difficulties arise when γ is integral. $M(\alpha, \gamma, x)$ is infinite if γ is zero or a negative integer, unless α is also zero or a negative integer and $|\alpha| < |\gamma|$. (If $\alpha = \gamma = -p$, p being zero or a positive integer, $M(\alpha, \gamma, x)$ may conveniently be taken to be either $\sum_0^p x^r/r!$ or e^x, while $x^{1-\gamma}M(\alpha-\gamma+1, 2-\gamma, x)$ is $(p+1)!\sum_{p+1}^{\infty} x^r/r!$.) $M(\alpha, \gamma, x)/\Gamma(\gamma)$ is finite, but merely a multiple of $x^{1-\gamma}M(\alpha-\gamma+1, 2-\gamma, x)$. It is perhaps simplest, however, to consider only the case $\gamma > 1$. In view of the transformed equation given above, this involves no loss of generality; it merely ensures that the solution $M(\alpha, \gamma, x)$ never fails, and that it is the "second" solution which (in general) becomes logarithmic for integral γ. The second solution is considered in Art. 22·57 below.

Various functions may be expressed in terms of confluent hypergeometric functions. Some of these, such as the exponential integral, incomplete gamma functions, the error integral, Hermite polynomials, parabolic cylinder functions and Bessel functions, have been dealt with in other sections of this *Index*. For analytical details, see Webb & Airey 1918, B.A. 1926 and Airey 1937, or (in terms of Whittaker's functions, see Art. 22·58 below) Whittaker & Watson, *Modern Analysis*, Chapter XVI. We may mention a few other functions which have not been indexed independently.

An important special class of confluent hypergeometric functions is that of the Laguerre polynomials, which occur in the theory of hydrogen-like atoms and in other physical problems. The ordinary Laguerre polynomial $L_n(x)$, defined as $e^x(d/dx)^n(x^n e^{-x})$, is equal to $n!M(-n, 1, x)$, and is identical with a function used by Lagrange. The mth derivative of this, $L_n^m(x)$, where $m < n$, is a multiple of $M(m-n, m+1, x)$. Various other notations for Laguerre polynomials exist. It may be noticed particularly that $x^{-m}e^x\left(\frac{d}{dx}\right)^n(x^{n+m}e^{-x})$ is a multiple of $M(-n, m+1, x)$, and hence of our $L_{n+m}^m(x)$. The Sonin polynomial $T_m^n(x)$ is also a multiple of $M(-n, m+1, x)$ or of $L_{n+m}^m(x)$. In general, we shall refer to cases in which γ is a positive integer and α zero or a negative integer as the Laguerre cases.

A programme involving the tabulation of (among other things) Laguerre polynomials was announced in Klose 1934. A manuscript table by J. C. P. Miller gives exact values of $L_n(x)$ for $n = 0(1)8$, $x = 0(1)20$. The tables of $M(\alpha, \gamma, x)$ mentioned below, in Webb & Airey 1918, B.A. 1927 and the Thompson manuscript, include a number of the Laguerre cases. H. Bethe, *Handbuch der Physik* (Zweite Auflage), 24/1, 284, 1933, gives explicit expressions for radial functions R_{nl}, proportional to $r^l e^{-r/n} L_{n+l}^{2l+1}(2r/n)$, for $n = 1(1)4$, $l = 0(1)n-1$, as well as some graphs, for which see also Condon & Morse, *Quantum Mechanics*, New York, 1929, page 65. Some zeros of Sonin polynomials are given in Burnett 1937. See also Bates 1938.

Another function expressible in terms of confluent hypergeometric functions is the Pearson–Cunningham function $\omega_{n,m}(x)$. See Whittaker & Watson, also Airey 1937. Another is the Toronto function $T(m, n, r)$, see Art. 22·91.

Webb & Airey 1918 tabulates $M(\alpha, \gamma, x)$ to 4 figures for $\gamma = 1(1)7$, $\alpha = -3(\frac{1}{2})+4$, $x = 1(1)6(2)10$.

B.A. 1926 (279) tabulates the function to 5 decimals or 6–7 figures for $\gamma = \pm\frac{1}{2}, \pm\frac{3}{2}$, $\alpha = -4(\frac{1}{2})+4$, $x = 0(\cdot 1)1(\cdot 2)3(\cdot 5)8$.

B.A. 1927 (221) gives $M(\alpha, \gamma, x)$ to 5 decimals or 6 figures for $\gamma = \pm\frac{1}{2}, \pm\frac{3}{2}$, $\alpha = -4(\frac{1}{2})+4$, $x = 0(\cdot 02)\cdot 08, \cdot 15(\cdot 1)\cdot 95, 1\cdot 1(\cdot 2)1\cdot 9$. B.A. 1927 (229) gives it to 5 decimals or 6–7 figures for

$$\gamma = 1(1)4, \quad \alpha = -4(\tfrac{1}{2})+4, \quad x = 0(\cdot 02)\cdot 1(\cdot 05)1(\cdot 1)2(\cdot 2)3(\cdot 5)8$$

Jahnke & Emde 1938 (276) gives diagrams of $M(\alpha, \gamma, x)$ based upon the tables in B.A. 1926, 1927.

A. J. Thompson has computed a large number of basic values of $M(\alpha, \gamma, x)$ and $N(\alpha, \gamma, x)$ (see Art. 22·57 below). Among these is a table of both functions for $\alpha = 0, 1$, $\gamma = 1(1)100$, and $\pm x = 1(1)10(10)50$. From these any detailed tabulation that may be required in the quadrant for which α is negative and γ positive could be derived. The same series of values would serve as a basis for detailed tabulations in the neighbourhood of the axis of γ in the quadrant for which α and γ are both positive; but, in this quadrant, it is generally better to proceed from basic values for which $\gamma = 2\alpha$. A series of these values, on a scale similar to that of the table just described, is now being computed (see Bessel Functions, Art. 18·43). Among the experimental working tables which have been constructed is one for $x = 10$: (i) $\pm\alpha = 0(\cdot 1)1\cdot 5$, $\gamma = 1(1)50$; (ii) $-\alpha = 0(\cdot 01)20(\cdot 1)100$, $\gamma = 1$; (iii) $\alpha = 1$, $\gamma = 0(\cdot 1)50$. (We are indebted to Dr. Thompson for this description of his tables.)

J. C. P. Miller has under construction tables of $M(\alpha, \gamma, 1)$ for $\alpha = -1(\cdot 1)+1$, $\gamma = -1(\cdot 1)+1$ to 11 working decimals.

Olsson 1937*a* tabulates to 4 decimals for $k = -2(\frac{1}{2})+2$, $\rho = 0(\cdot 1)1$ the functions

$$\begin{array}{llll} M(\cdot 65, 2, k\rho^2), & M(1\cdot 65, 3, k\rho^2), & M(-\cdot 35, 1, k\rho^2) & \\ M(\cdot 5, 2, k\rho^2), & M(1\cdot 5, 3, k\rho^2), & M(-\cdot 5, 1, k\rho^2) & \\ M(\cdot 325, 1\cdot 5, k\rho^4), & M(1\cdot 325, 2\cdot 5, k\rho^4), & M(-\cdot 675, \cdot 5, k\rho^4), & M(\cdot 825, 1\cdot 5, k\rho^4) \end{array}$$

The double-entry form of tabulation for given α and γ is used with a view to applications (see other papers by Olsson in the same volume, particularly the paper immediately preceding Olsson 1937*a*), but militates against general use of the tables.

Olsson 1937*b* tabulates to 4 decimals for $x = \cdot 02(\cdot 02)\cdot 1(\cdot 05)1(\cdot 1)2\cdot 5$ the functions $M(1\cdot 3, 3, x)$, $M(\cdot 65, 2, x)$, $M(\cdot 325, 1\cdot 5, x)$, $x^{-\frac{1}{2}}M(-\cdot 175, \cdot 5, x)$, and other expressions involving confluent hypergeometric functions.

22·57. Second Solutions of $xy'' + (\gamma - x)y' - \alpha y = 0$

We have seen that if γ is not integral, two independent solutions of the above equation are $M(\alpha, \gamma, x)$ and $x^{1-\gamma}M(\alpha - \gamma + 1, 2 - \gamma, x)$. We shall confine ourselves to the case $\gamma \geqslant 1$. If $\gamma = 1$, the two solutions are identical. If γ is a positive integer greater than 1, the second solution breaks down, unless α is also a positive integer less than γ.

The nature of the second solution when γ is a positive integer has been the subject of misconception. If we write

$$N(\alpha, \gamma, x) = \frac{\alpha}{\gamma}\left(\frac{1}{\alpha} - \frac{1}{\gamma} - 1\right)x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - 1 - \tfrac{1}{2}\right)\frac{x^2}{2!} + \dots$$

Webb & Airey 1918 gave the second solution as $M(\alpha, \gamma, x)\log_e x + N(\alpha, \gamma, x)$. This is true only when $\gamma = 1$. When γ is a positive integer greater than 1, the solution contains also terms in $x^{-1}, x^{-2}, \dots, x^{1-\gamma}$.

The error has been pointed out, and the correct solution given, by R. Stoneley, *Mon. Not. R.A.S., Geoph. Suppl.*, **3**, 222–232, 1934.

Olsson 1937*a*, *b* has tabulated $M \log x + N$ in several cases, apparently under the impression that it is a solution.

The extensive tables in preparation by A. J. Thompson include many values of $N(\alpha, \gamma, x)$, which remains a useful auxiliary function.

22·58. Other Forms of Confluent Hypergeometric Function

Other confluent hypergeometric functions are dealt with in Whittaker & Watson, *Modern Analysis*, Chapter XVI. If $2m$ is not integral, the differential equation

$$y'' + \left\{-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - m^2}{x^2}\right\}y = 0$$

has independent solutions

$$M_{k,m}(x) = x^{\frac{1}{2}+m}e^{-\frac{1}{2}x}M(\tfrac{1}{2} - k + m, 1 + 2m, x)$$

and

$$M_{k,-m}(x) = x^{\frac{1}{2}-m}e^{-\frac{1}{2}x}M(\tfrac{1}{2} - k - m, 1 - 2m, x)$$

Whittaker's solutions $W_{k,m}(x)$ and $W_{-k,m}(-x)$, which are available for all values of k and m, are also defined.

In the case of hydrogen-like atoms the equation of the above kind which occurs is

$$y'' + \left\{-\frac{1}{4} + \frac{n}{x} - \frac{l(l+1)}{x^2}\right\}y = 0$$

where $n = 1, 2, 3, \dots$ and $l = 0, 1, 2, \dots$ are two quantum numbers ($l < n$). Thus $k = n$, $m = l + \frac{1}{2}$, corresponding to $\alpha = l + 1 - n$, $\gamma = 2l + 2$, a Laguerre case. On the wave functions, finite at the origin and expressed in terms of $r = \frac{1}{2}nx$, see Art. 22·56 above. We have not encountered any general tabulation of $M_{k,m}(x)$ or $W_{k,m}(x)$ for real x; it is evident that the tables described in Art. 22·56 are a partial substitute. Mention may also be made of the functions G and H used by D. R. Hartree in atomic calculations (*Proc. Camb. Phil. Soc.*, **24**, 426–437, 1928 and **25**, 310–314, 1929); see also Bates & Massey 1941.

Yost, Wheeler & Breit 1935 tabulates as Coulomb wave functions for repulsive fields solutions of

$$y'' + \left\{1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right\} y = 0$$

for $L = 0$ and 1. Putting $2i\rho = x$, this becomes Whittaker's equation with $k = i\eta$, $m = L + \frac{1}{2}$, corresponding to $a = L + 1 - i\eta$, $\gamma = 2L + 2$. See also the paper of 1936 by the same authors. Wicher 1936 extends the calculations to the case $L = 2$.

Lowan & Horenstein 1942 gives 7-figure tables of the real functions

$$H(m, a, x) = e^{-ix} M(m + 1 - ia, 2m + 2, 2ix)$$

and $$\frac{dH(m, a, x)}{dx} \quad \text{for} \quad x = 0(1)10, \; a = 0(1)10, \; m = 0(1)3$$

See also Airey 1937.

22·6. Debye Functions, Radiation Integrals, Fermi-Dirac Functions

We consider first $\int_0^x \frac{x^n dx}{e^x - 1}$ and $\int_x^\infty \frac{x^n dx}{e^x - 1}$, where n is positive. Arts. 22·61–22·63 relate to the cases $n = 1, 2, 3$; for the case $n = 4$, see Grüneisen 1933. It may be noticed that when $n > 0$, $\int_0^\infty \frac{x^n dx}{e^x - 1} = n!\zeta(n+1)$, where ζ is the Riemann zeta function, see Art. 22·1; for integral n, $\zeta(n+1)$ is often written as S_{n+1}, see Art. 4·6111. For Einstein functions, $x/(e^x - 1)$, etc., see Art. 10·38.

22·61. $\int_0^x \frac{x\,dx}{e^x - 1}$

Debye 1913 (86) tabulates $\frac{1}{x}\int_0^x \frac{x\,dx}{e^x - 1}$ to 3 figures without differences for $x = 0(\cdot 2)2(\cdot 5)3(1)10(2)16, 20$. The table is reproduced in Blake 1933 (176). See also Art. 22·84.

22·62. $\int_0^x \frac{x^2 dx}{e^x - 1}$, etc.

Zanstra 1931*a* gives a small table containing 19 values of $\int_x^\infty \frac{x^2 dx}{e^x - 1}$ to 4–5 decimals. The table is reproduced in Zanstra 1931*b*. These supersede a short preliminary table in Zanstra 1927 which contains errors in the last two figures.

22·63. $\int_0^x \frac{x^3 dx}{e^x - 1}$, etc.

Expressions involving this integral occur in connection with Debye's theory of specific heats. Using a common notation, the energy and specific-heat functions are

$$\frac{U}{3RT} = \frac{3}{x^3}\int_0^x \frac{x^3 dx}{e^x - 1}$$

$$\frac{C_v}{3R} = -x^2 \frac{d}{dx}\left(\frac{3}{x^4}\int_0^x \frac{x^3 dx}{e^x - 1}\right) = \frac{12}{x^3}\int_0^x \frac{x^3 dx}{e^x - 1} - \frac{3x}{e^x - 1} = f(x)$$

where $x = \Theta/T$ = (characteristic temperature)/(temperature).

Beattie 1926 tabulates both $U/3RT$ and $C_v/3R = f(x)$ to 6 figures without differences for $x = 0(\cdot 01)24$.

Bronson & Wilson 1936 tabulates $\log_{10} f(x)$ to 5 decimals for $x = 0(\cdot 01)2\cdot 09$, with first differences and some proportional parts.

Jeans 1925 (398) tabulates $f(x)$ to 3 figures for various values of $1/x = T/\Theta$, from the original table in Debye 1912 (803). Small tables of this sort must be common.

Zanstra 1931*a*, *b* gives 19 values of $\int_x^\infty \frac{x^3 dx}{e^x - 1}$ to 3–4 decimals. A small table in Eddington 1926 (112) essentially involves $\int_0^x \frac{x^3 dx}{e^x - 1}$.

Tables involving the experimentally determined constant R in calories per degree centigrade are given in Nernst 1924 ($R = 1\cdot 985$) and MacDougall 1926 ($R = 1\cdot 987$). Nernst gives $3Rf(x)$, mainly to 3 figures, for $x = 0(\cdot 1)16(1)30$ without differences, and U/T to 3 decimals for $x = 0(\cdot 01)2$ and to 4 decimals for $x = 0(\cdot 1)16$, in each case with first differences. Another function involving the same integral is also tabulated. MacDougall's tables are almost identical, except that they are based on the different value of R and give no differences. Allowing for the difference in the constant, the tables are not always in good agreement. Nernst's table of $3Rf(x)$ may also be found in Nernst 1912, which gives only brief tables of the other two functions. Landolt & Börnstein 1927 (1. Ergbd., 705) gives recomputed Debye tables.

22·65. Radiation Integrals, $\int_x^\infty \frac{dx}{x^n(e^x - 1)}$

Airey 1938 tabulates these to 6 decimals without differences for $n = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 2$ and $x = 0(\cdot 1)10$.

22·66. $\frac{15}{4\pi^4}\int_0^x \frac{x^7 e^{-x} dx}{(1 - e^{-x})^3}$

5–4 dec.	$x = 0(\cdot 5)3(\cdot 2)5(\cdot 1)9(\text{var.})\infty$	No Δ	Morse 1940
2; 1 dec.	$x = 0(\cdot 1)5;\ 5\cdot 1(\cdot 1)12(1)16,\ \infty$	Δ	Strömgren 1932

22·67. Fermi–Dirac Functions

McDougall & Stoner 1939 consider the functions $F_k(\eta) = \int_0^\infty \frac{x^k dx}{e^{x-\eta} + 1}$. They tabulate the five functions

$$\tfrac{2}{3}F_{\frac{3}{2}}(\eta),\quad F = F_{\frac{1}{2}}(\eta) = \frac{d}{d\eta}\{\tfrac{2}{3}F_{\frac{3}{2}}(\eta)\},\quad \frac{1}{10}\frac{dF}{d\eta},\quad \frac{1}{100}\frac{d^2F}{d\eta^2},\quad \frac{1}{1000}\frac{d^3F}{d\eta^3}$$

The tables are all to 6 decimals for $\eta = -4(\cdot 1)+4$ and to 5 decimals for $\eta = 4(\cdot 1)20$. (The last of the five functions is negligible after $\eta = 8$, and is not tabulated.)

Stoner 1939 puts $F_{\frac{1}{2}}(\eta) = \frac{2}{3}\tau^{-\frac{3}{2}}$, and tabulates to 5 decimals without differences for $\tau = 0(\cdot 05)2$ the four functions

$$\tau\eta,\quad \frac{\tau F_{\frac{3}{2}}}{F_{\frac{1}{2}}},\quad \tau\left(\eta - \frac{2}{3}\frac{F_{\frac{3}{2}}}{F_{\frac{1}{2}}}\right),\quad \frac{5}{3}\frac{F_{\frac{3}{2}}}{F_{\frac{1}{2}}} - \eta$$

where $F_{\frac{1}{2}}$, $F_{\frac{3}{2}}$ stand for $F_{\frac{1}{2}}(\eta)$, $F_{\frac{3}{2}}(\eta)$.

22·7. Miscellaneous Functions

22·71. Chebyshev Polynomials $C_n(x)$ and $S_n(x)$

The Chebyshev polynomials $C_n(x)$ may be defined by $C_n = 2\cos n\theta$, $x = 2\cos\theta$. They satisfy the recurrence relation $C_{n+1} = xC_n - C_{n-1}$. The allied polynomials $S_n(x)$ are defined by $S_n = \dfrac{\sin(n+1)\theta}{\sin\theta}$, $x = 2\cos\theta$, and satisfy the same recurrence relation. The first few polynomials are

$$C_0 = 2,\quad C_1 = x,\quad C_2 = x^2-2,\quad C_3 = x^3-3x,\quad C_4 = x^4-4x^2+2,\quad \ldots$$
$$S_0 = 1,\quad S_1 = x,\quad S_2 = x^2-1,\quad S_3 = x^3-2x,\quad S_4 = x^4-3x^2+1,\quad \ldots$$

and the series may be continued with great ease.

A New York W.P.A. volume (in press) gives $C_n(x)$ and $S_n(x)$ to 12 decimals for $n = 1(1)12$ and $x = 0(\cdot 001)2$. A manuscript by C. W. Jones and J. C. P. Miller gives $C_n(x)$ for $n = 1(1)12$, $x = 0(\cdot 02)2$ (exact to $n = 7$, then 10 decimals); this has been combined for publication with an independent table of similar scope (but exact to $n = 10$) by J. F. C. Conn and R. C. Pankhurst. Algebraic expressions in powers of x for $C_n(x)$ and $S_n(x)$ up to $n = 20$ are also given. Miller (MS.) has exact values of $C_n(x)$ and $S_n(x)$ for $n = 0(1)11$, $x = 0(\cdot 1)3\cdot 5$.

The polynomials as defined above are adjusted to the range $(-2, +2)$ in x. They may also be adjusted to other ranges, e.g. $(-1, +1)$ or $(0, 1)$. Jones, etc. also give algebraic expressions to $n = 20$ for the range $(-1, +1)$; coefficients for $(0, 1)$ are identical with (for C_n) or half (for S_n) those for even n with the range $(-1, +1)$. Lanczos 1938 gives the C-coefficients to $n = 10$ for the range $(0, 1)$, while Prévost 1933 (136*) goes up to C_{12} and S_{11} for the range $(-1, +1)$.

Functions related to the Chebyshev S polynomials are tabulated in Harzer 1886.

22·72. Pearson Integrals

The notation is

$$G(r, \nu) = \int_0^{\pi} \sin^r\theta\, e^{\nu\theta}\, d\theta,\quad F(r, \nu) = e^{-\frac{1}{2}\nu\pi} G(r, \nu),\quad H(r, \nu) = \frac{(r-1)^{\frac{1}{2}} e^{-\nu\phi} F(r, \nu)}{\cos^{r+1}\phi}$$

where $\nu = r\tan\phi$.

B.A. 1896 tabulates certain auxiliary functions χ defined by

$$\chi_{2m+1}(\phi) = \frac{B_{2m+1}}{(2m+1)(2m+2)}\{(\tfrac{1}{2})^{2m+2} - \cos^{2m+1}\phi\cos(2m+1)\phi\}$$

For $m = 0(1)3$ and $\phi = 0(1^\circ)90^\circ$, the tables give to 7 decimals $\log(\pm\chi_{2m+1})$ without differences and χ_{2m+1} with Δ^2. Eighth and ninth decimals of the natural values and their differences are given in brackets.

B.A. 1899 (Pearson & Lee) gives 7-decimal tables for $r = 1(1)50$. For $r = 1$, $\log F(r, \nu)$ is given with Δ^2 for $\phi = 0(1^\circ)45^\circ$. For $r = 2(1)50$, $\log F(r, \nu)$ is given without differences and $\log H(r, \nu)$ with Δ^2, all for $\phi = 0(1^\circ)45^\circ$. The whole table from B.A. 1899 is reproduced in Pearson 1930 (126).

Jahnke & Emde 1909 (44) gives $\log F(r, \nu)$ to 4 decimals without differences for $r = 1(1)50$, $\phi = 0(5^\circ)45^\circ$.

$$22\cdot73.\quad S(k,n)=\int_0^1 x^k \sin n\pi x\,dx \text{ and } C(k,n)=\int_0^1 x^k \cos n\pi x\,dx$$

Ten-decimal tables (based on a 20-decimal New York W.P.A. manuscript) are given for $k = 0(1)10$, $n = 0(1)100$ in Lowan & Laderman 1943. See page 110.

$$22\cdot74.\quad \int_a^{\frac{1}{2}\pi+a} e^{-u\sin x}\,dx$$

Tables for various values of a and u are given in L. D. Rosenberg 1942. See *M.T.A.C.*, **1**, 192, 1944.

$$22\cdot75.\quad \int_{-\infty}^{\infty} e^{-y^2z^2-z-xe^{-z}}\,dz$$

This is tabulated as a function of x and y in Jahnke & Emde 1933 (113), 1938 (39). The table is reproduced from Wagner 1913.

$$22\cdot752.\quad \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-t^2-xe^{-t^2}}\,dt=\sum_0^{\infty}\frac{(-x)^n}{n!\sqrt{n+1}}$$

Small tables of this and related integrals are given in Chapman 1939. See also Mitchell & Zemansky 1934 (324).

$$22\cdot76.\quad \int_0^{\infty}\frac{u^s du}{\sinh 2u \pm 2u} \text{ and } \int_0^{\infty}\frac{u^s e^{-2u} du}{\sinh 2u \pm 2u}$$

These integrals, as well as their products by $2^s/s!$, are tabulated for integral values of s up to about 20 in the case of the positive signs in Howland 1930 and in the case of the negative signs in Howland & Stevenson 1933.

$$22\cdot77.\quad f_r(z)=(-)^r\left(\int_z^{\infty}\right)^r \log_e \tanh \tfrac{1}{2}z\,(dz)^r$$

The functions $f_r(z)$ defined by

$$f_0(z)=\log_e \tanh \tfrac{1}{2}z \qquad f_r(z)=-\int_z^{\infty} f_{r-1}(z)\,dz$$

are considered in Rosenhead & Miller 1937, and are tabulated to 4 decimals for $r = 0, 2, 4$ and $x = 0(\cdot01)\cdot1(\cdot02)\cdot3(\cdot05)\cdot5(\cdot1)1(\cdot2)3(\cdot5)5(1)10$.

$$22\cdot78.\quad \int_0^{\frac{\pi}{2}}(1+k^{-1}\operatorname{cosec}^2\phi \log \sec\phi)^{-\frac{1}{2}}\,d\phi$$

Pidduck 1937 tabulates this in degrees and minutes for $k = \cdot05(\cdot05)\cdot5(\cdot1)1\cdot5(\text{var.})5$. Other small tables are given.

$$22\cdot79.\quad \int_{-1}^{1}\frac{t^{2n+1}}{t-y}\,dt \text{ and } \int_{-1}^{1}\frac{t^{2n+2}}{1-yt}\,dt$$

These integrals, which can be expressed in terms of elementary functions, are given in Squire 1939 to 4 decimals without differences for $n = 0(1)2$, $y = 0(\cdot1)\cdot9, \cdot95$.

22·80. Mittag-Leffler Polynomials

These are the functions $g_n(x)$ defined by

$$(1+t)^x(1-t)^{-x} = \sum_0^\infty g_n(x)t^n, \qquad |t| < 1$$

Bateman 1940 tabulates them as exact integers for $n = 0(1)10$ and $x = 0(1)10$. Explicit algebraic expressions for $g_0(x)$ to $g_6(x)$ are also given.

22·81. Solutions of $u\,e^{-u} = e^{-t-1}$

Watson 1929 tabulates the two solutions, $u < 1$ and $U > 1$, to 7–8 decimals for $t = 0(\cdot 02)1(\cdot 1)11$. In Wilson & Burke 1942 the argument is u/U, say λ, and the tabulated quantities are $u = -\lambda(1-\lambda)^{-1}\log_e \lambda$ and $U = -(1-\lambda)\log_e \lambda$.

22·82. $\dfrac{2}{\pi\beta}\displaystyle\int_0^\infty \exp\left\{-\left(\frac{x}{\beta}\right)^{\frac{3}{2}}\right\} x \sin x\,dx$

This function is tabulated in Chandrasekhar 1943 (73).

22·83. Boundary Layer Equations

A full discussion of the equations of the boundary layer in viscous motion lies outside the scope of the present work, but we may allude to the fact that the non-linear equation (in Hartree's notation)

$$y''' + yy'' = \beta(y'^2 - 1)$$

where β is a parameter, has occurred in much numerical work. When $\beta \geqslant 0$ the boundary conditions are $y = y' = 0$ at $x = 0$ and (say) $y' \to 1$ as $x \to \infty$. On the case $\beta < 0$ see Hartree 1937*b*, in which y' is tabulated against x for a range of values of β between $-\cdot 1988$ and $+2\cdot 4$; for each value of β, $y''(0)$ is also given. See also Falkner & Skan 1930, 1931. The cases $\beta = 0$ and $\beta = 1$ are specially important, the equation $y''' + yy'' = 0$ of Blasius being classic. Falkner 1936 evaluates a solution (in slightly variant form) for the case $\beta = 0$; this case also occurs in, for example, Howarth 1938 and Goldstein 1930 (ψ_0' on page 24 is Hartree's y' for $\beta = 0$, putting Hartree's x equal to $\sqrt{2}$ times Goldstein's y). For the case $\beta = 1$, values of y, y' and y'' are tabulated to 4 decimals for $x = 0(\cdot 1)4\cdot 3$ in Howarth 1934 (functions f_1, f_1' and f_1'' on page 370 or 372), as well as to 8 decimals for $x = 0(\cdot 1)5$ in Bickley (MS.). Both Goldstein and Howarth tabulate a whole series of more complicated functions. Details and further references will be found in the papers mentioned.

22·84. Spence's Integrals

Spence 1809 investigates functions which we may call $L_n(u)$. The n is placed by Spence above the L; our notation in the present article will lead to no confusion with the L_n used for Laguerre polynomials in Art. 22·56. The functions may be defined by

$$L_n(u) = \int_1^u \frac{L_{n-1}(u)}{u-1}\,du \qquad L_0(u) = \frac{u-1}{u}$$

which give

$$L_1(u) = \log_e u \qquad L_2(u) = \int_1^u \frac{\log_e u}{u-1}\,du \qquad L_3(u) = \int_1^u \frac{L_2(u)}{u-1}\,du$$

and so on, L_0 and L_1 being elementary functions and the remainder transcendental. It is easily seen that if $|x| < 1$ we have

$$L_n(1+x) = x - 2^{-n}x^2 + 3^{-n}x^3 - 4^{-n}x^4 + \dots$$

and that in the notation of Art. 4·022 we have

$$L_n(0) = -S_n \qquad L_n(2) = s_n$$

and in particular $L_2(0) = -\frac{1}{6}\pi^2$, $L_2(2) = \frac{1}{12}\pi^2$. Spence 1809 tabulates L_2 on page 24 and L_3 on page 41, in both cases to 9 decimals without differences for $u = 1(1)100$. Other numerical values are given in the text; we may mention particularly 12-decimal values of $L_2(\frac{1}{2})$, $L_2(\frac{3}{2})$, $L_3(\frac{1}{2})$ and $L_3(\frac{3}{2})$ on pages 15 and 35. See also Sang 1884*b* (609). Spence also considers a similar series of functions $C_n(x)$, and tabulates on page 63 the function $C_1(x)$, which is simply $\tan^{-1} x$ expressed in radians; see Art. 9·12.

Undoubtedly the most important of Spence's transcendents is L_2, which following F. W. Newman we may henceforward call simply L, dropping the suffix 2. It is connected with the Debye function of Art. 22·61, since we have

$$\int_0^t \frac{t\,dt}{e^t - 1} = -L(e^{-t}) = L(e^t) - \tfrac{1}{2}t^2$$

$L(u)$ has the properties

$$L(u) + L(u^{-1}) = \tfrac{1}{2}(\log_e u)^2$$
$$L(u) + L(1-u) = \log_e u \log_e (1-u) - \tfrac{1}{6}\pi^2$$

Newman 1892 (64–65) tabulates $L(1+x)$ and $-L(1-x)$ to 12 decimals without differences for $x = 0(\cdot01)\cdot50$; in $-L(\cdot55)$, for 5043 read 5143. The values are derived from tables of $\phi(x)$ and $\psi(x)$, where

$$\phi(x) = \tfrac{1}{2}L(1+x) - \tfrac{1}{2}L(1-x) = x + 3^{-2}x^3 + \dots$$
$$\psi(x) = -\tfrac{1}{2}L(1+x) - \tfrac{1}{2}L(1-x) = 2^{-2}x^2 + 4^{-2}x^4 + \dots$$

These tables, which were checked and corrected by J. C. Adams, are also to 12 decimals without differences for $x = 0(\cdot01)\cdot50$, and are given on pages 103–104. In view of the two properties of $L(u)$ mentioned above, Newman's tables are more than sufficient to enable $L(u)$ to be found with little labour anywhere in the range from $u = 0$ to $u = \infty$. A few values of L are also given in Hill 1828 (159).

Powell 1943 gives $L(u)$ to 7 decimals with second differences for $u = 0(\cdot01)2(\cdot02)6$. This paper uses the notation Rl instead of L. See Fletcher 1944 for a comparison of Powell with Newman and Spence, and a list of errata in Newman.

22·85. Clausen's Integral

The function

$$\mathrm{Cl}\, x = -\int_0^x \log_e (2 \sin \tfrac{1}{2}x)\,dx$$
$$= \sin x + 2^{-2} \sin 2x + 3^{-2} \sin 3x + \dots$$

is tabulated to 16 decimals without differences for $x = 0(1°)180°$ in Clausen 1832. The table is reprinted in Newman 1892 (85).

22·86. Thomas–Fermi Equation

Solutions of the Thomas–Fermi equation $\phi'' = x^{-\frac{1}{2}}\phi^{\frac{3}{2}}$, with the conditions $\phi(0) = 1$ and $\phi(\infty) = 0$, are given in Baker 1930 and in Bush & Caldwell 1931. On putting $\phi = xy$, the equation becomes $y'' + 2y'/x - y^{\frac{3}{2}} = 0$; compare Hartree 1937*a* in Art. 22·32.

22·87. Langmuir's Equation

This equation, which arises in connection with currents limited by space charge between coaxial cylinders, is

$$3\beta\frac{d^2\beta}{d\gamma^2} + \left(\frac{d\beta}{d\gamma}\right)^2 + 4\beta\frac{d\beta}{d\gamma} + \beta^2 - 1 = 0$$

with initial conditions $\beta = 0$ and $d\beta/d\gamma = 1$ at $\gamma = 0$. On putting $\beta = e^{-\frac{1}{2}\gamma}\theta$, the equation becomes

$$3\theta\frac{d^2\theta}{d\gamma^2} + \left(\frac{d\theta}{d\gamma}\right)^2 - e^{\gamma} = 0$$

with $\theta = 0$ and $d\theta/d\gamma = 1$ at $\gamma = 0$. For tables and references to earlier work, see Langmuir & Blodgett 1923.

22·88. $\dfrac{d^2\phi}{d\xi^2} + \left(a + b\dfrac{e^{-\xi}}{\xi}\right)\phi = 0$

Solutions with the conditions $\phi(0) = 0$ and ϕ everywhere finite are given for 5 pairs of values of a and b in Wilson 1938.

22·89. Damped Vibrations

A number of tables relating to solutions of the equation

$$\frac{d^2x}{dt^2} + \frac{1}{2}\frac{dx}{dt}\left|\frac{dx}{dt}\right| + 2b\frac{dx}{dt} + x = 0$$

are given in W. E. Milne 1923, 1929 and 1935. With suitable choice of the units of length and time, the equation applies to any one-dimensional motion under a restoring force proportional to the displacement and a resistance proportional partly to the speed and partly to the square of the speed.

22·90. $\displaystyle\int_0^\infty y^s\{\log(1+y)\}^n e^{-ay^2}\,dy$

Orr 1942 tabulates this integral to 6 decimals without differences for $a = \cdot 1(\cdot 1)1\cdot 5$ and

$$\begin{cases} s = 0, & n = 1, 2 \\ s = 1, & n = 1(1)3 \end{cases}$$

Various coefficients needed are also tabulated.

22·91. Toronto Functions

Heatley 1943 tabulates functions $T(m, n, r)$ defined by

$$T(m, n, r) = r^{2n-m+1} e^{-r^2} \Gamma(\tfrac{1}{2}m + \tfrac{1}{2}) M(\tfrac{1}{2}m + \tfrac{1}{2}, n+1, r^2)/\Gamma(n+1)$$

where M is the confluent hypergeometric function of Art. 22·56. The tables relate to the cases $m = -\frac{1}{2}(\frac{1}{2}) + 1$, $n = -2(\frac{1}{2}) + 2$. In some of these cases there are gaps in the tabulation between $r = 2$ and $r = 4$ or 5, power series having been used for $r < 2$ and asymptotic expansions for the higher values of r (up to 50). Some values of $M(a, \gamma, x)$ and $e^{-x} M(a, \gamma, x)$ are also given. See Art. 20·861.

SECTION 23

INTERPOLATION, NUMERICAL DIFFERENTIATION AND INTEGRATION, CURVE-FITTING

23·0. Introduction

This section deals mainly with interpolation, numerical differentiation and numerical integration, principally in relation to tables with a constant interval in the argument. For trigonometric interpolation, see Section 24 (harmonic analysis). The coefficients needed for differentiation at, or integration between, tabular points or mid-points have in many cases already been dealt with in Section 4 (Arts. 4·9 ff.) in connection with Bernoulli numbers and polynomials, and we give references to the particular articles in the appropriate places. Differentiation at general points and integration between general limits will be the principal topics in Arts. 23·4 and 23·6. The section closes with a brief account of some curve-fitting tables.

For general references to works on numerical mathematics, see the Introduction.

We use Δ for forward differences, δ for central differences, μ for means, D for derivatives, w for the interval of the argument and n for the fraction of the interval.

The main subdivisions of the present section are as follows:

23·1. Interpolation Coefficients

A good collection is given in *N.A.* 1937 (Comrie). For a few tables relating to sexagesimal interpolation, see Arts. 9·8 ff.

23·11. Lagrange Interpolation Coefficients

These are for interpolation without differences. For example, a 4-point formula is

$$f(a+nw) = L_{-1}f(a-w) + L_0 f(a) + L_1 f(a+w) + L_2 f(a+2w)$$

or, more concisely,

$$y_n = L_{-1}y_{-1} + L_0 y_0 + L_1 y_1 + L_2 y_2$$

where the four Lagrange coefficients L may be tabulated, the interval in n being usually ·1, ·01 or ·001. Tables to assist the computation of odd-point Lagrange coefficients are given in Rutledge 1928.

Exact tables of the coefficients, at intervals ·1, ·01 and ·001 in n, require the following numbers of decimals:

	3-point	4-point	5-point	6-point	7-point	8-point
·1	3	4	7	8	10	11
·01	5	7	11	13	16	18
·001	7	10	15	18	22	25

For Lagrange coefficients for harmonic interpolation, see Comrie & Hartley 1941.

23·111. Lagrange Coefficients for an Even Number of Ordinates

We take the suffix 0 to refer to the left end of the middle interval, so that the ordinates are y_{-1} to y_2, y_{-2} to y_3, y_{-3} to y_4 and y_{-4} to y_5 in formulæ for 4, 6, 8 and 10 points respectively.

Tables normally relate to interpolation in the middle interval, but **New York W.P.A. 1944*b***, Huntington 1929 and Pearson 1920*a* cater for interpolation anywhere between the extreme ordinates. The latter contains more numerical matter than can be indexed fully here. Its exact values are given in various forms, so that we do not always specify a number of decimals.

Coefficients for the first half of the range are easily used for the second half, so that coefficients for the latter are not always given explicitly.

The following tables give no differences, except that first differences are given in the 4-point table in *N.A.* 1935, etc.

10-point	10 dec.	$n = -4(\cdot 1)+5$	**New York W.P.A. 1944*b***
10-point	10 dec.	$n = -4(\cdot 2)+5$	Pearson 1920*a* (T. C)
4-, 6-, 8-, 10-pt.	Fractions	$n = \frac{1}{2}, 1\frac{1}{2}$, etc.	Pearson 1920*a*

In equation (θ) (iii) on page 46, for +16170 read −16170 (Miller). The 6-point results are also given in Pearson 1931 (ccxlviii).

8-point	10 dec.	$n = 0(\cdot 001)1$	**New York W.P.A. 1944*b***
	10 dec.	$n = -3(\cdot 1)+4$	
	Fractions	$n = -3(\frac{1}{12})+4$	
8-point	11 dec.	$n = 0(\cdot 1)1$	Kelley 1938
8-point	Exact	$n = 0(\cdot 1)1$	Pearson 1920*a* (50)
8-point	8 dec.	$n = -3(\cdot 2)+4$	Pearson 1920*a* (T. B)
6-point	10 dec.	$n = 0(\cdot 001)1$	**New York W.P.A. 1944*b***
	10 dec.	$n = -2(\cdot 01)0,\ 1(\cdot 01)3$	
	Fractions	$n = -2(\frac{1}{12})+3$	
6-point	10 dec.	$n = 0(\cdot 001)1$	Comrie (MS. 1930)
6-point	10 dec.	$n = 0(\cdot 01)1$	Kelley 1938
6-point	8 dec.	$n = -2(\cdot 01)+3$	Huntington 1929
6-point	6 dec.	$n = -2(\cdot 2)+3$	Pearson 1920*a* (T. A)
6-point	6 dec.	$n = 0(\cdot 2)1$	Comrie 1928*b*
6-point	Fractions	$n = 0(\frac{1}{8})1,\ 0(\frac{1}{12})1$	Bickley (MS. 1941)
4-point	10 dec.	$n = 0(\cdot 0001)1$	**New York W.P.A. 1944*b***
	10 dec.	$n = -1(\cdot 001)0,\ 1(\cdot 001)2$	
	Fractions	$n = -1(\frac{1}{12})+2$	
4-point	10 dec.	$n = 0(\cdot 001)1$	Comrie (MS. 1930)
4-point	10 dec.	$n = 0(\cdot 001)1$	**Kelley 1938**

Page 121, in L_0 at $n = \cdot 155$, for 79375 read 69375 (Comrie, *M.T.A.C.*, 1, 161, 1944).

4-point	9 dec.	$n = -1(\cdot 002)0$	Little 1927

Gushee 1942 has first-interval interpolation tables for 4(1)8 points.

4-point	7 dec.	$n = -1(\cdot 01)+2$	Huntington 1929
4-point	7 dec.	$n = 0(\cdot 01)1$	B.A. **2**, 1932 (34)
4-point	5 dec.	$n = 0(\cdot 01)1$	*N.A.* 1935 (747), 1937 (785); N.A.O. 1933, 1939*b*
4-point	For interpolation to tenths		Pearson 1920*a* (31)
4-point	Fractions	$n = 0(\frac{1}{8})1, 0(\frac{1}{12})1$	Bickley (MS. 1941)

23·112. Lagrange Coefficients for an Odd Number of Ordinates

We take the suffix 0 to refer to the middle point, so that the ordinates are y_{-1} to y_1, y_{-2} to y_2 and y_{-3} to y_3 in formulæ for 3, 5 and 7 points respectively. The exact values in Pearson 1920*a* are given in various forms, so that we do not specify a number of decimals. No differences are given in the following tables.

11-point	10 dec.	$\pm n = 0(\cdot 1)5$	**New York W.P.A. 1944*b***
11-point	Exact	$\pm n = 0(\cdot 1)\cdot 5(1)4\cdot 5$	Pearson 1920*a* (29)
9-point	10 dec.	$\pm n = 0(\cdot 1)4$	**New York W.P.A. 1944*b***
9-point	Exact	$\pm n = 0(\cdot 1)\cdot 5(1)3\cdot 5$	Pearson 1920*a* (28)
7-point	10 dec.; Fractions	$\pm n = 0(\cdot 001)1(\cdot 01)3$; $\pm n = 0(\frac{1}{12})3$	**New York W.P.A. 1944*b***
7-point	16 dec.	$\pm n = \cdot 25(\cdot 50)2\cdot 75$	Hildebrand & Crout 1941
7-point	10 dec.	$\pm n = 0(\cdot 1)3$	Crout 1940
7-point	10 dec.	$\pm n = 0(\cdot 1)1, 4, 5$	Rutledge & Crout 1930
7-point	7 dec.	$\pm n = 0(\cdot 1)1$	Dwight 1941
7-point	Exact	$\pm n = 0(\cdot 1)\cdot 5(1)2\cdot 5$	Pearson 1920*a* (26)
5-point	10 dec.; Fractions	$\pm n = 0(\cdot 001)2$; $\pm n = 0(\frac{1}{12})2$	**New York W.P.A. 1944*b***
5-point	11 dec.	$\pm n = \cdot 25(\cdot 50)1\cdot 75$	Hildebrand & Crout 1941
5-point	10 dec.	$\pm n = 0(\cdot 001)1$	New York W.P.A. 1943*a*
5-point	7 dec.	$\pm n = 0(\cdot 001)2$	New York W.P.A. 1942*c*
5-point	7 dec.	$\pm n = 0(\cdot 1)2$	Crout 1940
5-point	7 dec.	$\pm n = 0(\cdot 1)1, 3, 4$	Rutledge & Crout 1930
5-point	7 dec.	$\pm n = 0(\cdot 1)1$	Dwight 1941
5-point	Exact	$\pm n = 0(\cdot 1)\cdot 5, 1\cdot 5$	Pearson 1920*a* (25)
5-point	Fractions	$\pm n = 0(\frac{1}{8})1, 0(\frac{1}{12})1$	Bickley (MS. 1941)
3-point	9 dec.; Fractions	$\pm n = 0(\cdot 0001)1$; $\pm n = 0(\frac{1}{12})1$	**New York W.P.A. 1944*b***
3-point	3 dec.	$\pm n = 0(\cdot 1)1, 2, 3$	Rutledge & Crout 1930
3-point	3 dec.	$\pm n = 0(\cdot 1)1$	Crout 1940; Dwight 1941
3-point	Fractions	$\pm n = 0(\frac{1}{8})1, 0(\frac{1}{12})1$	Bickley (MS. 1941)

23·116. Cumulative Lagrange Coefficients

Huntington 1929 gives exact 4-point and 6-point coefficients at interval ·2 in n for finding the difference between two successive ordinates, $y_{n+\cdot 2}-y_n$, anywhere between the extreme ordinates.

23·12. Taylor Interpolation Coefficients, $n^r/r!$

These are for use with the formula

$$f(a+nw) = f(a) + n \times wf'(a) + \frac{n^2}{2!} \times w^2 f''(a) + \dots$$

where primes denote derivatives.

8 dec.	$r = 2(1)6$, $n = 0(\cdot 01)1$	No Δ	B.A. 2, 1932 (32)

Miller (MS. 1941) has the same functions to 20+ decimals for $n = 0(\cdot 03)\cdot 48$, $r = 0(1)9$ to 17, according to the value of n.

23·13. Gregory–Newton Interpolation Coefficients

These are the binomial coefficients ${}^nC_2 = \frac{n(n-1)}{2!}$, etc. which occur in the formula

$$f(a+nw) = f(a) + n\Delta f(a) + {}^nC_2\Delta^2 f(a) + {}^nC_3\Delta^3 f(a) + \dots$$

where Δ denotes a forward difference. Tables are very common.

It should be noticed that all Gauss and Everett coefficients, and half the Stirling and Bessel coefficients, are Gregory–Newton coefficients for various n, not confined to the range 0 to 1.

Tables of the coefficients are exact, for intervals in n of ·1, ·01 and ·001, in the following numbers of decimals:

	2nd	3rd	4th	5th	6th
·1	3	4	7	8	10
·01	5	7	11	13	16
·001	7	10	15	18	22

For n at sexagesimal intervals, see Art. 9·84.

11 dec.	2nd–6th	$n = 0(\cdot 01)1$	Δ^n	Davis 1933 (102)
7 dec.	2nd	$n = 0(\cdot 001)1$	No Δ	Chappell 1929 New York W.P.A. 1943*b*
7 dec.	2nd–6th	$n = 0(\cdot 01)1$	No Δ	Dwight 1941 Vega 1797 Barlow 1814 (254) Vega–Hülsse 1849
6 dec.	2nd–5th	$n = 0(\cdot 01)1$	No Δ	Köhler 1847
6; 5; 4; 3 dec.	2nd; 3rd; 4th; 5th	$n = 0(\cdot 01)1$	No Δ	Schorr 1916
5 dec.	2nd–6th	$n = 0(\cdot 01)1$	No Δ	Potin 1925 (857)
5 dec.	2nd–5th	$n = 0(\cdot 01)1$	Δ	Rice 1899 Glover 1930 (414)
5 dec.	2nd–5th	$n = 0(\cdot 01)1$	No Δ	G. W. Jones 1893 Loomis 1873 (393)
5; 4 d.	2nd; 3rd–6th	$n = -\cdot 5(\cdot 01)+\cdot 5$	v^2; Δ	Lindow 1928 (158)
5; 4 d.	2nd; 3rd–5th	$n = 0(\cdot 01)1$	Δ	Wirtz 1918 (194)
5; 4 d.	2nd; 3rd, 4th	$n = 0(\cdot 01)1$	No Δ	Bauschinger 1901 (138) Bauschinger–Stracke 1934 (185)

4 dec.	2nd–7th	$n = 0(\cdot 1)1$	No Δ	Rice 1899 (51)
4 dec.	2nd–6th	$n = -1(\cdot 1)+1$	No Δ	Bruns 1903 (43)
4; 3 d.	2nd; 3rd, 4th	$n = 0(\cdot 01)1$	No Δ	Desvallées 1920
3 dec.	2nd, 3rd	$n = 0(\cdot 01)1$	No Δ	Dale 1903; Pryde 1930 (437)

Several of the New York W.P.A. volumes contain a 6-decimal table of $n(1-n)$ for $n = 0(\cdot 001)\cdot 5$.

23·14. Gauss Interpolation Coefficients

These are the coefficients in the formula

$$f(a+nw) = f(a) + n\Delta f(a) + \frac{n(n-1)}{2!}\Delta^2 f(a-w) + \frac{(n+1)n(n-1)}{3!}\Delta^3 f(a-w) + \ldots$$

in which the differences used may also be written $\Delta f(a)$, $\delta^2 f(a)$, $\Delta\delta^2 f(a)$, ...

Exact tables require the same numbers of decimals as in the case of Gregory–Newton coefficients.

7–10 d.	2nd–6th	$n = 0(\cdot 001)1$	No Δ	Chappell 1929
Exact	2nd–6th	$n = 0(\cdot 01)1$		
5; 4 d.	2nd; 3rd–6th	$n = -\cdot 25(\cdot 01)+\cdot 25$	v^2; (Δ)	Lindow 1928 (160)
4 dec.	2nd–6th	$n = -1(\cdot 1)+1$	No Δ	Bruns 1903 (43)

23·15. Stirling Interpolation Coefficients

These are the coefficients in the formula

$$y_n = y_0 + n\mu\delta y_0 + \frac{n^2}{2!}\delta^2 y_0 + \frac{n(n^2-1)}{3!}\mu\delta^3 y_0 + \frac{n^2(n^2-1)}{4!}\delta^4 y_0 + \ldots$$

which involves central even and mean odd differences. The odd coefficients are the same as in Gauss's formula.

The odd coefficients may however be given as coefficients of the double mean odd differences, and are then $\frac{1}{2}\cdot\frac{n(n^2-1)}{3!}$, $\frac{1}{2}\cdot\frac{n(n^2-1)(n^2-4)}{5!}$, etc.

23·151. Stirling Coefficients for Even and Mean Odd Differences

Tables of the coefficients are exact in the following numbers of decimals:

	2nd	3rd	4th	5th	6th
·1	3	4	7	8	10
·01	5	7	11	13	16
·001	7	10	15	18	22

Exact	2nd–6th	$\pm n = 0(\cdot 01)\cdot 55$	No Δ	Jäderin 1915 (50)
12 dec.	2nd–6th	$n = 0(\cdot 01)1$	Δ^n	Davis 1933 (110)
Exact	2nd–5th	$\pm n = 0(\cdot 1)3$	No Δ	Jäderin 1915 (51)
6 dec.	6th–12th			
5 dec.	2nd–5th	$n = 0(\cdot 01)1$	Δ	Rice 1899; Glover 1930 (416)
5; 4 d.	2nd; 3rd–6th	$\pm n = 0(\cdot 01)\cdot 25$	v^2; (Δ)	Lindow 1928 (162)
5; 4 d.	2nd; 3rd, 4th	$\pm n = 0(\cdot 01)\cdot 50$	No Δ	Bauschinger 1901 (139); Bauschinger–Stracke 1934 (186)
4 dec.	2nd–6th	$n = 0(\cdot 1)1$	No Δ	Bruns 1903 (43)

23·152. Stirling Coefficients for Even and Double Mean Odd Differences

Tables of the coefficients are exact in the following numbers of decimals:

	2nd	3rd	4th	5th	6th
·1	3	5	7	9	10
·01	5	8	11	14	16
·001	7	11	15	19	22

5; 4 d.	2nd; 3rd	$n = 0(\cdot 01)1$	Δ	C. F. W. Peters 1871 (104)
4 dec.	4th, 5th	$n = 0(\cdot 01)1$	No Δ	C. F. W. Peters 1871 (104)
5; 4 d.	3rd, 4th; 5th	$n = 0(\cdot 01)1$	Δ; no Δ	Hansen 1857 (69)

Last-figure errors exist.

23·16. Bessel Interpolation Coefficients

These are the coefficients in the formula

$$y_n = \tfrac{1}{2}(y_0+y_1) + (n-\tfrac{1}{2})\Delta y_0 + \frac{n(n-1)}{2!}\,\frac{\delta^2 y_0+\delta^2 y_1}{2} + \frac{n(n-1)(n-\frac{1}{2})}{3!}\Delta\delta^2 y_0 + \frac{(n+1)n(n-1)(n-2)}{4!}\,\frac{\delta^4 y_0+\delta^4 y_1}{2} + \ldots$$

involving odd and mean even differences situated centrally in the interval under consideration. The even coefficients may however be tabulated as coefficients of the double mean even differences, $\delta^2 y_0+\delta^2 y_1$, $\delta^4 y_0+\delta^4 y_1$, etc., and are then

$$\frac{1}{2}\cdot\frac{n(n-1)}{2!},\quad \frac{1}{2}\cdot\frac{(n+1)n(n-1)(n-2)}{4!},\ \text{etc.}$$

23·161. Bessel Coefficients for Odd and Mean Even Differences

Tables of the coefficients are exact in the following numbers of decimals:

	2nd	3rd	4th	5th	6th
·1	3	3	7	7	10
·01	5	7	11	13	16
·001	7	10	15	18	22

10–11 d.	2nd–6th	$n = 0(\cdot 01)1$	Δ^n	Davis 1933 (118)
10 dec.	2nd, 4th, 6th, 8th	$n = 0(\cdot 001)1$	No Δ	Jordan 1932*a*

The coefficients are tabulated in connection with a method of interpolation without differences.

8 dec.	2nd, 4th, 6th	$n = 0(\cdot 001)1$	No Δ	Chappell 1929
Exact	2nd, 4th, 6th	$n = 0(\cdot 01)1$	No Δ	Chappell 1929
6; 5; 4 d.	2nd; 3rd; 4th	$n = 0(\cdot 01)1$	Δ	Albrecht 1908 (316)
5 dec.	2nd–5th	$n = 0(\cdot 01)1$	Δ	Rice 1899 Glover 1930 (418)
5 dec.	2nd–5th	$n = 0(\cdot 01)1$	No Δ	G. W. Jones 1893 Loomis 1873 (392)
5; 4 d.	2nd, 3rd; 4th	$n = 0(\cdot 01)1$	Δ	Schorr 1916 Ist. Idrog. 1916

5; 4 d.	2nd; 3rd–6th	$n = \cdot 25(\cdot 01)\cdot 75$	v^2; no Δ	Lindow 1928 (163)
5; 4 d.	2nd; 3rd–5th	$n = 0(\cdot 01)1$	Δ	Wirtz 1918 (193)
5; 4 d.	2nd; 3rd, 4th	$n = 0(\cdot 01)1$	No Δ	Bauschinger 1901 (139); Bauschinger–Stracke 1934 (186)
4 dec.	2nd–6th	$n = -1(\cdot 1)+1$	No Δ	Bruns 1903 (43)
3 dec.	2nd–4th	Critical tables		*N.A.* 1931 (726)

23·162. Bessel Coefficients for Odd and Double Mean Even Differences

Tables of the coefficients are exact in the following numbers of decimals:

	2nd	3rd	4th	5th	6th
·1	4	3	8	7	11
·01	6	7	12	13	17
·001	8	10	16	18	23

Exact; 7; 6 d.	2nd–5th; 6th; 7th–9th	$n = 0(\cdot 1)1$	No Δ	Comrie 1928*b*

Same, omitting 9th, in *N.A.* 1931 (828).

8 dec.	6th	$n = 0(\cdot 01)1$	No Δ	*N.A.* 1937 (800)
6 dec.	2nd	$n = 0(\cdot 001)1$	Δ	*N.A.* 1937 (795)
5; 4 d.	3rd; 4th	$n = 0(\cdot 001)1$	No Δ	*N.A.* 1937 (795)
5; 4 d.	2nd; 3rd, 4th	$n = 0(\cdot 01)1$	Δ; no Δ	C. F. W. Peters 1871 (103)
5 dec.	2nd, 3rd	Critical tables		Johnston & Comrie (MS. 1941)

An extensive table calculated by S. Johnston and arranged by L. J. Comrie.

4 dec.	2nd, 3rd	Critical tables	*N.A.* 1937 (790)
4 dec.	2nd	Critical table	*N.A.* 1935 (742); B.A. **6**, 1937 (286)
3 dec.	2nd–4th	Critical tables	*N.A.* 1935 (746); *N.A.* 1937 (784)
3 dec.	2nd	Critical table	MT.C. 1931 (212); B.A. **6**, 1937 (288); Comrie 1942 (18)

B.A. **2**, 1932 (34) gives a 2-decimal critical table of $5B^{iv}$, which is substantially a 3-decimal critical table of the coefficient of the double fourth difference.

23·17. Everett Interpolation Coefficients

These are the coefficients $\dfrac{n(n^2-1)}{3!}, \dfrac{n(n^2-1)(n^2-4)}{5!}, \ldots$ which occur for the second, fourth, ... differences in Everett's formula

$$y_n = my_0 + \frac{m(m^2-1)}{3!}\delta^2 y_0 + \frac{m(m^2-1)(m^2-4)}{5!}\delta^4 y_0 + \ldots$$

$$+ ny_1 + \frac{n(n^2-1)}{3!}\delta^2 y_1 + \frac{n(n^2-1)(n^2-4)}{5!}\delta^4 y_1 + \ldots$$

where $m = 1 - n$.

It should be noted that extreme (first and last) even-point Lagrange coefficients (see Art. 23·111, especially New York W.P.A. 1944*b*) are also Everett coefficients.

Tables of the coefficients are exact in the following numbers of decimals:

	2nd	4th	6th	8th	10th	12th
·1	4	8	11	16	19	23
·01	7	13	18	25		
·001	10	18	25			

Exact	2nd–8th	$n = 0(\cdot 01)1$	All δ^{2n}	Thompson 1943
Exact	2nd–12th	$n = 0(\cdot 1)1$	All δ^{2n}	Thompson 1943
9–13 dec.	2nd–16th	$n = 0(\cdot 2)1$	All δ^{2n}	Thompson 1943
10–13 dec.	2nd–12th	$n = -5(\cdot 1)+6$	No Δ	Thompson 1943
11 dec.	6th	$n = 0(\cdot 1)1$	No Δ	Kelley 1938
11 dec.	6th	$n = 0(\cdot 1)1$	No Δ	Thompson 1943
10 dec.	2nd–8th	$n = 0(\cdot 001)1$	δ^2	Thompson 1943

Same, but without coefficients of 8th differences, in Thompson 1921.

10 dec.	2nd, 4th, 6th	$n = 0(\cdot 01)1$	δ^{2n}	Davis 1933 (126)
9 dec.	2nd, 4th, 6th	$n = 0(\frac{1}{12})1$	No Δ	Sheppard 1910 (708)
7; 10; 8 d.	2nd; 4th; 6th	$n = 0(\cdot 01)1$	No Δ	*N.A.* 1937 (800)
10; 8 d.	2nd; 4th	$n = 0(\cdot 001)1$	No Δ	Chappell 1929
7; 13 d.	2nd; 4th	$n = 0(\cdot 01)1$	No Δ	Chappell 1929
7 dec.	2nd	$n = 0(\cdot 001)1$	No Δ	Mursi 1941 (282)
7 dec.	2nd	$n = 0(\cdot 01)1$	δ^2	*B.A. Card*, No. 1 (in press)
7 dec.	2nd	$n = 0(\cdot 01)1$	No Δ	B.A. **6**, 1937 (285)
6 dec.	2nd	$n = 0(\cdot 001)1$	No Δ	New York W.P.A.

This table is provided in several New York W.P.A. volumes (1940*c*, *d*, 1942*a*, *d*, 1943*b*). The quantity tabulated is $\frac{1}{2}n(1-n^2)$, i.e. the negative sign is omitted.

5 dec.	2nd, 4th, 6th	$n = 0(\cdot 01)1$	Δ	Glover 1930 (420)
5 dec.	2nd	$n = 0(\cdot 01)1$	Δ	*N.A.* 1935 (747); *N.A.* 1937 (785)
4 dec.	2nd	$n = 0(\cdot 01)1$	No Δ	*N.A.* 1931 (727)
3 dec.	2nd	Critical table		*N.A.* 1935 (748); *N.A.* 1937 (786)

23·18. Other Interpolation Coefficients

Becker & Van Orstrand 1924 (273) tabulates to 4 decimals with Δ, for $n = 0(\cdot 01)\cdot 50$, coefficients in the formula

$$f(a+nw) = f(a) + nw\left\{f'(a) + \frac{n}{2}\mu\delta f'(a) + \frac{n^2}{6}\delta^2 f'(a) + \frac{n}{12}\left(\frac{n^2}{2}-1\right)\mu\delta^3 f'(a) + \ldots\right\}$$

involving differences of the first derivative, up to the coefficient of $\mu\delta^3 f'(a)$. The same quantities are given to 4 decimals with first differences for $n = 0(\cdot 02)\cdot 6$ in Rice 1899.

23·2. Miscellaneous (Second-Difference Correction, Throw-Back, Functions of Two or more Variables, etc.)

23·21. Values of the Second-Difference Correction $\dfrac{n(n-1)}{4}(\delta^2 y_0 + \delta^2 y_1)$

These are given to the nearest unit for interval ·01 or greater in n and for $\delta^2 y_0 + \delta^2 y_1 = 10(5)200$ in *N.A.* 1937 (788).

23·22. Values of the Second-Difference Correction $\frac{n(n-1)}{2}\delta^2$

These are given to 1 decimal for $n = 0(\cdot 01)1$, $\delta^2 = 1(1)44$ in Peters 1922 (xiv), and also to 1 decimal for $n = 0(\cdot 1)1$, $\delta^2 = 4(1)100$ in Norén & Raab 1900.

23·23. Correction $\frac{1-n}{2}\delta^2$ to the First Difference

This is for use with the formula $y_n = y_0 + n\Delta'$, where $\Delta' = \Delta y_0 + \frac{n-1}{2}\delta^2 y_0$ is a corrected first difference. Peters 1919*b* (4) gives the correction to the nearest unit for $n = 0(\cdot 01)1$, $\delta^2 = 0(10)1000$.

23·25. Throw-Back

On the theory of this process of modification of differences, see Comrie 1928*b*, *N.A.* 1931 (especially p. 832), B.A. **1**, 1931 (xi), *N.A.* 1935 (852), *N.A.* 1937 (928) and Lidstone 1937 (286), etc.

23·251. $0{\cdot}184\delta^4$ (Throw-Back from Fourth to Second Differences)

This may be used when δ^4 is less than about 1000. A critical table to the nearest integer up to $\delta^2 = 1024$ is given in *N.A.* 1935 (749), *N.A.* 1937 (787), and up to $\delta^2 = 1002$ in *N.A.* 1931 (851).

23·252. Residual Coefficient of Fourth Difference after Throw-Back

The residual coefficient of the double fourth difference is the quantity denoted by T^{iv} and given to 5 decimals for $n = 0(\cdot 01)1$ in *N.A.* 1937 (800). It is given virtually to 6 decimals for $n = 0(\cdot 01)1$ in *B.A. Card*, No. 1 (in press), where $T^4 = 1000\,T^{iv}$ is given to 3 decimals. The card will also give residual Everett coefficients of the fourth differences to the same accuracy and over the same range. The manuscript table by Johnston & Comrie mentioned in Art. 23·162 gives 5-decimal values of T^{iv}.

23·27. Inverse Interpolation

Tables of coefficients for inverse interpolation with central and advancing differences are given in Salzer 1943*b* and Salzer 1944*b* respectively.

23·28. Interpolation in Tables with Two or more Arguments

See Pearson 1920*b*, 1922 (x), 1930 (xiv), 1931 (xvii, liii), 1934, Willers 1928 (99), Scarborough 1930 (96), Davis 1933 (95) and Comrie 1940*a* (16), 1942 (29).

23·3. Subtabulation

In addition to the references given below, see Hayashi 1926, 1930*a*, 1933.

23·31. End-Figure Method

For a description of this with extensive auxiliary tables, see *N.A.* 1931 (828). An explanation without the tables is given in Comrie 1928*b*.

23·32. Subdivided Differences in Terms of Original Differences

See Leverrier 1855 (151), King 1902 (442), Thiele 1909 (88), Jäderin 1915, Pearson 1920*a* (52), Steffensen 1927 (73), Glover 1930 (430), Davis 1933 (90), Bower 1933, 1935 and *N.A.* 1937 (807).

The coefficients in the expressions for subdivided central differences in terms of original central differences (that is, the coefficients given on page 807 of *N.A.* 1937) are tabulated *in extenso* in Jäderin 1915 (31–49). Coefficients are given, for the subdivision of the interval into any number of parts from 2 to 20, for and of all differences up to the twelfth inclusive. Every coefficient is given in three forms, as an exact vulgar fraction, as a decimal to 15 places, and logarithmically to 7 decimals.

23·4. Numerical Differentiation

In accordance with what has been said in Art. 23·0, we consider here only numerical differentiation at any point in the interval, except that derivatives at tabular points, using ordinates, are dealt with in Art. 23·48.

For derivatives:

$\left.\begin{array}{l}\text{At tabular points}\\ \text{At mid-points}\end{array}\right\}$ in terms of central differences, see Arts. 4·9322, 4·9323

At tabular points in terms of forward differences, see Art. 4·9232

At mid-points in terms of forward differences, see Art. 4·943

In terms of mixed differences, see Art. 4·951

The tables for differentiation at any point of the interval which we shall consider are contained in Oppolzer 1880, *N.A.* 1937 (Comrie), Lindow 1928, Rice 1899, Bauschinger 1901, Bauschinger–Stracke 1934, Davis 1933 and Glover 1930. Of these Oppolzer is the most extensive. Oppolzer, Comrie and Lindow consider both first and higher derivatives.

Oppolzer and Comrie cater directly for first and second differences. Notation apart, they use the same formulæ, which involve central differences of Stirling or Bessel type; and, as far as the arguments of the two authors overlap, Oppolzer's values are the logarithms of the numbers given naturally by Comrie.

Oppolzer 1880 (515–544) gives 6-decimal logarithms, with first differences, for finding first and second derivatives from differences up to the tenth inclusive. For n = fraction of interval $= \pm 0(\cdot 001)\cdot 250$ Stirling-type differences are used, and Bessel-type differences are used for $m = n - \frac{1}{2} = \pm 0(\cdot 001)\cdot 250$.

N.A. 1937 (804–805) gives to 4–6 decimals, without differences, the natural values required for finding first and second derivatives from differences up to the sixth inclusive. The arguments are $n = \pm 0(\cdot 01)\cdot 50$ for Stirling-type differences and $m = \pm 0(\cdot 01)\cdot 50$ for Bessel-type, so that either set of differences may be used anywhere in the interval. The simple formulæ for obtaining the corresponding coefficients for finding third and fourth derivatives are printed below the tables.

Both Oppolzer and Comrie remove factors, n in Stirling-type formulæ and m in Bessel-type, from coefficients of even differences in forming the first derivative and those of odd differences in forming the second derivative. Thus their values are not quite the same as those listed among Arts. 23·41–23·44 below, which are simply the derivatives with respect to n of the corresponding interpolation coefficients.

Lindow 1928 also caters directly for first and second derivatives, but does not remove the factor n or m.

23·41. First Derivatives from Gregory-Newton Differences

The following tabulate coefficients of forward differences.

Exact	2nd–7th	$n = 0(\cdot 01)1$	Δ^n	Davis 1933 (130)

Coefficients of 2nd–6th differences given as recurring decimals; that of 7th difference to 17 decimals with provision for extension.

5 dec.	2nd–5th	$n = 0(\cdot 01)1$	Δ	Rice 1899; Glover 1930 (422)

23·42. First Derivatives from Stirling Differences

The following tabulate coefficients of even and mean odd central differences. See also Art. 23·4.

5 dec.	2nd–5th	$n = 0(\cdot 01)1$	Δ	Davis 1933 (140)
5 dec.	2nd–5th	$n = 0(\cdot 01)1$	Δ	Rice 1899; Glover 1930 (424)
5; 4 dec.	3rd; 4th–6th	$\pm n = 0(\cdot 01)\cdot 25$	v^2; (Δ)	Lindow 1928 (166)

23·422. Second Derivatives from Stirling Differences

The following tabulates coefficients of even and mean odd central differences. See also Art. 23·4.

5; 4 dec.	4th; 5th, 6th	$\pm n = 0(\cdot 01)\cdot 25$	v^2; (Δ)	Lindow 1928 (168)

23·43. First Derivatives from Bessel Differences

The following tabulate coefficients of odd and mean even central differences. See also Art. 23·4.

5 dec.	2nd–5th	$n = 0(\cdot 01)1$	Δ	Davis 1933 (142)
5 dec.	2nd–5th	$n = 0(\cdot 01)1$	Δ	Rice 1899; Glover 1930 (426)
5 dec.	2nd–4th	$n = 0(\cdot 01)1$	No Δ	Bauschinger 1901 (140); Bauschinger–Stracke 1934 (187)
5; 4 dec.	3rd; 4th–6th	$n = \cdot 25(\cdot 01)\cdot 75$	v^2; Δ	Lindow 1928 (167)

23·432. Second Derivatives from Bessel Differences

The following tabulates coefficients of odd and mean even central differences. See also Art. 23·4.

5; 4 dec.	4th; 5th, 6th	$n = \cdot 25(\cdot 01)\cdot 75$	v^2; Δ	Lindow 1928 (169)

23·44. First Derivatives from Everett Differences

The following tabulate coefficients of even central differences.

10 dec.	2nd, 4th, 6th	$n = 0(\cdot 01)1$	δ^{2n}	Davis 1933 (144)
5 dec.	2nd, 4th, 6th	$n = 0(\cdot 01)1$	Δ	Glover 1930 (428)

23·48. Derivatives at Tabular Points from Ordinates

Bickley 1941 gives formulæ for all the derivatives at tabular points which can be found from 3(1)7 equidistant ordinates, and all the derivatives of the first four orders at tabular points in the cases of 9 and 11 such ordinates. If

$$\frac{w^m D^m y_p}{m!} = \frac{1}{n!} \sum_{r=0}^{n} {}_{mn}A_{pr} y_r$$

the quantities ${}_{mn}A_{pr}$, which are integral, are given for $n = 2(1)6$, $m = 1(1)n$, $p = 0(1)n$, $r = 0(1)n$, and for $n = 8$ and 10, $m = 1(1)4$, $p = 0(1)n$, $r = 0(1)n$. Error terms are also given.

The same quantities are given in Southwell 1940 (244) for $n = 6$, $m = 1(1)6$, $p = 0(1)6$ and $r = 0(1)6$.

Rutledge 1922 gives explicitly the polynomials taking given values at $x = -nh$, $-(n-1)h, \ldots, (n-1)h, nh$ for $n = 1(1)4$. The coefficients of the various powers of x enable one to write down immediately expressions for all the derivatives at a tabular point which can be obtained from 3, 5, 7 or 9 ordinates centred at the point. Crout 1940 gives coefficients for such polynomials and derivatives (up to the third inclusive) from 3, 5 or 7 ordinates.

23·5. Numerical Integration, using Ordinates

It is not our purpose to give here lists of formulæ which, like Simpson's rule, give approximate values of definite integrals in terms of ordinates. A very useful list is given in Bickley 1939; this includes expressions for integrals over ranges shorter and longer than that covered by the ordinates. Our Part II includes various works, such as Moors 1905 and Irwin 1923, dealing with the subject; others may be found in the extensive bibliography of Bennett, Milne & Bateman 1933 (38). A number of integration formulæ may be found in many textbooks, such as Whittaker & Robinson 1926. See also Pearson 1931 (xv, ccvii, 234, ccxlviii), Elderton 1938 (26), Fisher & Yates 1938 (61), David 1938 (ix–xi), Blanch & Rhodes 1943 and New York W.P.A. 1944*b*.

Here we consider chiefly several well-known types of formulæ which can be applied to any number of ordinates and for which the constants involved have been evaluated systematically up to some limit. We have not stated whether or not error terms are given, as they usually are in the more important tables.

23·51. Cotes's Formulæ

These give $\int_0^{nw} y\,dx$ in terms of the ordinates $y_0, y_1, \ldots, y_n$ at $x = 0, w, \ldots, nw$, n being the number of strips of equal width. We do not distinguish between the various trivially different ways in which the results may be given. The coefficients are all rational, and are given as follows:

$n = 1(1)20$	**W. W. Johnson 1915** (also contains historical matter)
$n = 1(1)12$	Moors 1905 (Table A)
$n = 1(1)10$	Merrifield 1880 (336, from Cotes), B.A. 1879 (50), Bertrand 1870 (333), Carr 1886 (438), Gauss 1816, Láska 1888–94 (233, 1060), Markoff 1896 (61), Steffensen 1927 (158, 166)

$n = 2(1)8, 10$	Bauschinger 1901 (140), Bauschinger–Stracke 1934 (187)
$n = 1(1)8$	Bennett, Milne & Bateman 1933 (29)
$n = 1(1)7$	Milne-Thomson 1933 (170)
$n = 1(1)6, 8, 10$	Bickley 1939
$n = 7, 9$	Blanch & Rhodes 1943
$n = 1(1)6, 8$	Whittaker & Robinson 1926 (155)

23·52. Partial-Range Formulæ

The following formulæ give $\int_0^{mw} y\,dx$ in terms of $y_0, y_1, \ldots, y_n$, where $m = 1(1)n-1$.

$n = 2(1)6, \quad m = 1(1)n-1$ Bickley 1939

The related coefficients for finding the areas of single strips from 3(1)11 equidistant ordinates have recently been tabulated as "Lagrangian integration coefficients" to 10 decimals in New York W.P.A. 1944*b* (390).

23·53. Formulæ for Forward Integration

These give $\int_0^{(n+1)w} y\,dx$ in terms of $y_0, y_1, \ldots, y_n$. They thus involve the initial but not the final ordinate of the range of integration, i.e. they are of "closed" type "on the left" and of "open" type "on the right". They are of great importance in the numerical solution of differential equations. See also Art. 23·54. Other suitable formulæ will be found in Bennett, Milne & Bateman 1933 and Levy & Baggott 1934.

$n = 1(1)5$ Bickley 1939

When $n = 2$, the formula for $\int_0^{4w} y\,dx$ is also given.

23·54. Steffensen's Formulæ

These are "open" type formulæ giving $\int_0^{nw} y\,dx$ in terms of $y_1, y_2, \ldots, y_{n-1}$. They thus use $n-1$ equidistant ordinates which divide the range into n strips, the ordinates at the ends of the range not being used. The formulæ are thus adapted to forward integration.

$n = 3(1)12$	Steffensen 1927 (158, 167)
$n = 2(1)10$	Bennett, Milne & Bateman 1933 (33)

23·55. Maclaurin's Formulæ

These express $\int_0^{nw} y\,dx$ in terms of $y_{1/2}, y_{3/2}, \ldots, y_{n-\frac{1}{2}}$, or, what amounts to the same thing, $\int_{\frac{1}{2}w}^{(n+\frac{1}{2})w} y\,dx$ in terms of $y_1, y_2, \ldots, y_n$.

$n = 1(1)10$	Moors 1905 (Table B)
$n = 1(1)5$	Milne-Thomson 1933 (199)

For the areas of the individual strips when $n = 5$, see Pearson 1931 (xv).

23·56. Filon's Formula

Filon 1929 tabulates coefficients in a special quadrature formula for integrals of the type $\int_a^b \sin kx : \psi(x)\,dx$.

23·57. Chebyshev's Formulæ

These are of the form $\int_a^b f(x)\,dx = \dfrac{b-a}{n}\{f(x_1) + f(x_2) + \ldots + f(x_n)\}$. The abscissæ $x_1, x_2, \ldots, x_n$ are determined to give maximum accuracy. The range (a, b) is usually taken to be $(-1, +1)$, $(-\frac{1}{2}, +\frac{1}{2})$ or $(0, 1)$ in stating the values of the abscissæ. The equal weighting of the ordinates is useful when these are liable to errors of measurement.

8 dec.	$n = 2(1)7, 9$	Range $-\frac{1}{2}$ to $+\frac{1}{2}$	Moors 1905 (Table F)
8 dec.	$n = 2(1)5$	Range -1 to $+1$	Milne-Thomson 1933 (180)
6 dec.	$n = 2(1)7$	Range -1 to $+1$	Chebyshev 1874

Most of Chebyshev's values are correct to only 4 decimals; that for $n = 2$ is quite wrong. They have sometimes been quoted; e.g. for $n = 5$ in Whittaker & Robinson 1926 (159).

6 dec.	$n = 2(1)5$	Range -1 to $+1$	Runge & König 1924 (273) Mader 1928 (629)
6 dec.	$n = 2(1)5$	Range $-\frac{1}{2}$ to $+\frac{1}{2}$	Willers 1923 (39)
5 dec.	$n = 2(1)7, 9$	Range -1 to $+1$	Karelitz 1935 (A 12)
5 dec.	$n = 2(1)5$	Range -1 to $+1$	Hütte 1931 (182)

23·58. Other Formulæ with Unequal Intervals

There are a number of other formulæ of the form

$$\int_a^b f(x)\,dx = A_1 f(x_1) + A_2 f(x_2) + \ldots + A_n f(x_n)$$

The formulæ of Gauss are obtained by choosing $A_1, A_2, \ldots, A_n, x_1, x_2, \ldots, x_n$ so that the formula is exact when the integrand is a polynomial of degree $(2n-1)$. They are thus very precise. Details have already been given in Art. 16·18 in connection with the Legendre polynomials.

For numerical constants relating to other formulæ with unequal intervals, see Radau 1880*a*, 1880*b*, Moors 1905 (Tables D, E, G, H), Lampe 1917 and Giraud 1924.

Burnett 1937 gives some zeros of Sonin (generalized Laguerre) polynomials, and corresponding weight factors for the evaluation of integrals of the form

$$\int_0^\infty x^m e^{-x} f(x)\,dx$$

23·59. Lagrangian Integration Polynomials

These are integrals of the Lagrangian interpolation polynomials, and may be used for integrating to any point in an interval (cf. Art. 23·6) by means of ordinates. Algebraic expressions are given in New York W.P.A. 1944*b* (386) for the cases of $3(1)11$ equidistant ordinates.

23·6. Integration and Summation, using Differences

In accordance with what has been said in Art. 23·0, this sub-section is mainly concerned with integration at any point of the interval; some miscellaneous items are added in Arts. 23·68 and 23·69.

For single integration, $\int^{x} f(x)\,dx$:

At tabular points, using central differences } see Arts. 4·9341, 4·9343
At mid-points, using central differences }
At tabular points, using forward differences, see Arts. 4·9251 (Gregory), 4·9221
At mid-points, using forward differences, see Art. 4·942.

For repeated integration, $\int^{x}\int^{x} f(x)\,dx^2$:

At tabular points, using central differences } see Arts. 4·9341, 4·9343
At mid-points, using central differences }

Lindow 1928 gives (pp. 94, 172) some forward-difference formulæ for $\iint f\,dx^2$ and (p. 96) some central-difference formulæ for $\iiint f\,dx^3$.

For double integrals, see Art. 23·7.
For Euler–Maclaurin coefficients, see Art. 4·18, also Carr 1886 (558) to $n = 5$.

23·61. Single Integrals Anywhere in the Interval

Oppolzer 1880 (546–564) gives logarithmically to 6 decimals with first differences the coefficients required in working out $\int^{a+nw} f(x)\,dx$, a being a tabular argument, from differences up to the tenth inclusive. Coefficients for Stirling differences (even and mean odd) are tabulated for $\pm n = 0(\cdot001)\cdot250$, and coefficients for Bessel differences (odd and mean even) for $\pm m = 0(\cdot001)\cdot250$, where $m = n - \frac{1}{2}$. Factors n^3 in the first case and m in the second are removed from coefficients of even differences.

Lindow 1928 (173–174) gives natural values of coefficients for Stirling differences for $\pm n = 0(\cdot01)\cdot25$ and of coefficients for Bessel differences for $\pm m = 0(\cdot01)\cdot25$. The coefficients of the first difference are to 5 decimals with v^2, those of the second to sixth differences to 4 decimals, with Δ when necessary. No factors are removed.

23·62. Repeated Integrals Anywhere in the Interval

Oppolzer 1880 (565–586) tabulates logarithmically to 6 decimals with first differences the coefficients required in working out $\iint^{a+nw} f(x)\,dx^2$ from differences up to the tenth inclusive. Coefficients of Stirling differences and Bessel differences are given for $\pm n = 0(\cdot001)\cdot250$ and $\pm m = 0(\cdot001)\cdot250$ respectively. Factors n and m respectively are removed from coefficients of odd differences.

Lindow 1928 (175–176) tabulates natural values of coefficients for Stirling and Bessel differences for $\pm n = 0(\cdot01)\cdot25$ and $\pm m = 0(\cdot01)\cdot25$ respectively. The coefficients of the function are to 5 decimals with v^2, those of the first to sixth differences to 4 decimals, with Δ when necessary. No factors are removed.

23·68. $\int_{-nw}^{nw} f(x)dx$ in Terms of $f(0)$ and its Even Central Differences

The coefficients are given as rational numbers for $n = 1(1)6$ in Merrifield 1880 (355), $\delta^{2n}f(0)$ being the highest difference considered for a given value of n.

The coefficients of $f(0)$, $\delta^2 f(0)$, ..., $\delta^{10}f(0)$ in the expressions for the average ordinate are given in decimal form for $n = 1(1)5$ in Fisher & Yates 1938 (61).

23·69. Lubbock Coefficients

These are the quantities $\dfrac{s^2-1}{12s}$, $\dfrac{s^2-1}{24s}$, $\dfrac{(s^2-1)(19s^2-1)}{720s^3}$, ... which occur as coefficients of 1st, 2nd, 3rd, ... sloping differences in Lubbock's formula.

12 dec.	1st–6th	$s = 2(1)100$	Davis 1935 (252)
10 dec.	1st–6th	$s = 2(1)11$	Steffensen 1927 (140)
8 dec.	1st–3rd	$s = 2(1)12$	Glover 1930 (432)
6 dec.	1st–6th	$s = 2(1)11$	Sprague 1874 (313)

Also 6-decimal logarithms. Both sets reproduced in King 1902 (471). Natural values reproduced in Galbrun 1924 (88).

Steffensen 1927 also gives tables of coefficients in two Lubbock-type formulæ involving central differences.

23·7. Double Integrals, etc.

See Irwin 1923, which deals mainly with the numerical evaluation of double integrals and gives further references.

See also Merrifield 1880 and Lindow 1928 (which deal also with triple integrals), Biermann 1905 (183), Whittaker & Robinson 1926 (374), Steffensen 1927 (224), Scarborough 1930 (146) and Pearson 1931 (xiv).

23·8. Curve-Fitting, etc.

We mention here a few curve-fitting tables which have come to our notice. Others doubtless exist, as curve-fitting is the subject of a considerable literature, not all of which has been available, so that the following remarks are intended only as suggestions which may be of use in some cases. A great amount of information about fitting, mainly by means of polynomials, is given in **Sasuly 1934**. Much of the information is in tabular form. We have made no attempt to index completely the contents of this comprehensive volume.

A very elementary treatment of the fitting of a number of kinds of curve is given in Running 1917. Scarborough 1930 (351) may be consulted.

On the fitting of catenaries, see Art. 10·47.

23·81. The Fitting of Ordinary Polynomials

The fitting of polynomials by least squares or similar criteria can be facilitated by special tables when the values of the dependent variable y are given with equal weight for n equidistant values of the independent variable x. The values of x are usually taken, without loss of generality, to range at unit interval from 1 to n, from 0 to $n-1$, or from $-\frac{1}{2}(n-1)$ to $+\frac{1}{2}(n-1)$; in the latter case the values of x are integral only if n is odd, consequently some writers take centred values ranging at interval 2 from $-(n-1)$ to $+(n-1)$ when n is even.

Davis 1935 gives two sets of tables for fitting polynomials by the method of least squares. The coefficients in the polynomial may be expressed as linear functions of the power moments $\Sigma x^r y$ of various orders; the coefficients in these linear expressions are functions of n, and are the quantities tabulated by Davis. (It is convenient in this article to use "moment" as meaning $\Sigma x^r y$ rather than this quantity divided by Σy.)

The first set of tables (pp. 325–359) relates to the range $-\frac{1}{2}(n-1)$ to $+\frac{1}{2}(n-1)$, the origin being thus placed centrally. It gives to 10 figures the coefficients required in fitting a linear expression $y = a_0 + a_1 x$ to any number of equidistant observations from 2 to 301, a quadratic $y = a_0 + a_1 x + a_2 x^2$ when $n = 3(1)201$, a cubic when $n = 4(1)101$, a quartic when $n = 5(1)51$, a quintic when $n = 6(1)51$, a sextic when $n = 7(1)51$ and a septimic when $n = 8(1)51$.

The second set of tables (pp. 369–385) relates to the range $x = 1(1)n$, so that the origin is placed one step outside an extreme abscissa. It gives to 10 figures the coefficients required in fitting a straight line to 2(1)240 observations, a parabola to 3(1)120 observations and a cubic to 4(1)60 observations. This set of tables is also given in Davis & Latshaw 1930. An abbreviation of the tables for straight line and parabola is given in Davis & Nelson 1935 (406–409).

Jäderin 1915 gives coefficients of power moments $\Sigma x^r y$ for fitting polynomials of degrees up to the fifth to $n = 2(1)21(2)51$ observations, the range in x being from $-\frac{1}{2}(n-1)$ to $+\frac{1}{2}(n-1)$ when n is odd and from $-(n-1)$ to $+(n-1)$ when n is even. Table 6 gives coefficients for determining $a_{0(1)5}$, and is thus similar to Davis's first set of tables, while Table 7 gives coefficients for determining the central differences at mid-range ($x = 0$).

Roeser 1920 gives tables for fitting a straight line or parabola to 5(1)31 observations. Hale 1882 gives tables for fitting $y = a_1 x + a_2 x^2$ to 5(1)26 points, including the origin at one end, and for fitting $y = a_1 x + a_2 x^2 + a_3 x^3$ to 9(1)26 points, including the origin at one end.

Kerawala 1941 gives tables for fitting polynomials of degrees up to the fifth to $n = 1(1)30$ observations, the range in x being from $-\frac{1}{2}(n-1)$ to $+\frac{1}{2}(n-1)$. The quantities tabulated are the coefficients of the ordinates in the expressions for the power-series coefficients.

Tables which we have not seen are contained in Birge & Shea 1927 (which uses central differences) and Pareto 1899. Further references are given in Davis 1935 (307).

N. Campbell 1930 gives tables for fitting polynomials by the method of zero sum.

23·82. Orthogonal Polynomials and their Fitting

The fitting of polynomial expressions may be greatly facilitated by the use of special polynomials constructed so as to be orthogonal over the discrete set of equidistant values taken by the independent variable x. These are called Chebyshev (or Gram) polynomials, and are not to be confused with the Chebyshev polynomials $C_n(x)$ and $S_n(x)$ dealt with in Section 22. We shall denote them in a general way by $T_r(x)$, or $T_r(x, n)$ if the number n of observations is to be made explicit; r is the degree of the polynomial, and can take any of the values $0(1)n-1$. The polynomials have different algebraic forms according as the ranges 1 to n, 0 to $n-1$, or $-\frac{1}{2}(n-1)$ to $+\frac{1}{2}(n-1)$ are considered, owing to the differences in the origin of x, but the numerical values for given n and r are in a definite ratio at the successive abscissæ. Each polynomial is, however, indeterminate to the extent of an arbitrary factor, until some conventional factor is adopted; Aitken 1933*a* (59) mentions that at least four different conventions have been adopted, and we shall not enter into this matter. Algebraic expressions for the various T_r as functions of x and n for the different ranges

of x are given in various places; e.g. Pareto 1899, Fisher 1936 (149), F. E. Allan, *Proc. Roy. Soc. Edin.*, **50**, 310–320, 1930, Aitken 1933*a* (67), Sasuly 1934 (35–37, 271), Davis 1935 (319–324) and C. Eisenhart (reviewing Anderson & Houseman 1942), *M.T.A.C.*, **1**, 148, 1944. Most of these give a number of further references to works dealing with orthogonal polynomials of the above kind.

Aitken 1933*a* gives (p. 60) the initial values and leading forward differences, and (p. 68) the central values and their central differences, of the various T_r for $n = 7(1)25$, $r = 0(1)5$ (also for $n = 4, 5, 6$ up to $r = 2, 3, 4$ respectively), together with ΣT_r^2, the summation being over all the abscissæ. From either set of values and differences a complete table of $T_r(x, n)$ for any of the values of r and n may easily be built up. A numerical table of all the polynomials for $n = 7$ is given in Fraser 1935 (149); see also Lidstone 1933*a* (137). Aitken 1933*a* explains the use of the tables with numerical examples. Aitken 1939 (114) may also be consulted.

The orthogonal polynomials themselves are tabulated for $n = 3(1)52$, $r = 1(1)5$, $r < n$ in Fisher & Yates 1938 (54). As with Aitken, the values are all integral; but Fisher & Yates remove the highest common factor in each case, so that the integers are as small as possible. Values of ΣT_r^2 are also given. (At $n = 39$, $r = 2$, for 496,388 read 4,496,388; Gertrude M. Cox, *M.T.A.C.*, **1**, 86, 1943.) Similar tables for $n = 3(1)104$, $r = 1(1)5$, $r < n$ are given in Anderson & Houseman 1942, the values for $n = 3(1)52$ being taken from Fisher & Yates 1938.

Numerical values useful in the calculation of fitted polynomial values by summation are given in R. A. Fisher 1936 (156), 1941 (149) and Goulden 1939 (235, 242).

Tables for approximation by orthogonal polynomials are given together with extensive theory and numerical examples in C. Jordan 1932*b* and **Sasuly 1934**.

23·83. The Fitting of Exponentials

Glover 1930 (471) gives a table for fitting $y = ar^x$ to n observed values of y corresponding to $x = 0(1)n - 1$. It consists of values of

$$M = \frac{\Sigma xy}{\Sigma y} = \frac{n}{1 - r^{-n}} - \frac{1}{1 - r^{-1}}$$

to 4 decimals without differences for $n = 2(1)40$, $r = 1(\cdot001)1\cdot1$. This table is used inversely to find r.

On the fitting of the transformed Gompertz function $a + be^{\beta x}$ and the transformed Makeham function $a + bx + ce^{\gamma x}$, see Sasuly 1934 (Chapter XII).

On the fitting of $y = ae^{\alpha x} + be^{\beta x} + ce^{\gamma x} + \ldots$, see Runge & König 1924 (231, 246), Whittaker & Robinson 1926 (369), Mader 1928 (533) and Willers 1928 (289).

Tables for fitting $y = k/(1 + e^{a+bx+cx^2+gx^3})$ are given in Bailey 1931.

23·84. The Fitting of the "Logarithmic Growth Curve"

For fitting $y = a + bx + c\log_{10} x$ to equidistant observations by the method of least squares, Holzinger 1928 (330) tabulates to 5 decimals

$$\sum_1^n (\log n), \qquad \sum_1^n (n \log n) \quad \text{and} \quad \sum_1^n (\log n)^2$$

for $n = 1(1)25$. A more extended table is given in Pearl 1923 (368).

23·85. The Fitting of Frequency Curves

In this and several following articles, we consider a few tables and other aids to the fitting of frequency curves. On methods see **Elderton 1938**, which devotes particular attention to Pearson curves.

The two volumes of *Tables for Statisticians and Biometricians* (**Pearson 1930, 1931**) contain numerous tables. Many have been indexed in various places, particularly in Section 15, which deals with the normal curve of error; we mention below a few which relate to other frequency curves; other tables in these important volumes are of a technical statistical character, and hardly fall within the scope of our *Index*. The tables in Pearson 1922, 1934 have been indexed in detail in Section 14, which may be consulted for tables of gamma and beta functions. The introductions to all four volumes contain a wealth of information.

Tables of the derivatives of the error function, Hermite polynomials, etc. are dealt with in Section 15. On the fitting of series of "Type A", see Charlier 1906, A. Fisher 1922 and Aitken 1939 (75).

23·86. Poisson Distribution

On the fitting of series of "Type B", see Charlier 1906 (22) and Aitken 1939 (76).

23·861. Individual Terms, $e^{-m}m^x/x!$

In the following tables, x takes all integral values for which the function does not vanish to the number of decimals tabulated; i.e. for each m the set of terms is complete for the number of decimals stated. Fry 1928 gives x all integral values for which the function is greater than $\cdot 10^{-6}$.

7 dec.	$m = \cdot 001(\cdot 001)\cdot 01(\cdot 01)\cdot 3,\ \cdot 4$	Molina 1942
6 dec.	$m = \cdot 5(\cdot 1)15(1)100$	
6 dec.	$m = \cdot 1(\cdot 1)15$	Soper 1914 Pearson 1930 (113)

The values for $m = \cdot 1(\cdot 1)1(1)11$ are reproduced in Darmois 1928 (308).

5 fig.	$m = \cdot 1(\cdot 1)1(1)20$	Fry 1928 (458)
4 dec.	$m = \cdot 1(\cdot 1)10$	Bortkiewicz 1898

Reprinted in Charlier 1906 (47). Soper 1914 reports last-figure errors.

23·862. Cumulative Sums, $P(c, m) = \sum_{x=c}^{\infty} e^{-m}m^x/x!$

The quantities tabulated may be the above cumulative values, or their complements $1 - P(c, m) = \sum_{x=0}^{c-1} e^{-m}m^x/x!$. In the following tables, c takes all integral values for which P is not 0 or 1 to the number of decimals stated. Fry 1928 tabulates P for all integral values of c for which P is greater than 10^{-6}.

It should be noted that Poisson sums are expressible in terms of the incomplete gamma function (see Art. 14·8), since

$$P(c, m) = \frac{1}{\Gamma(c)}\int_0^m e^{-m}m^{c-1}dm$$

7 dec.	P	$m = \cdot 001(\cdot 001)\cdot 01(\cdot 01)\cdot 3,\ \cdot 4$	Molina 1942
6 dec.	P	$m = \cdot 5(\cdot 1)15(1)100$	
5 dec.	P or $1-P$	$m = 1(1)30$	L. Whitaker 1914 Pearson 1930 (122)
5 fig.	P	$m = \cdot 1(\cdot 1)1(1)20$	Fry 1928 (463)

23·865. Inverse Values

Molina 1942 mentions tables for $P = \cdot 0001$, $\cdot 001$ and $\cdot 01$, with $m = \cdot 0001\,(\text{var.})\,928$, in Molina 1913. G. A. Campbell 1923 gives tables and graphs, including a table of m for $P = \cdot 000001, \cdot 0001, \cdot 01, \cdot 1, \cdot 25, \cdot 5, \cdot 75, \cdot 9, \cdot 99, \cdot 9999, \cdot 999999$ and $c = 1\,(1)\,101$. Tables of χ^2 (see Art. 15·7) may also be put to use for finding m, given P and c; see Fisher & Yates 1938 (or 1943), pages 1–2. Some numerical values of m are included in the table by W. L. Stevens on page 44 of the 1943 edition (only).

23·87. Pearson Curves

For an account of Pearson curves, see Elderton 1938, also Kelley 1923. Craig 1936, which we have not seen, gives a new exposition and chart.

In fitting, use may be made of the various diagrams which show the type in terms of β_1 and β_2, where in the usual notation $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$. Such diagrams are given as follows:

Up to $\beta_1 = 1{\cdot}8$, $\beta_2 = 8$; Up to $\beta_1 = 60$, $\beta_2 = 240$	Rhind 1909, Pearson 1930 (66)
Up to $\beta_1 = 2$, $\beta_2 = 7$	Kelley 1923 (127)
Up to $\beta_1 = 10$, $\beta_2 = 24$	Pearson 1916, Pearson 1931 (frontispiece)

Tables of the higher constants $\beta_3, \beta_4, \beta_5, \beta_6$, which for Pearson curves are functions of β_1 and β_2, are given in Rhind 1909 and Pearson 1930 (78), and in extended form in Yasukawa 1926, reprinted with corrections in Pearson 1931 (148).

Elderton 1903*a* and Pearson 1930 (37) contain a table of $-\log_{10}\{(1+x)e^{-x}\}$ in connection with Pearson Type III curves. For details, see Art. 10·35. Salvosa 1930 contains extensive tables (over 180 pages) relating to the same curves. The form used is

$$y = y_0(1 + \tfrac{1}{2}\alpha_3 t)^{\frac{4}{\alpha_3^2}-1} e^{-\frac{2t}{\alpha_3}}$$

the origin being at the mean. Areas, ordinates and the first six derivatives are given. Burgess 1927 (296) gives 4-decimal ordinates and 5-decimal areas of the curves

$$y = \frac{e^{-n} n^{n-\frac{1}{2}}}{(n-1)!}\left(1 + \frac{x}{\sqrt{n}}\right)^{n-1} e^{-x\sqrt{n}}$$

for $n = 4, 9, 16$ and $x = -4\,(\text{var.}) - 2\,(\cdot 1) + 2\,(\text{var.}) + 5$. The values are based on the tables in Pearson 1922 (see Art. 14·81). Ordinates and areas of the normal curve are given alongside for comparison.

Comrie 1938 gives tables of $\log_{10}(1+x^2)$ and $\tan^{-1} x$ in connection with Pearson Type IV curves. For details, see Arts. 9·32 and 9·12 respectively.

23·89. Graduation or Smoothing

On general methods of graduation, see Rhodes 1921, Whittaker & Robinson 1926 (285), Macaulay 1931 and Sasuly 1934; all of which give further references. Aitken 1926, 1933*a* and Wolfenden 1942 may also be consulted.

On the tabulation of a function from observations of its mean values over finite intervals, see Greenspan 1939.

SECTION 24

TABLES AND SCHEDULES FOR HARMONIC ANALYSIS, ETC.

24·0. Introduction

In this section we are concerned with the numerical operations involved in harmonic analysis and synthesis. We do not particularly consider periodogram analysis, but some of the aids to computation which we mention will be of assistance in that connection, and it is to be noted that a number of the tables in Stumpff 1939 were inserted with a view to periodogram calculations. In accordance with our policy throughout the *Index*, we do not consider graphical and mechanical methods; information about the latter may be found in Stumpff 1937 and in a number of the works on machines and instruments listed in the Introduction—for example, in Horsburgh 1914.

Harmonic analysis and related topics are dealt with in special works such as Carse & Shearer 1915, Eagle 1925, Stumpff 1927 and **Stumpff 1937**. The introductory text of **Pollak 1929** and the exercises in **Stumpff 1939** may also be consulted. Stumpff 1937 contains an extensive bibliography (to which Stumpff 1939 adds a small supplement), while Stumpff 1927 and Pollak 1929 also give a notable number of references. A certain amount of information may be found in less specialized works, such as Whittaker & Robinson 1926, Brunt 1931, Runge & König 1924, Horsburgh 1914 (Carse & Urquhart). See also Chapter XVI of S. Chapman & J. Bartels, *Geomagnetism*, Oxford 1940. Burkhardt 1904 gives a valuable account of older work.

Much the largest tabular works with which we are acquainted are **Pollak 1926, Pollak 1929** and **Stumpff 1939**, all of which are considerably more extensive than the well-known Turner 1913. Each of Pollak's works consists principally of one very large table, while Stumpff gives a collection of tables of small or medium size, some of which have already been indexed in earlier sections. Besides tables, we consider schedules, etc. for harmonic analysis. Indeed, there is no clear-cut division between aids designed to lighten the arithmetic of harmonic analysis directly and those designed to systematize the arrangement, although some works, such as Pollak 1926, evidently belong to the former class, and others, such as the schedules in Whittaker & Robinson 1926, belong to the latter.

We shall be concerned mainly with the exact representation by numerical Fourier analysis of a number of values of a periodic function given at equal intervals throughout the period. A great diversity of notation exists; we adopt the following.

We shall suppose that s ordinates y_r, where $r = 0(1)s-1$, corresponding to $x_r = 2r\pi/s = ra$, where $a = 2\pi/s$, are to be represented by a Fourier series

$$a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$$

where, if s is odd, n takes values from 1 to $\frac{1}{2}(s-1)$, and, if s is even, n takes values from 1 to $\frac{1}{2}s$ in the cosine terms and from 1 to $\frac{1}{2}s-1$ in the sine terms. Thus s is the number of ordinates, r is the ordinate number, n is the wave or harmonic number, and $a = 2\pi/s$ is the phase interval.

We then know that

$$a_0 = \frac{1}{s}\sum_{r=0}^{s-1} y_r \qquad a_n = \frac{2}{s}\sum_{r=0}^{s-1} y_r \cos rn\alpha \qquad b_n = \frac{2}{s}\sum_{r=0}^{s-1} y_r \sin rn\alpha$$

except that, when s is even,

$$a_{\frac{1}{2}s} = \frac{1}{s}\sum_{r=0}^{s-1} y_r \cos r\pi = \frac{1}{s}\sum_{r=0}^{s-1} (-)^r y_r$$

The general arrangement of the section is as follows:

24·1. Angles

For values of $\alpha = 2\pi/s$, see Art. 7·851.

24·11. $r\alpha = 2r\pi/s$

Pollak 1929 gives all the angles (reduced when necessary to the first quadrant by altering $\pm r\alpha$ by multiples of 90°) which occur for $s = 3(1)40$, to 0″·01 and also to 8 decimals of 1° (correct to about 6 decimals of 1°). Pollak 1926 gives the same values to 0″·01 only.

Vogel 1943 gives all the values of $2r\pi/s$ for $s = 2(1)200$ to 0″·001.

Stumpff 1939 gives values for $s = 2(1)40$ to 0°·01.

24·2. Cosines and Sines

24·21. Cosines and Sines of $r\alpha = 2r\pi/s$

These are the main requisite in numerical harmonic analysis. With them the computer can quickly make his own schedule if he so desires. The values may be given, with signs attached, for $r = 0(1)s-1$, or only the various numerical values which occur may be given. Most schedules incorporate values of the relevant cosines and sines; we consider here mainly tables which give them systematically up to some limit in s.

Pollak 1929 gives the whole set of cosines and sines to 7 decimals with signs for $s = 3(1)40$. For each s, r goes from 0 to $s-1$ (case $n = 1$ in Art. 24·23). The same numbers also suffice for various higher values of s, for instance, all even values between 40 and 80 except multiples of 8. The various cosines and sines for $s = 48, 64, 72, 80$ may be read without interpolation from ordinary trigonometrical tables; we have not encountered any table of the cosines and sines for $s = 56$.

Pollak 1926 gives the same numbers to 4 decimals (see also Art. 24·3). Five-decimal logarithms of the cosines and sines are also given. For $s = 3(1)24$, 7-decimal values of the cosines and sines are given (in the headings of Table II).

Stumpff 1939 gives the same numbers to 3 decimals.

Turner 1913 gives the values to 3 decimals for $s = 9(1)21$.

Fritsche 1903 gives 5-decimal logarithms of the cosines and sines for $s = 52, 73$.

24·22. $\frac{4}{s}\cos r\alpha$ and $\frac{4}{s}\sin r\alpha$

Eagle 1925 (90) gives a small table containing to 4 decimals all the numbers, apart from sign, occurring in the above quantities when $s = 8(4)48$.

24·23. Cosines and Sines of $rn\alpha = 2rn\pi/s$

These are all contained in the cosines and sines of $r\alpha = 2r\pi/s$ considered in Art. 24·21, but they occur in various orders for various values of n. **Pollak 1929** gives them explicitly to 7 decimals, with signs, for $s = 3(1)40$, $n = 1(1)\frac{1}{2}s$ or $1(1)\frac{1}{2}s-\frac{1}{2}$, $r = 0(1)s-1$. The arrangement is first in order of s; for each s a separate table is given for the various values of n; in each table, $r = 0(1)s-1$. The values of r are unfortunately not printed, but the columns of cosines and sines are to be multiplied by $y_0, y_1, \ldots, y_{s-1}$, the ordinates being always used in this same order. This 98-page table and an ordinary calculating machine provide a perfectly uniform and unsophisticated process for dealing with any harmonic for any number of ordinates from 3 to 40. For those without a calculating machine, there is printed against every cosine and sine the number of a table; if $s = 3(1)24, 26, 28, 30, 34, 36, 38$, the number always refers to a table in Pollak 1926 giving a thousand multiples (see Art. 24·3).

It may be added that **Stumpff 1939** changes the order of the ordinates instead of the order of the cosines and sines in finding the various harmonics. Against the cosines and sines of $r\alpha$ mentioned in Art. 24·21 are given the numbers of the appropriate ordinates, if n is prime to s; if n and s have a common factor, grouping of ordinates according to the method of Buys-Ballot is used.

24·3. Multiples of Cosines and Sines

These are useful in forming $\Sigma y_r \cos rn\alpha$ and similar expressions without a calculating machine.

Pollak 1926 consists mainly of a 240-page table giving the first thousand multiples to 3 decimals of the 120 different numerical values (excluding 0, $\frac{1}{2}$, 1, and apart from sign) of cosines and sines which occur when $s = 3(1)24$; these also suffice for various higher values of s, for example, all even values less than 48 except 32 and 40, which are multiples of 8. The volume is about the size of "Crelle", which served as a model in arranging the table.

Turner 1913 gives to the nearest integer the first hundred multiples of the cosines and sines occurring when $s = 9(1)21$.

The ordinary traverse tables listed in Art. 9·21 may be of use for certain values of s. Thus multiples of the sines of 15°, (30°), 45°, 60° and 75° suffice for harmonic analysis with 3, 6, 8, 12 or 24 ordinates, and many tables give several hundred multiples of these sines.

Stumpff 1939 (26) gives to 1 decimal the first 1000 multiples of the cosines of 15°, $22\frac{1}{2}$°, 30°, 45°, $67\frac{1}{2}$° and 75°, as required when $s = 3, 6, 8, 12, 16$ or 24. These are followed by a table of $1(1)100 \times \sin 1°(1°)90°$ to 2 decimals.

Jelinek 1910 gives 3-decimal tables of 100(1)999 times the sines of 15°, 45°, 60° and 75°. The tables appear to have originated in Kämtz 1860. Strachey 1887*b* gives 0(·001)1·009 times the same sines to 4 decimals. Baird 1886 gives to 6 decimals 0(·001)1 times the 5-decimal sines of 15°, 45°, 60° and 75°. Naturally the values are useful to only about 5 decimals.

Grover 1913 (639) gives $1(1)100 \times \sin 10°(10°)80°$, 15°, 45°, 75° to 3 decimals (the fourth decimal is given when it is 5).

24·4. Schedules, etc.

We give below a number of references to places in which schemes, schedules, etc. for harmonic analysis with a definite number of ordinates are to be found. The skilled computer may prefer to make his own scheme, but the following list may prove acceptable to many.

The list has been formed by supplementing a number of references to modern works with a few taken from the list in Burkhardt 1904 (686), which should be consulted on older work. It has already been noted that Pollak and Stumpff provide systematically for any number of ordinates not exceeding 40. Consequently we have not attempted to make the list complete for such values of s, but we have not omitted to mention any scheme, etc. for $s > 40$ which has come to our notice. Enormous numbers of 12 and 24 ordinate schemes must exist, including many in meteorological and tidal literature, so that we have mentioned only a very limited number. On tidal analysis see Sir George Darwin, *Scientific Papers*, Vol. 1, Cambridge 1907, and A. T. Doodson, *Phil. Trans.*, (A) **227**, 223–279, 1928.

It should be noted that harmonic analysis with an even number, s, of ordinates can be performed by making two analyses, of $y_0, y_2, y_4, \ldots$ and of $y_1, y_3, y_5, \ldots$, with half that number of ordinates. This is very simple if s is not divisible by 4, and almost as simple if s is divisible by 4 but not by 8. On such or similar cases of decomposition into partial series, see Pollak 1929 (12*), Stumpff 1939 (142, 144), Stumpff 1937 (66), Runge & König 1924 (226), and Runge & Emde 1913.

Most of the following items relate merely to places in which some explicit arrangement for the value of s in question is suggested. But the Edinburgh 12 and 24 ordinate schedules as given in Whittaker & Robinson 1926 may be purchased as blank forms in bulk. So may the 24 ordinate transparent forms of Zipperer 1922; the error in the underlay for a_4, mentioned on the errata slip, must be corrected. The 24 ordinate "masks" of Terebesi 1930 provide for an unlimited number of analyses (and syntheses).

On the systematic construction of schedules, see Bruns 1903 (119) and Taylor 1915. For a method involving non-equidistant ordinates, see S. P. Thompson 1911.

$s = 9$	Fritsche 1903
$s = 10$	Running 1917 (80), Fritsche 1903
$s = 11$	Stumpff 1937 (62)
$s = 12$	Pipes 1940, Hussmann 1938, Scarborough 1930 (388), Willers 1928 (283), Whittaker & Robinson 1926 (267), Runge & König 1924 (218), Plummer 1918 (337), Running 1917 (81), Carse & Shearer 1915 (16), Grover 1913 (645), Runge 1903, and many others
$s = 16$	Andoyer 1923 (242), Running 1917 (82)
$s = 18$	Pernot 1918 (331), Grover 1913 (646), Fritsche 1903
$s = 20$	Eagle 1925 (84, 86), Lohmann 1921 (masks), Running 1917 (84)
$s = 24$	Hussmann 1938, Terebesi 1930, Scarborough 1930 (398), Willers 1928 (286), Whittaker & Robinson 1926 (273), Zipperer 1922, Running 1917 (85), Carse & Shearer 1915 (21), Taylor 1915, Horsburgh 1914 (234), Runge & Emde 1913, Runge 1905, and many others
$s = 30$	Kunzek 1830 (according to Burkhardt 1904)
$s = 32$	Weyer 1888, Hansen 1831 (50)
$s = 36$	Hussmann 1938, Pernot 1918 (328), Runge 1903, Fritsche 1903
$s = 40$	Hermann 1890 (description of masks)
$s = 42$	Stumpff 1939 (20)
$s = 45$	Stumpff 1939 (23)

$s = 48$	Stumpff 1939 (22)
$s = 52$	Fritsche 1903
$s = 60$	Stumpff 1939 (20)
$s = 72$	Stumpff 1939 (21), Hussmann 1938, Pernot 1918 (325), Strachey 1887*a* (up to a_4, b_4)
$s = 73$	Fritsche 1903, Strachey 1887*a*, *b* (up to a_4, b_4; approximate only)
$s = 121$	Stumpff 1939 (112, 149, Darwin–Börgen scheme; approximate only)

24·5. Special Cases

There are three special cases of importance. (1) Harmonic analysis of even functions; sine terms absent. (2) Harmonic analysis of odd functions; only sine terms present. (3) Harmonic analysis of "odd-harmonic" functions, such that $f(x+\pi) = -f(x)$. This case occurs in connection with alternating currents. Only odd harmonics are present, and the mean value of the function is zero.

Ordinary schedules for general analysis may in some cases be used fairly conveniently, but the following relate explicitly to the various particular cases.

24·51. Even Functions

If the interval is $2\pi/s$, the number of given ordinates is $\frac{1}{2}s+1$, the first and last corresponding to $x = 0$ and $x = \pi$ respectively.

s	Ordinates	Interval	
64	33	5° 37½′	Hansen 1835 (339)
32	17	11° 15′	Leverrier 1855 (145)
20	11	18°	Eagle 1925 (86)
16	9	22° 30′	Leverrier 1855 (144)
12	7	30°	Brown & Brouwer 1932 (147) Brown & Shook 1933 (296)
8	5	45°	Brown & Brouwer 1932 (146) Brown & Shook 1933 (295)
8	5	45°	Leverrier 1855 (143)
6	4	60°	Brown & Brouwer 1932 (146) Brown & Shook 1933 (294)

Means of using value at $x = 90°$ is also provided.

24·52. Odd Functions

If the interval is $2\pi/s$, the number of given ordinates (other than the zero ordinates at $x = 0$ and $x = \pi$) is $\frac{1}{2}s-1$.

s	Ordinates	Interval	
36	17	10°	Gauss 1866 (325)
20	9	18°	Eagle 1925 (84)
32	15	11° 15′	Leverrier 1855 (137)
16	7	22° 30′	Leverrier 1855 (137)
8	3	45°	Leverrier 1855 (137)

Schedules for $s = 8, 12$ in Brown & Brouwer 1932 and in Brown & Shook 1933 make use of the derivatives of the function at $x = 0$ and $x = \pi$, where the function itself vanishes.

24·53. Odd-harmonic Functions

If the interval is $2\pi/s$, where s is an even integer, the number of ordinates given is in general $\frac{1}{2}s$, and they belong to $x = 0, 2\pi/s, \ldots, (s-2)\pi/s$; but if the origin is taken at a zero of the function, only $\frac{1}{2}s - 1$ other ordinates are used.

s	Ordinates	Interval	
72	36	5°	Pernot 1918 (326)
36	18	10°	Martens 1911 Grover 1913 (581) Pernot 1918 (329)
28	13	$\pi/14$	Eagle 1925 (88)
24	12	15°	Grover 1913 (580) Hütte 1920 (128)
24	11	15°	S. P. Thompson 1905
12	6	30°	Grover 1913 (579)
12	5	30°	S. P. Thompson 1905

Reproduced in Horsburgh 1914 (245, Carse & Urquhart).

24·6. Double Fourier Series

On double harmonic analysis, i.e. the analysis of functions periodic in each of two variables, see Brown & Brouwer 1932 (148), where a numerical example is given, or Brown & Shook 1933 (298). See also Brown 1930. Leverrier 1855 (147) gives a numerical example with 16×16 ordinates. Hansen 1831 (50) gives a scheme for 32×16 ordinates.

Double Fourier series are also of importance in crystallography; references will be found in the papers of Beevers and Lipson.

24·7. Harmonic Synthesis

Use might be found for the 5-decimal tables of $\cos nx$ and $\sin nx$ for $n = 1(1)8$, $x = 0(1°)90°$ given in Hayashi 1930*b* (54) and Prévost 1933. Prévost gives first differences, but Hayashi's arrangement is probably more convenient, though the column for $\cos x$ needs to be filled in.

Buerger 1941 tabulates $\cos 2\pi nx$ and $\sin 2\pi nx$ to 3 decimals for $n = 1(1)30$, $x = 0(\cdot001)1$.

On the general topic of harmonic synthesis, see Runge & König 1924 (223). Stumpff 1937 (70) gives a scheme for synthesis when $s = 12$.

A number of the schedules mentioned in Art. 24·4 (e.g. those of Hussmann 1938) are designed so as to be usable for synthesis as well as analysis.

On an inexpensive form of apparatus for synthesis, see Beevers & Lipson 1934 and Lipson & Beevers 1936. They use strips giving to the nearest integer the first 99 multiples of $\cos nx$ and $\sin nx$, where $n = 1(1)20$, $x = 0(6°)90°$.

On a proposed special machine, see Beevers 1939. Harmonic synthesis may be performed on existing punched-card machines (Hollerith, etc.)—see Comrie 1932*c*. See also Robertson 1936.

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The following tables have been prepared by the Mathematical Tables Project of the Work Projects Administration for the City of New York, under the technical direction of Dr. A. N. Lowan. For other work of the same origin, see the papers by Lowan and collaborators given in this Bibliography. A large number of tables in progress are also mentioned in our Part I. (In view of recent changes of organization, it would perhaps have been better if we had referred to New York M.T.P., but we have not thought it desirable to try to alter our mode of reference at a late stage in the printing of our *Index*.)

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1940*a* *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments.* (Our copy has 1939 on the title-page and 1940 on the binding.)
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1941*a* *Table of Natural Logarithms.* Vol. I. Logarithms of the integers from 1 to 50,000.
1941*b* *Table of Natural Logarithms.* Vol. II. Logarithms of the integers from 50,000 to 100,000.
1941*c* *Table of Natural Logarithms.* Vol. III. Logarithms of the decimal numbers from 0·0001 to 5·0000.
1941*d* *Table of Natural Logarithms.* Vol. IV. Logarithms of the decimal numbers from 5·0000 to 10·0000.
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1941*f* *Miscellaneous Physical Tables.* Planck's Radiation Functions and Electronic Functions. (The radiation functions are reproduced from Lowan and Blanch 1940.)
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INDEX TO PART I

The numbers refer to the articles, and the notation is strictly decimal. The integral part is the number of the section in which the article occurs; the decimal parts increase in magnitude as a given section progresses. Thus Art. 7·11 occurs much earlier than Art. 7·9 in Section 7. All or some decimals may be suppressed; thus 19 denotes that a topic is dealt with in many or all articles in Section 19, and 9·1 may be used for a topic treated in Arts. 9·1, 9·11, 9·12, . . ., 9·19.

A

B

C

D

E

F

G

CORRIGENDUM

The statement near the end of Art. 4·6111 on page 57 is incorrect. In fact, the table of S_n in Glaisher 1914*b* is original only for $n > 34$, the values for $n < 34$ being taken from Stieltjes 1887. A check was applied by Glaisher to the table as a whole; this could not find errors of only 2 or 3 units in the final digit, so that Stieltjes values are, to this extent, incompletely checked. A comparison with Peters & Stein 1922 shows several discrepancies of 2 units in the last, 32nd, decimal in the early part of the table, but none exceeding 2 units, and none for $n > 34$. We are indebted to Professor R. C. Archibald for pointing this out to us.